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ELSEVIER

Differential Geometry and its Applications 18 (2003) 271–283

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

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On the structure of the moduli of jets of G-structures with a linear connection ☆

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Received 17 September 1999; received in revised form 12 September 2001

Communicated by A.M. Vinogradov

Abstract

The moduli space of jets of G-structures admitting a canonical linear connection is shown to be isomorphic to the quotient by G of a natural G-module.

MSC: primary 53A55; secondary 53B30

Keywords: Canonical linear connection; Differential invariant; G-structure; Jet bundles; Moduli of G-structures

1. Introduction

Given a smooth n-dimensional manifold M and a closed subgroup $G \subset GL(n, \mathbb{R})$, a G-structure on M is a reduced bundle $P(M, G)$, with structure group G, of the bundle of linear frames $\pi : FM \to M$. The main types of geometries correspond to different choices of G. For example, there is a one-to-one correspondence between the set of Riemannian metrics on M and the set of $O(n)$-structures on M. Analogously, almost Hermitian geometries correspond to $U(n/2)$-structures, almost symplectic geometries to $Sp(n/2)$-structures, and so on.

☆ Partially supported by DGICYT, Spain, under grant PB98-533.

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0926-2245/02/$ – see front matter
doi:10.1016/S0926-2245(02)00162-6

As it is well known (cf. [5, I.1]), G-structures on M are in one-to-one correspondence with smooth sections $s \in \Gamma(FM/G)$ of the quotient bundle $\bar{\pi}: FM/G \to M$. If $\pi_G: FM \to FM/G$ denotes the natural projection, then the fibre over a point $x \in M$ of the G-structure $P_s \to M$ associated to $s \in \Gamma(FM/G)$ is given by $(P_s)_x = \{u = (X_1, \ldots, X_n) \in F_x M \mid \pi_G(u) = s(x)\}$.

The group of diffeomorphisms of M acts in a natural way on G-structures as follows. Each diffeomorphism $f: M \to M'$ defines a principal bundle isomorphism $\tilde{f}: FM \to FM'$, $\tilde{f}(X_1, \ldots, X_n) = (f_\star(X_1), \ldots, f_\star(X_n))$ (cf. [6, VI.1]), and $\tilde{f}$ induces $\bar{f}: FM/G \to FM'/G$ determined by the relation $\bar{f} \circ \pi_G = \pi_G \circ \tilde{f}$. The action of $\mathrm{Diff}(M)$ on $\Gamma(FM/G)$ is defined by $f \cdot s = \bar{f} \circ s \circ f^{-1}$, where $f \in \mathrm{Diff}(M)$, and $s \in \Gamma(FM/G)$. Two G-structures s and s' are said to be equivalent if they are related by a diffeomorphism $f \in \mathrm{Diff}(M)$, which amounts to the fact that $\tilde{f}(P_s) = P_{s'}$. The quotient $\mathfrak{M}_G(M) = \Gamma(FM/G)/\mathrm{Diff}(M)$ is called the *moduli space of G-structures* on M. The description of this space is a basic problem in Differential Geometry, which in general requires the use of topological methods.

The same problem can be stated only locally, its solution being a prerequisite for the solution of the global problem. Two G-structures are said to be locally equivalent at points $x \in M$ and $x' \in M'$ if there exist open neighborhoods $\mathcal{U}$ of x and $\mathcal{U}'$ of x' such that the restricted G-structures on $\mathcal{U}$ and $\mathcal{U}'$ are equivalent by a diffeomorphism which maps x to x'.

In order to study the local equivalence problem of G-structures up to a finite order, it is natural to introduce the spaces $J^r(FM/G)$ of jets of G-structures. The action of $\mathrm{Diff}(M)$ on $\Gamma(FM/G)$ induces a natural action of the groupoid $J^{r+1}_{\mathrm{inv}}(M, M)$ of $(r+1)$-jets of local diffeomorphisms of M on the space $J^r(FM/G)$ as follows: $(j^{r+1}_x f) \cdot (j^r_x s) = j^r_{f(x)}(f \cdot s)$, with $j^{r+1}_x f \in J^{r+1}_{\mathrm{inv}}(M, M)$, $j^r_x s \in J^r(FM/G)$.

The quotient $\mathfrak{M}^r_G(M) = J^r(FM/G)/J^{r+1}_{\mathrm{inv}}(M, M)$ is called the *moduli space of r-jets of G-structures* on M. There are natural projections $\mathfrak{M}^r_G(M) \to \mathfrak{M}^k_G(M)$, $r \geqslant k$, and one can define the moduli space of jets of G-structures as the projective limit $\mathfrak{M}^\infty_G(M) = \varprojlim \mathfrak{M}^r_G(M)$.

The local equivalence problem for analytic G-structures can be reduced to the study of these moduli spaces.

The aim of this paper is to describe, to some extent, the structure of the moduli spaces of jets of G-structures in the particular case of $G \subset GL(n, \mathbb{R})$ being such that each G-structure has a canonical adapted linear connection, as defined in Section 2. We will prove that each moduli space $\mathfrak{M}^r_G(M)$ is canonically isomorphic to the quotient by G of a G-module $\mathbf{S}^r$, i.e., a real finite-dimensional vector space $\mathbf{S}^r$ endowed with a linear action of G. More precisely, we prove the following result:

Theorem 1.1. *Let M be a n-dimensional smooth manifold. Assume that $G \subset GL(n, \mathbb{R})$ is a closed subgroup such that $GL(n, \mathbb{R})/G$ is reductive and that each G-structure admits an associated canonical linear connection. Then, there exist a family of G-modules $\mathbf{S}^r$ and homomorphisms $\mathbf{S}^r \to \mathbf{S}^k$, $r \geqslant k$, such that each space $\mathfrak{M}^r_G(M)$ of r-jets of G-structures is canonically isomorphic to the quotient $\mathbf{S}^r/G$. The moduli space $\mathfrak{M}^\infty_G(M)$ is then canonically isomorphic to the quotient $\mathbf{S}^\infty/G$, where $\mathbf{S}^\infty = \varprojlim \mathbf{S}^r$.*

We remark that the condition of $GL(n, \mathbb{R})/G$ being reductive, (which means that the Lie algebra $\mathfrak{g}$ admits an Ad_G-invariant supplementary $\mathfrak{f}$ in $\mathfrak{gl}(n, \mathbb{R})$) is not essential and it is included only because it simplifies the exposition.

As an application of Theorem 1.1, we will confirm that the Poincaré series of such a moduli space is a rational function (see [1]) and we will compute it explicitly for some particular choices of G.

The moduli spaces of jets of geometrical structures, and in particular of G-structures, were first introduced and studied by A.M. Vinogradov. A description of the moduli spaces of jets of homogeneous geometrical structures, under the additional assumption of the existence of $n = \dim M$ independent scalar differential invariants, can be found in [12]. Our result represents an alternative approach which can simplify the use of moduli spaces in some respects. For example, one can define characteristic classes of G-structures as follows [11,12]. Let $\mathfrak{M}^\infty_G(M)_{\mathrm{reg}}$ denote the smooth regular part of $\mathfrak{M}^\infty_G(M)$. Given a G-structure $s \in \Gamma(FM/G)$ it is possible to define, using a suitable perturbation argument, a map $\bar{s} : M \to \mathfrak{M}^\infty_G(M)_{\mathrm{reg}}$ which is well defined up to a homotopy. This gives a well-defined homomorphism $\bar{s}^\star : H^\bullet(\mathfrak{M}^\infty_G(M)_{\mathrm{reg}}) \to H^\bullet(M)$ between the cohomology rings, whose image is the set of characteristic classes of the G-structure. In a forthcoming paper, we will show how the isomorphism $\mathfrak{M}^\infty_G(M) \approx \mathbf{S}^\infty/G$ can be used to compute these characteristic classes in specific examples.

2. Preliminaries

In this section we state some previous results about jet bundles of G-structures and the corresponding moduli spaces. We also introduce the notion of a canonical linear connection associated to a G-structure, and we obtain certain conditions on G guaranteeing the existence of such a connection. From now on, we set $V = \mathbb{R}^n$.

2.1. A simpler description of the moduli spaces

In this subsection we only need to assume that $G \subset GL(n, \mathbb{R})$ is a closed subgroup.

First of all, we must point out that, due to the fact that $\mathrm{Diff}(M)$ acts transitively on M, the description of the moduli spaces is a local problem and it does not depend on the base manifold M.

Let us denote by $\mathfrak{G}^{r+1}_0 \subset J^{r+1}_{\mathrm{inv}}(V, V)$ the Lie group of $(r + 1)$-jets of local diffeomorphisms of V which leave the origin $0 \in V$ fixed. The restriction to $\mathfrak{G}^{r+1}_0$ of the action of $J^{r+1}_{\mathrm{inv}}(V, V)$ on $J^r(FV/G)$ defines an action of $\mathfrak{G}^{r+1}_0$ on $J^r_0(FV/G)$, and the following lemma holds (see, e.g., [12]).

Lemma 2.1. *There exists a canonical identification* $\mathfrak{M}^r_G(M) \cong J^r_0(FV/G)/\mathfrak{G}^{r+1}_0$.

Proof. For each $[j^r_x s] \in \mathfrak{M}^r_G(M)$, let us choose a chart $\varphi : \mathcal{U} \subset M \to V$, centered at $x \in M$, and define the element $(j^{r+1}_x \varphi) \cdot (j^r_x s) = j^r_0(\bar{\varphi} \circ s \circ \varphi^{-1}) \in J^r_0(FV/G)$. If we take another representative $(j^{r+1}_x f) \cdot (j^r_x s)$ of $[j^r_x s]$, where $j^{r+1}_x f \in J^{r+1}_{\mathrm{inv}}(M, M)$, and a chart $(\mathcal{U}', \varphi')$ centered at $f(x) \in M$, we have:

$$\left(j^{r+1}_{f(x)}\varphi'\right) \cdot \left(\left(j^{r+1}_x f\right) \cdot \left(j^r_x s\right)\right) = \left(j^{r+1}_0\left(\varphi' \circ f \circ \varphi^{-1}\right)\right) \cdot \left(\left(j^{r+1}_x \varphi\right) \cdot \left(j^r_x s\right)\right).$$

This means that the element of $J^r_0(FV/G)$ assigned to $[j^r_x s]$ in this way is determined up to a $(r + 1)$-jet of the form $j^{r+1}_0(\varphi' \circ f \circ \varphi^{-1})$, which belongs to $\mathfrak{G}^{r+1}_0$. Hence there is a well defined map $\mathfrak{M}^r_G(M) \longrightarrow J^r_0(FV/G)/\mathfrak{G}^{r+1}_0$.

In order to find the inverse of this map, let us consider any class $[j^r_0 t] \in J^r_0(FV/G)/\mathfrak{G}^{r+1}_0$ and take a chart $\varphi : \mathcal{U} \subset M \to V$ centered at $x \in M$. Then, the class $[(j^{r+1}_0 \varphi^{-1}) \cdot (j^r_0 t)] \in J^r(FM/G)/J^{r+1}_{\mathrm{inv}}(M, M)$ is well-defined, and it is easily seen that the map $[j^r_0 t] \mapsto [(j^{r+1}_0 \varphi^{-1}) \cdot (j^r_0 t)]$ is the desired inverse $J^r_0(FV/G)/\mathfrak{G}^{r+1}_0 \to \mathfrak{M}^r_G(M)$. $\square$

Now, notice that $\mathfrak{G}_0^1 \cong GL(n,\mathbb{R})$ acts transitively on $J_0^0(FV/G) \cong F_0V/G$, so that one can further simplify the description of the moduli space by fixing the target $s(0)$. More precisely, let us denote by $u^0 : V \to T_0V$ the canonical frame and by $[u^0] = \pi_G(u^0)$ its projection onto F_0V/G. For any class $[j_0^r s] \in J_0^r(FV/G)/\mathfrak{G}_0^{r+1}$, let us take $u \in \pi_G^{-1}(s(0))$ and consider the linear isomorphism $f = u^{-1} \circ u^0 : V \to V$. Then, $\tilde{f}(u) = u^0$ and $(f \cdot s)(0) = \bar{f}(s(0)) = \bar{f}(\pi_G(u)) = \pi_G(\tilde{f}(u)) = [u^0]$. Therefore, any class contains a representative in the closed submanifold:

$$J_0^r(FV/G)_{[u^0]} = \left\{ j_0^r s \in J_0^r(FV/G) \mid s(0) = [u^0] \right\}.$$

Moreover, any two r-jets in $J_0^r(FV/G)_{[u^0]}$ belong to the same orbit if and only if they are related by a $j_0^{r+1} f \in \mathfrak{G}_0^{r+1}$ such that the usual differential $Df(0)$ of the (local) diffeomorphism f at $0 \in V$, belongs to G. For if $s(0) = [u^0]$, then $(f \cdot s)(0) = \bar{f}(s(0)) = \pi_G(\tilde{f}(u^0))$ equals $[u^0]$ if and only if there exists $g \in G$ such that $\tilde{f}(u^0) = u^0 \circ g$, i.e., if and only if $Df(0) = (u^0)^{-1} \circ \tilde{f}(u^0)$ belongs to G.

Thus, denoting by $q_k^r : \mathfrak{G}_0^r \to \mathfrak{G}_0^k$, $r \geqslant k$, the natural projections, and identifying G with its image under the isomorphism $GL(n,\mathbb{R}) \cong \mathfrak{G}_0^1$, there is a canonical identification

$$\mathfrak{M}_G^r(M) \cong \frac{J_0^r(FV/G)_{[u^0]}}{(q_1^{r+1})^{-1}(G)}.$$

Now, let us consider the subgroup $G^{r+1} = \{ j_0^{r+1} g \in \mathfrak{G}_0^{r+1} \mid g \in G \}$ of $(q_1^{r+1})^{-1}(G)$. The homomorphism $G \to G^{r+1}$ defined by $g \mapsto j_0^{r+1} g$ splits the projection $(q_1^{r+1})^{-1}(G) \to G$, which is a surjective group homomorphism. So, $(q_1^{r+1})^{-1}(G)$ is a semidirect product of G^{r+1} and the normal subgroup $\ker q_1^{r+1}$.

It is clear that there is a well-defined action of $G^{r+1} \cong G$ on the quotient $\mathbf{S}^r = J_0^r(FV/G)_{[u^0]}/\ker q_1^{r+1}$, and this leads to a canonical identification $\mathfrak{M}_G^r(M) \cong \mathbf{S}^r/G$. The projections $\bar{\pi}_k^r : J_0^r(FV/G)_{[u^0]} \to J_0^k(FV/G)_{[u^0]}$ induce projections $\mathbf{S}^r \to \mathbf{S}^k$, $r \geqslant k$, which allow to define $\mathbf{S}^\infty$ as the projective limit of the family $\{\mathbf{S}^r\}$. The action of G induces an action on the limit $\mathbf{S}^\infty$ in a natural way, and the following proposition holds.

Proposition 2.2. *For any $r \in \mathbb{N} \cup \{\infty\}$, there is a natural identification between the moduli space $\mathfrak{M}_G^r(M)$ and the quotient $\mathbf{S}^r/G$.*

2.2. The bundles $\bar{\pi}_r^{r+1} : J_0^{r+1}(FV/G)_{[u^0]} \to J_0^r(FV/G)_{[u^0]}$

Let us denote by F the quotient $GL(n,\mathbb{R})/G$, by $q : GL(n,\mathbb{R}) \to F$ the natural projection and by I the identity transformation of V. In this subsection we assume that $GL(n,\mathbb{R})/G$ is reductive. As previously mentioned, this condition is not essential, but it considerably simplifies the exposition.

We have an isomorphism $(\pi,\phi) : FV \to V \times GL(n,\mathbb{R})$ defined as follows. If $\sigma_0 : V \to FV$ is the standard linear frame, $\sigma_0(x) = ((\partial/\partial x^1)_x, \ldots, (\partial/\partial x^n)_x)$, so that $\sigma_0(0) = u^0$, then we define $\phi(u)$ as the unique matrix such that $u = \sigma_0(\pi(u)) \cdot \phi(u)$. The isomorphism (π,ϕ) induces a diffeomorphism of fibred manifolds over V, $(\bar{\pi}, \phi_G) : FV/G \to V \times F$, by setting $\phi_G \circ \pi_G = q \circ \phi$.

Furthermore, the exponential map determines a diffeomorphism between an open neighborhood V_0 of the origin in $\mathfrak{gl}(n,\mathbb{R})$ and an open neighborhood $\mathcal{U}_I$ of the identity map I in $GL(n,\mathbb{R})$. Let $\mathfrak{f}$ be an Ad_G-invariant supplementary of $\mathfrak{g}$ in the matrix algebra; that is, $\mathfrak{gl}(n,\mathbb{R}) = \mathfrak{g} \oplus \mathfrak{f}$. By passing to the quotient, the exponential map $\exp : V_0 \to \mathcal{U}_I$ induces a diffeomorphism $e : V_0 \cap \mathfrak{f} \to q(\mathcal{U}_I)$ which is G-equivariant with respect the adjoint action of the group and the natural left action of G on F.

Let $s: U \to FV/G$ be a G-structure defined on an open neighborhood of the origin such that $s(0) = [u^0]$. By composing s with ϕ_G and shrinking U if necessary, we obtain a map taking values in $q(\mathcal{U}_I)$, so that we can consider the map

$$\begin{aligned} \bar{s}: U &\to V_0 \cap \mathfrak{f} \subseteq \mathfrak{f}, \\ \bar{s} &= e^{-1} \circ \phi_G \circ s. \end{aligned}$$

In this way we have identified the section s with a $\mathfrak{f}$-valued function on U. Let us consider the canonical flat connections on V and $\mathfrak{f}$ and the map

$$\begin{aligned} \left(\bar{\pi}_r^{r+1}, \Phi^{r+1}\right): J_0^{r+1}(FV/G)_{[u^0]} &\to J_0^r(FV/G)_{[u^0]} \times \left(S^{r+1}(V^\star) \otimes \mathfrak{f}\right), \\ j_0^{r+1} s &\mapsto \left(j_0^r s, \left(\nabla^{r+1}\bar{s}\right)_0\right) \equiv \left(j_0^r s, D^{r+1}\bar{s}(0)\right) \end{aligned} \tag{1}$$

where $(\nabla^{r+1}\bar{s})_0$ denotes the $(r+1)$th covariant differential of $\bar{s}$ with respect to these flat connections, $D^{r+1}\bar{s}(0)$ denotes its usual $(r+1)$th order differential and we have identified $T_0 V^\star \equiv V^\star$. These identifications make sense because the connections involved are the canonical flat connections on the linear spaces V and $\mathfrak{f}$. From [8, IV. Theorem 6] it follows that (1) is a G-equivariant diffeomorphism of fibred manifolds over $J_0^r(FV/G)_{[u^0]}$.

Using this trivialization, each moduli space can be described from the preceding one. In Section 3, we will prove that, under the conditions stated in Theorem 1.1, each $\mathbf{S}^{r+1} \to \mathbf{S}^r$ can be endowed with the structure of a trivial vector bundle where G acts by vector bundle automorphisms. Then, the conclusion of the theorem will follow.

2.3. *Canonical linear connections*

Let us introduce now our notion of canonical linear connections. We will denote by $C(M) \xrightarrow{\rho} M$ the affine bundle of linear connections on a manifold M. Recall that $P_s \subset FM$ is the total space of the G-structures attached to the section $s \in \Gamma(FM/G)$.

Definition 2.3. Assume that, for each n-dimensional manifold M, there exists an operator $\nabla: \Gamma(FM/G) \to \Gamma(C(M))$ satisfying:

(1) $\nabla(s)$ is reducible to P_s for each $s \in \Gamma(FM/G)$; i.e., horizontal spaces of $\nabla(s)$ are tangent to P_s.
(2) ∇ is natural, i.e., for each diffeomorphism $f: M \to M'$, the direct image of $\nabla(s)$ onto $\tilde{f}(P_s)$ is $\nabla(f \cdot s)$.
(3) ∇ is a first order differential operator, i.e., if $j_x^1 s = j_x^1 s'$, then $\nabla(s)(x) = \nabla(s')(x)$ and ∇ defines a bundle morphism:

$$\begin{array}{ccc} J^1(FM/G) & \xrightarrow{F_\nabla} & C(M) \\ \downarrow{\scriptstyle \bar{\pi}_1} & & \downarrow{\scriptstyle \rho} \\ M & \xrightarrow{\text{id}} & M \end{array}$$

by $F_\nabla(j_x^1 s) = \nabla(s)(x)$.

Under these hypotheses we say that there is a canonical linear connection $\nabla(s)$ associated to each G-structure s.

Remark 2.4. Let us denote by $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ the Lie algebra of G. The alternation operator $\delta : V^\star \otimes \mathfrak{g} \to \bigwedge^2 V^\star \otimes V$ is defined as

$$\delta\tau(u, v) = \tau(u)v - \tau(v)u, \quad \tau \in V^\star \otimes \mathfrak{g}, u, v \in V.$$

The natural action of G on V defines linear G-actions on the spaces $V^\star \otimes \mathfrak{g}$ and $\bigwedge^2 V^\star \otimes V$, with respect to which the operator δ is a homomorphism of G-modules.

The first prolongation of $\mathfrak{g}$ is defined as the vector space $\mathfrak{g}^{(1)} := \ker\delta$. It can be proved (see [10]) that the conditions of Definition 2.3 imply that this prolongation is trivial.

Besides, it is well known (see, e.g., [2] and [4]) that, if $\mathfrak{g}^{(1)} = \{0\}$ and the image of δ admits a supplementary G-submodule W:

$$\bigwedge^2 V^\star \otimes V = \delta(V^\star \otimes \mathfrak{g}) \oplus W,$$

then, for each G-structure s there is a unique linear connection $\nabla(s)$ reducible to s, which is characterized by the condition that its torsion is a section of the vector bundle $P_s \times_G W \subset P_s \times_G (\bigwedge^2 V^\star \otimes V) \cong \bigwedge^2 T^\star M \otimes TM$.

It is straightforward to check that the map $s \mapsto \nabla(s)$ also satisfies the conditions (2) and (3) of Definition 2.3.

3. Proof of Theorem 1.1

From now on, we will assume that $GL(n, \mathbb{R})/G$ is reductive and there is a canonical linear connection associated to each G-structure. We will prove that this makes each $J_0^r(FV/G)_{[u^0]}$ a trivial principal bundle over $\mathbf{S}^r$ with structure group $\ker q_1^{r+1}$ and allows to identify $\mathbf{S}^r$ to a closed G-invariant submanifold $\mathcal{E}^r(V, G)$ of $J_0^r(FV/G)_{[u^0]}$. Then, we will show that the trivialization (1) of $\bar{\pi}_r^{r+1} : J_0^{r+1}(FV/G)_{[u^0]} \to J_0^r(FV/G)_{[u^0]}$ induces a G-equivariant global trivialization of the bundle $\mathcal{E}^{r+1}(V, G) \to \mathcal{E}^r(V, G)$ where the typical fibre is a G-module. These trivializations will be used to define inductively a G-module structure on each $\mathcal{E}^r(V, G)$, and hence on each $\mathbf{S}^r$.

3.1. The bundles $J_0^r(FV/G)_{[u^0]} \to \mathbf{S}^r$

Let $s : \mathcal{U} \subset V \to FV/G$ be a local section defined on an open neighborhood $\mathcal{U}$ of $0 \in V$. The connection $\nabla(s)$ provides an exponential mapping $\exp_s : \mathcal{W}_0 \subset T_0V \to V$ defined on some neighborhood $\mathcal{W}_0$ of $0 \in T_0V$. The composition of this map with the isomorphism $u^0 : V \to T_0V$ yields a diffeomorphism defined on a neighborhood of $0 \in V$, which is nothing but the set of normal coordinates associated to the connection $\nabla(s)$ and the canonical frame u^0.

Lemma 3.1. *Let* $\mathrm{Exp}^r : J_0^r(FV/G)_{[u^0]} \to \ker q_1^{r+1}$ *be defined by*

$$\mathrm{Exp}^r\left(j_0^r s\right) = j_0^{r+1}\left(\exp_s \circ u^0\right).$$

For each r, Exp^r *is a well-defined smooth map. Moreover, it is equivariant with respect to the action of* $\ker q_1^{r+1}$ *on* $J_0^r(FV/G)_{[u^0]}$ *defined above and the natural left action of* $\ker q_1^{r+1}$ *on itself.*

Proof. Let $(V,(x^1,\dots,x^n))$ be the standard chart of V, and let us denote $f=\exp_s\circ u^0$. The map $t\mapsto f(tx)$ is the geodesic from $0\in V$ with initial velocity $u^0(x)$, i.e., the solution of the system of second order differential equations:

$$\frac{d^2f^k(tx)}{dt^2}=-\sum_{j_1,j_2}\Gamma^k_{j_1j_2}\big(f(tx)\big)\frac{df^{j_1}(tx)}{dt}\frac{df^{j_2}(tx)}{dt},\quad 1\leqslant k\leqslant n,$$

with initial conditions $f(0)=0$, $\frac{df(tx)}{dt}\big|_0=u^0(x)$.

We can write this system in an equivalent way as follows:

$$j^1_0f=j^1_0I,$$
$$\sum_{i_1,i_2}x^{i_1}x^{i_2}\frac{\partial^2f^k}{\partial x^{i_1}\partial x^{i_2}}(tx)=-\sum_{i_1,i_2,j_1,j_2}x^{i_1}x^{i_2}\Gamma^k_{j_1j_2}\big(f(tx)\big)\frac{\partial f^{j_1}}{\partial x^{i_1}}(tx)\frac{\partial f^{j_2}}{\partial x^{i_2}}(tx)$$

with $k=1,\dots,n$. Taking the $(r-1)$th order derivative ($r\geqslant 1$) with respect to t of the second equation and evaluating it at $t=0$ we conclude that each $\frac{\partial^{r+1}f^k}{\partial x^{i_1}\dots\partial x^{i_{r+1}}}(0)$ is a polynomial in the derivatives of the Christoffel symbols at $x=0$ up to order $r-1$, and the derivatives at $x=0$ of the components f^k up to order r. Since the map $s\mapsto\nabla(s)$ is a first order differential operator (see Definition 2.3), the derivatives up to order $r-1$ of the Christoffel symbols $\Gamma^k_{ij}(x)$ are smooth functions of j^r_xs. Thus, the $(r+1)$th order derivatives at $x=0$ of f are smooth functions of j^r_0s and j^r_0f. Finally, using induction, it follows that Exp^r is a well-defined smooth map.

Moreover, due to the naturality of the map $s\mapsto\nabla(s)$, the equality $h\circ\exp_s=\exp_{h\cdot s}\circ T_0h$ holds for each diffeomorphism $h:M\to M'$ and each G-structure $s\in\Gamma(FM/G)$, and hence:

$$\begin{aligned}\mathrm{Exp}^r\big(\big(j^{r+1}_0h\big)\cdot j^r_0s\big)&=\mathrm{Exp}^r\big(j^r_0(h\cdot s)\big)=j^{r+1}_0\exp_{h\cdot s}\circ u^0\\&=j^{r+1}_0\big(h\circ\exp_s\circ u^0\big)=\big(j^{r+1}_0h\big)\cdot\mathrm{Exp}^r\big(j^r_0s\big),\end{aligned}$$

for each $j^{r+1}_0h\in\ker q^{r+1}_1$ and each $j^r_0s\in J^r_0(FV/G)_{[u^0]}$. Thus, the map Exp^r is equivariant. □

Proposition 3.2. *For each $r\in\mathbb{N}$, the space $J^r_0(FV/G)_{[u^0]}$ is the total space of a trivial principal bundle with structure group $\ker q^{r+1}_1$. In particular, $\mathbf{S}^r=J^r_0(FV/G)_{[u^0]}/\ker q^{r+1}_1$ acquires a quotient manifold structure.*

Proof. From the equivariance of the map Exp^r it follows that it is surjective, because the image of each orbit in $J^r_0(FV/G)_{[u^0]}$ is an orbit of left translations in the group, that is, $\ker q^{r+1}_1$. Moreover, it is a submersion, because if $j^r_0s\in J^r_0(FV/G)_{[u^0]}$ were a critical point of Exp^r, then every element of its orbit would also be a critical point, and Exp^r would not have any regular value, contradicting Sard's Theorem.

Thus, each fibre of Exp^r, and in particular $\mathcal{E}^r(V,G)=(\mathrm{Exp}^r)^{-1}(j^{r+1}_0I)$, is a smooth submanifold of $J^r_0(FV/G)_{[u^0]}$.

Besides, the map

$$\begin{aligned}\big(p^r,\mathrm{Exp}^r\big):J^r_0(FV/G)_{[u^0]}&\to\mathcal{E}^r(V,G)\times\ker q^{r+1}_1\\ j^r_0s&\mapsto\Big(\big(\mathrm{Exp}^r\big(j^r_0s\big)\big)^{-1}\cdot j^r_0s,\mathrm{Exp}^r\big(j^r_0s\big)\Big)\end{aligned}$$

is a $(\ker q^{r+1}_1)$-equivariant diffeomorphism (with respect to the obvious actions of $\ker q^{r+1}_1$ on both sides), whose inverse is given by $(p^r,\mathrm{Exp}^r)^{-1}(j^r_0s,j^{r+1}_0f)=(j^{r+1}_0f)\cdot(j^r_0s)$.

This diffeomorphism makes $p^r : J_0^r(FV/G)_{[u^0]} \to \mathcal{E}^r(V,G)$ a trivial principal $(\ker q_1^{r+1})$-bundle, and the first assertion in the proposition follows.

As a consequence, $\mathbf{S}^r = J_0^r(FV/G)_{[u^0]}/\ker q_1^{r+1}$ acquires a quotient manifold structure and the map:

$$\begin{aligned}\bar{p}^r &: \mathbf{S}^r \to \mathcal{E}^r(V,G)\\ \big[j_0^r s\big] &\mapsto p^r\big(j_0^r s\big)\end{aligned}$$

is a diffeomorphism. Its inverse is the composition: $\mathcal{E}^r(V,G) \hookrightarrow J_0^r(FV/G)_{[u^0]} \to \mathbf{S}^r$ of the inclusion with the natural projection. □

Again, the naturality of the map $s \mapsto \nabla(s)$ implies that $\exp_{g\cdot s} \circ u^0 \circ g = g \circ \exp_s \circ u^0$ for each G-structure s and each $g \in G$, whence:

$$\mathrm{Exp}^r\big(\big(j_0^{r+1} g\big) \cdot j_0^r s\big) = \big(j_0^{r+1} g\big) \cdot \mathrm{Exp}^r\big(j_0^r s\big) \cdot \big(j_0^{r+1} g^{-1}\big).$$

From this, it follows easily that the submanifold $\mathcal{E}^r(V,G)$ is invariant under the action of G on $J_0^r(FV/G)_{[u^0]}$, and that the projection $p^r : J_0^r(FV/G)_{[u^0]} \to \mathcal{E}^r(V,G)$, and hence the diffeomorphism $\bar{p}^r : \mathbf{S}^r \to \mathcal{E}^r(V,G)$, are G-equivariant.

Finally, it is immediate that the diffeomorphisms $\bar{p}^r : \mathbf{S}^r \to \mathcal{E}^r(V,G)$ commute with the projections $\mathbf{S}^r \to \mathbf{S}^k$ and $\mathcal{E}^r(V,G) \to \mathcal{E}^k(V,G)$, so that we can identify $\mathbf{S}^r/G$ with $\mathcal{E}^r(V,G)/G$ for each $r \in \mathbb{N} \cup \{\infty\}$, where $\mathcal{E}^\infty(V,G) = \varprojlim \mathcal{E}^r(V,G)/G$.

3.2. *Definition of the G-module structures*

Consider the global trivialization $(\bar{\pi}_r^{r+1}, \Phi^{r+1})$ defined in (1). The action of $\ker q_{r+1}^{r+2}$ on fibres of $J_0^{r+1}(FV/G)_{[u^0]} \to J_0^r(FV/G)_{[u^0]}$ induces an action on $S^{r+1}(V^*) \otimes \mathfrak{f}$ by means of this trivialization. An easy computation shows that this action is as follows. Let us consider the linear injective map (see the remark below)

$$L^{r+2} : S^{r+2}(V^*) \otimes V \hookrightarrow S^{r+1}(V^*) \otimes (V^* \otimes V) \to S^{r+1}(V^*) \otimes \mathfrak{f}, \tag{2}$$

where the first map is the natural embedding and the second one is induced by the projection

$$V^* \otimes V \cong \mathfrak{gl}(n,\mathbb{R}) = \mathfrak{g} \oplus \mathfrak{f} \to \mathfrak{f}.$$

Then, for each $j_0^{r+2} f \in \ker q_{r+1}^{r+2}$ and each $p \in S^{r+1}(V^*) \otimes \mathfrak{f}$, we have $(j_0^{r+2} f) \cdot p = p + L^{r+2}(D^{r+2} f(0))$. On the other hand, it is easily seen that the map L^{r+2} is equivariant with respect to the natural actions of G on $S^{r+2}(V^*) \otimes V$ and $S^{r+1}(V^*) \otimes \mathfrak{f}$.

So, the orbit space $\frac{S^{r+1}(V^*)\otimes\mathfrak{f}}{\ker q_{r+1}^{r+2}}$ is identified with the quotient vector space $\frac{S^{r+1}(V^*)\otimes\mathfrak{f}}{\operatorname{im} L^{r+2}}$ and the action of G on $S^{r+1}(V^*) \otimes \mathfrak{f}$ induces a well-defined linear action on this quotient.

Remark 3.3. The map L^{r+2} is injective as a corollary of Proposition 3.2. However, this fact can be established directly by computing its kernel from (2). The kernel $\ker L^{r+2}$ turns to be (isomorphic to) the $(r+1)$th prolongation $\mathfrak{g}^{(r+1)}$ of the Lie algebra of G. By induction on r, it can be proved that the only condition $\mathfrak{g}^{(1)} = \{0\}$ is equivalent to the fact that the action of $\ker q_1^{r+1}$ on $J_0^r(FV/G)_{[u^0]}$ is free for all r.

Now, we are ready to prove the following proposition:

Proposition 3.4. *For each $r \in \mathbb{N}$, $\mathcal{E}^{r+1}(V,G) \to \mathcal{E}^r(V,G)$ is a trivial vector bundle with fibre $\frac{S^{r+1}(V^*)\otimes\mathfrak{f}}{\operatorname{im} L^{r+2}}$ admitting a G-equivariant trivialization with respect to the actions previously defined.*

Proof. From the definition of the maps Exp^r, it follows that, for each $r \in \mathbb{N}$, the pair $(\mathrm{Exp}^{r+1}, \mathrm{Exp}^r)$ is a bundle morphism, i.e., the following diagram commutes.

$$\begin{array}{ccc} J_0^{r+1}(FV/G)_{[u^0]} & \xrightarrow{\mathrm{Exp}^{r+1}} & \ker q_1^{r+2} \\ \downarrow{\scriptstyle \bar{\pi}_r^{r+1}} & & \downarrow{\scriptstyle q_{r+1}^{r+2}} \\ J_0^r(FV/G)_{[u^0]} & \xrightarrow{\mathrm{Exp}^r} & \ker q_1^{r+1} \end{array}$$

Therefore, the restriction to $\mathcal{E}^r(V,G)$ of the trivial bundle $J_0^{r+1}(FV/G)_{[u^0]} \to J_0^r(FV/G)_{[u^0]}$ is the trivial bundle $(\mathrm{Exp}^{r+1})^{-1}(\ker q_{r+1}^{r+2}) \to \mathcal{E}^r(V,G)$. On the other hand, the image of $(\mathrm{Exp}^{r+1})^{-1}(\ker q_{r+1}^{r+2})$ by $(p^{r+1}, \mathrm{Exp}^{r+1})$ is the product $\mathcal{E}^{r+1}(V,G) \times \ker q_{r+1}^{r+2}$. Thus, $(\mathrm{Exp}^{r+1})^{-1}(\ker q_{r+1}^{r+2}) \xrightarrow{p^{r+1}} \mathcal{E}^{r+1}(V,G)$ is a trivial principal bundle with structure group $\ker q_{r+1}^{r+2}$.

Taking the trivialization (1), the restriction of the composed map $p^{r+1} \circ (\bar{\pi}_r^{r+1}, \Phi^{r+1})^{-1}$ to $\mathcal{E}^r(V,G) \times (S^{r+1}(V^*) \otimes \mathfrak{f})$:

$$p^{r+1} \circ \left(\bar{\pi}_r^{r+1}, \Phi^{r+1}\right)^{-1} : \mathcal{E}^r(V,G) \times \left(S^{r+1}(V^*) \otimes \mathfrak{f}\right) \to \mathcal{E}^{r+1}(V,G)$$

is a trivial principal bundle with structure group $\ker q_{r+1}^{r+2}$. Hence, there is a diffeomorphism $(\phi^{r+1})^{-1} : \mathcal{E}^r(V,G) \times \frac{S^{r+1}(V^*)\otimes\mathfrak{f}}{\operatorname{im} L^{r+2}} \to \mathcal{E}^{r+1}(V,G)$, given by

$$\left(j_0^r s, p + \operatorname{im} L^{r+2}\right) \mapsto p^{r+1} \circ \left(\bar{\pi}_r^{r+1}, \Phi^{r+1}\right)^{-1}\left(j_0^r s, p\right),$$

whose inverse is

$$\begin{aligned} \phi^{r+1} : \mathcal{E}^{r+1}(V,G) &\to \mathcal{E}^r(V,G) \times \frac{S^{r+1}(V^*) \otimes \mathfrak{f}}{\operatorname{im} L^{r+2}}, \\ j_0^{r+1} s &\mapsto \left(j_0^r s, \Phi^{r+1}\left(j_0^{r+1} s\right) + \operatorname{im} L^{r+2}\right). \end{aligned} \tag{3}$$

Moreover, the diffeomorphism ϕ^{r+1} is G-equivariant. This endows $\mathcal{E}^{r+1}(V,G)$ with the structure of a trivial vector bundle over $\mathcal{E}^r(V,G)$ where G acts by vector bundle automorphisms. □

Now, we will define a G-module structure on each $\mathcal{E}^r(V,G)$, inductively on r, such that the projections are homomorphisms of G-modules. The corresponding G-manifolds $\mathbf{S}^r = J_0^r(FV/G)_{[u^0]}/\ker q_1^{r+1}$ will inherit G-module structures, and the natural projections between them will also be homomorphisms of G-modules. Therefore, we will obtain a G-module structure on the projective limit $\mathbf{S}^\infty = \varprojlim \mathbf{S}^r$.

The case $r = 0$ is trivial because $\mathcal{E}^0(V,G)$ has a unique element, and so it admits the trivial G-module structure.

Now, assume that a G-module structure has been already defined on $\mathcal{E}^r(V,G)$. Then, we define the G-module structure on $\mathcal{E}^{r+1}(V,G)$ as the one that makes linear the trivialization (3) where we are considering the direct sum G-module structure on the right hand side.

It is clear that the projections $\bar{\pi}_r^{r+1}$ are linear. Using induction, as well as the fact that Ψ^{r+1} is G-equivariant, we conclude that the action of G on each $\mathcal{E}^r(V,G)$ is also linear.

More precisely, the G-module structure on $\mathcal{E}^{r+1}(V,G)$ is isomorphic to the one on $\mathcal{E}^r(V,G)\oplus W^{r+1}$, where $W^{r+1}=\frac{S^{r+1}(V^*)\otimes\mathfrak{f}}{\operatorname{im}L^{r+2}}$. This fact (together with the obvious identity $\mathcal{E}^0(V,G)=\{0\}$) yields an isomorphism of G-modules $\mathbf{S}^r\cong W^1\oplus W^2\oplus\cdots\oplus W^r$, and the G-module structure of $\mathbf{S}^r$ is determined by the one of the spaces W^k, $k=1,\ldots,r$.

4. Some examples and applications

The previous description of the moduli spaces simplifies the study of problems such as the determination of the Poincaré series of the moduli space, the computation of characteristic classes associated to G-structures, etc. In this section, we will treat briefly the first problem, and we will make explicit computations in some particular examples.

Let us denote by ν_r the dimension of the regular part of $\mathfrak{M}^r_G$. Then, the Poincaré series of the moduli space $\mathfrak{M}^\infty_G$ is defined as:

$$P(t)=\nu_0+\sum_{r=1}^{\infty}(\nu_r-\nu_{r-1})t^r.$$

In [1], Arnold conjectured that the Poincaré series of the moduli space is a rational function in many local problems of analysis. Our results confirm his conjecture for the moduli space of jets of G-structures satisfying the conditions of Theorem 1.1.

The action of G on $\mathcal{E}^r(V,G)$ is clearly analytic. Thus, on each finite-dimensional linear space $\mathcal{E}^r(V,G)$, the union of all orbits which have the maximal dimension m_r forms an everywhere dense open submanifold $\mathcal{E}^r(V,G)_{\mathrm{reg}}$, and the quotient $\mathcal{E}^r(V,G)_{\mathrm{reg}}/G$ is a smooth connected manifold, so that $\nu_r=\dim\mathcal{E}^r(V,G)-m_r$.

Since the projections $\mathcal{E}^r(V,G)\to\mathcal{E}^{r-1}(V,G)$ are equivariant submersions, it follows that $m_{r-1}\leqslant m_r$ for all r. Then, there exists a r_0 such that $m_r=m_{r_0}$ for all $r\geqslant r_0$. The Poincaré series can be written:

$$\begin{aligned}P(t)&=\nu_0+\sum_{r=1}^{r_0}(m_{r-1}-m_r)t^r+\sum_{r=1}^{\infty}\bigl(\dim\mathcal{E}^r(V,G)-\dim\mathcal{E}^{r-1}(V,G)\bigr)t^r\\&=\sum_{r=1}^{r_0}(m_{r-1}-m_r)t^r+\sum_{r=1}^{\infty}\bigl(\dim W^r\bigr)t^r.\end{aligned}$$

The dimension of W^r can be easily computed as

$$\begin{aligned}\dim W^r&=\dim\bigl(S^r(V^*)\otimes\mathfrak{f}\bigr)-\dim\bigl(S^{r+1}(V^*)\otimes V\bigr)\\&=\binom{n+r-1}{r}\bigl(n^2-\dim G\bigr)-\binom{n+r}{r+1}n\\&=\binom{n+r-1}{r+1}nr-\binom{n+r-1}{r}\dim G.\end{aligned}$$

It is a polynomial in r of degree $n-1$, which implies that the Poincaré series is rational.

In each of the following examples, we will describe explicitly the structure of the moduli space and we will compute its Poincaré series. All of them are known to satisfy the conditions of Theorem 1.1.

Example 4.1 ($\{e\}$-*structures*). For complete parallelisms or $\{e\}$-structures, it is

$$W^r = \frac{S^r(V^*) \otimes (V^* \otimes V)}{\operatorname{im} L^{r+1}}.$$

Denoting by $\operatorname{sym}^r : S^r(V^*) \otimes (V^* \otimes V) \to S^{r+1}(V^*) \otimes V$ the obvious symetrization operator, there is a G-invariant decomposition:

$$S^r(V^*) \otimes (V^* \otimes V) \cong \operatorname{im} L^{r+1} \oplus \ker \operatorname{sym}^r,$$

so that one can identify $W^r \cong \ker \operatorname{sym}^r$ for all $r \in N$.

The space W^r is then identified with the subspace of tensors $p \in S^r(V^*) \otimes (V^* \otimes V)$ satisfying the symmetry condition:

$$\underset{(i_1 \ldots i_{r+1})}{\mathfrak{S}} p^k_{i_1 \ldots i_{r+1}} = 0, \quad 1 \leqslant i_1, \ldots, i_{r+1}, k \leqslant n, \tag{4}$$

where $\mathfrak{S}$ stands for the cyclic sum with respect to the corresponding indices.

The Poincaré series reduces to:

$$P(t) = \sum_{r=1}^{\infty} nr \binom{n+r-1}{r+1} t^r = n\left(\frac{n-1}{(1-t)^n} + \frac{1}{t}\left(1 - \frac{1}{(1-t)^{n-1}}\right)\right).$$

Example 4.2 ($O(n)$-*structures*). For $G = O(n)$, we can take $\mathfrak{f}$ as the subspace of all symmetric matrices. On the other hand, it is not hard to check that $\ker(\operatorname{sym}^r \circ L^{r+1}) = \{0\}$, so that there is a $O(n)$-invariant decomposition:

$$S^r(V^*) \otimes \mathfrak{f} \cong \operatorname{im} L^{r+1} \oplus \ker\big(\operatorname{sym}^r |_{S^r(V^*) \otimes \mathfrak{f}}\big). \tag{5}$$

This allows us to identify $W^r \cong \ker(\operatorname{sym}^r |_{S^r(V^*) \otimes \mathfrak{f}})$ for all $r \in \mathbb{N}$.

Thus, W^r is isomorphic to the subspace of tensors $p \in S^r(V^*) \otimes (V^* \otimes V)$ satisfying the symmetry conditions (4) and $p^k_{i_1 \ldots i_r j} = p^j_{i_1 \ldots i_r k}$ for any indices $1 \leqslant i_1, \ldots, i_r, j, k \leqslant n$.

It should be pointed that this description of the moduli space of jets of $O(n)$-structures corresponds to that given in [3] starting from the Taylor series in normal coordinates of the components g_{ij} of a Riemannian metric g.

The Poincaré series of the metric structures is also well known (e.g., see [7] and [9]). However, it can be easily computed from the description given above. The results are the following:

The Poincaré series is given by:

$$\begin{aligned} P(t) &= -\binom{n}{2} t^2 + \sum_{r=1}^{\infty} \left(nr \binom{n+r-1}{r+1} - \binom{n}{2} \binom{n+r-1}{r} \right) t^r \\ &= \frac{(1-t)^n (n(n-1)t(1-t^2) + 2n) + n(n+1)t - 2n}{2t(1-t)^n}, \end{aligned}$$

if $n \geqslant 3$, and by the Taylor series at $t = 0$ of the rational function:

$$P(t) = \frac{t^2(-t^3 + 2t^2 - t + 1)}{(t-1)^2}, \quad \text{if } n = 2.$$

Example 4.3 ($O(n_1) \times O(n_2)$*-structures*). In this case, we can take the invariant supplementary $\mathfrak{f}$ of

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{gl}(n, \mathbb{R}) \mid A \in o(n_1), B \in o(n_2) \right\}$$

as the subspace of all matrices of the form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where A and D are symmetric matrices of orders n_1 and n_2, respectively.

Again, it can be seen that $\ker(\mathrm{sym}^r \circ L^{r+1}) = \{0\}$, and a G-invariant decomposition like (5) holds. Thus, W^r is isomorphic to the subspace of tensors $p \in S^r(V^*) \otimes (V^* \otimes V)$ satisfying (4) and $p^k_{i_1 \dots i_r i_{r+1}} - p^{i_{r+1}}_{i_1 \dots i_r k} = 0$ for any indices $1 \leqslant i_1, \dots, i_r \leqslant n$, and $1 \leqslant i_{r+2}, k \leqslant n_1$ or $n_1 + 1 \leqslant i_{r+2}, k \leqslant n$.

In this case the principal isotropy groups are trivial for any r. To see this, it is enough to find an element $p \in W^1$ whose isotropy group is trivial, because then $m_r = m_1 = \dim G$ for every r. Let us take the tensor p defined by

$$p^{n_1+j}_{i n_1+j} = -p^{n_1+j}_{n_1+j i} = j, \quad 1 \leqslant i \leqslant n_1,\ 1 \leqslant j \leqslant n_2,$$
$$p^{i}_{i n_1+j} = -p^{i}_{n_1+j i} = i, \quad 1 \leqslant i \leqslant n_1,\ 1 \leqslant j \leqslant n_2,$$

with all the remaining components vanishing. It is easy to check that this p has a trivial isotropy group.

So, in this case, the Poincaré series is given by

$$\begin{aligned} P(t) &= -(\dim G)t + \sum_{r=1}^{\infty} \left(nr \binom{n+r-1}{r+1} - (\dim G) \binom{n+r-1}{r} \right) t^r \\ &= n \left(\frac{n-1}{(1-t)^n} + \frac{1}{t} \left(1 - \frac{1}{(1-t)^{n-1}} \right) \right) - \left(\binom{n_1}{2} + \binom{n_2}{2} \right) \left(\frac{1}{(1-t)^n} - 1 + t \right). \end{aligned}$$

Example 4.4 ($\mathbb{R}^*$*-structures*). We can take the invariant supplementary $\mathfrak{f}$ of $\mathfrak{g} = \mathrm{Span}\{I\}$ in $\mathfrak{gl}(n, \mathbb{R})$ as the subspace of all traceless matrices.

Once again, it can be seen that a decomposition like (5) holds. The resulting equations characterizing W^r are then (4) and

$$\sum_k p^k_{i_1 \dots i_r k} = 0, \quad 1 \leqslant k, i_1, \dots, i_r \leqslant n.$$

The action of G on W^r is given by $(a \cdot p) = \frac{1}{a^r} p$, $a \in \mathbb{R}^*$, $p \in W^r$, and so the principal isotropy groups are trivial.

If $n = 2$ then $W^1 = \{0\}$, whence $m_1 = 0$. For any other $r \geqslant 2$ it is $m_r = 1$, and the Poincaré series is given by

$$P(t) = -t^2 + \sum_{r=1}^{\infty} (r-1) t^r = -t^2 + \frac{t^2}{(t-1)^2}.$$

If $n \geqslant 3$ then $m_r = 1$ for every $r \geqslant 1$, and the Poincaré series is given by

$$\begin{aligned} P(t) &= -t + \sum_{r=1}^{\infty} \left(nr \binom{n+r-1}{r+1} - \binom{n+r-1}{r} \right) t^r \\ &= 1 - t + \frac{n(n-1)-1}{(1-t)^n} + \frac{n}{t} \left(1 - \frac{1}{(1-t)^{n-1}} \right). \end{aligned}$$

Acknowledgements

The authors would like to thank A.M. Vinogradov, whose comments greatly helped to improve the manuscript.

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