GLOBAL AND ROBUST STABILITY OF INTERVAL HOPFIELD NEURAL NETWORKS WITH TIME-VARYING DELAYS

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In this paper, we investigate the problem of global and robust stability of a class of interval Hopfield neural networks that have time-varying delays. Some criteria for the global and robust stability of such networks are derived, by means of constructing suitable Lyapunov functionals for the networks. As a by-product, for the conventional Hopfield neural networks with time-varying delays, we also obtain some new criteria for their global and asymptotic stability.

Keywords: Interval Hopfield neural networks; time-varying delays; robust stability; Lyapunov functionals.

1. Introduction

Time-delays, both constant and time-varying, are often encountered in various engineering, biological, and economical systems. Due to the finite speed of information processing, their existence frequently causes oscillations and instability in neural networks. In recent years, the stability issue in time-delay neural networks has become a topic of great theoretical and also practical importance. This issue has gained increasing interest in potential applications in signal and image processing, artificial intelligence, and industrial automation, to name a few. As a result, many criteria for testing the global stability of constant time-delay neural networks have been devised (see, e.g., Refs. 2–6, 8–11, and the references therein).

On the other hand, it is unavoidable to involve some uncertainty and disturbance during the implementation of time-delay networks by VLSI chips, due to the existence of modeling errors, external perturbation, and parameter fluctuation. Therefore, it is important to investigate the stability and robustness
of the networks against such errors and uncertainties. In the case of uncertain neural networks with time-varying delays, the problem of robust stability has not been fully investigated, despite the fact that significant results on the analysis and synthesis of such neural networks have been obtained in recent years.

In the study of uncertain and time-delay neural networks, Liao and Yu extended the model of time-delayed Hopfield neural networks to interval networks that have various time delays, thereby obtaining the so-called interval delayed Hopfield neural networks (IDHNN). Some robust stability criteria for these systems have been derived, when the delay time is constant. The main objective of the present paper is to furthermore investigate the robust stability issue for such interval Hopfield neural networks but with time-varying delays. The basic approach taken is by means of constructing suitable Lyapunov functionals for the networks.

The organization of the remaining part of the paper is as follows. In Sec. 2, we first introduce the interval time-delayed Hopfield neural networks. Some mathematical definitions and preliminary lemma are also presented therein. In Sec. 3, we employ the interval dynamics approach to analyze the existence and uniqueness of the equilibrium point for the interval Hopfield neural networks with time-varying delays (IHNTVD). By constructing several Lyapunov functionals, the global and robust stability of IHNTVD is analyzed in detail. Our analysis method does not require the symmetry of interconnection weight matrix in the network structure, and the monotonicity or smoothness condition of the activation functions are also released. We conclude the paper by Sec. 4.

2. Preliminaries

Consider the following Hopfield neural networks with time-varying delay:

\[
\frac{du_i(t)}{dt} = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) + \sum_{j=1}^{n} w_{ij}^\tau f_j(u_j(t - \tau_{ij}(t))) + I_i
\]

where \(a_i\) and \(\tau_{ij}\) are nonnegative numbers representing the neuron charging time constants and axonal signal transmission delays, respectively; \(w_{ij}\) and \(w_{ij}^\tau\) stand for the weights of the neuron interconnections, and \(f_j\) and \(I_i\) are the activation function of the neurons and the external constant inputs, respectively.

Usually, basic assumption on this network model is that the activation functions are continuous, differentiable, and monotonically increasing and bounded (such as the sigmoid-type functions). In this paper, however, we do not assume the monotonicity or smoothness of the activation functions \(f_j, j = 1, 2, \ldots, n\). Instead, we assume the following weaker conditions:

(H1) There exist constants \(L_j; 0 < L_j < +\infty, j = 1, 2, \ldots, n\), such that the incremental ratio for \(f_j; R \rightarrow R \) satisfies

\[
f_j: R \rightarrow R \text{ satisfies } 0 \leq \frac{f_j(x_j) - f_j(y_j)}{x_j - y_j} \leq L_j, \quad x_j \neq y_j;
\]

(H2) \(|f_j(x)| \leq M_j, x \in R^n \) and \(M_j > 0, j = 1, 2, \ldots, n\).

In Ref. 9, we considered the deviations and perturbations of the neuron charging time constants and the weights of interconnections are unknown but bounded. We may henceforth intervalize the above network model by defining the following set of interval matrices:

\[
\begin{align*}
A_l & \triangleq \{ A = \text{diag}(a_i)_{n \times n}; A \leq \bar{A}, \text{i.e., } a_i \leq \bar{a}_i, i = 1, 2, \ldots, n, \forall A \in A_l \} \\
W_l & \triangleq \{ W = \text{diag}(w_{ij})_{n \times n}; W \leq \bar{W}, \text{i.e., } w_{ij} \leq \bar{w}_{ij}, i, j = 1, 2, \ldots, n, \forall W \in W_l \} \\
W_l^\tau & \triangleq \{ W^\tau = \text{diag}(w_{ij}^\tau)_{n \times n}; W^\tau \leq \bar{W}^\tau, \text{i.e., } w_{ij}^\tau \leq \bar{w}_{ij}^\tau, i, j = 1, 2, \ldots, n, \forall W^\tau \in W_l^\tau \}
\end{align*}
\]

Compared with Ref. 9, we do not intervalize the time delay constants \(\tau_{ij}\) in this paper but consider the delays across the network are all time-varying yet may be continuous and differentiable. More precisely, we assume that the time-varying delays are as follows:
Remark 1

The assumption of $\tau_{ij}(t) \leq R < 1$ stems from the need to bound the growth of variations in the delay factor as a time function. It may be considered restrictive but in many applications it is realistic and holds for a wide class of IDHNN systems such as those with bounded steady-state delays.

Hence, hypothesis (H3) ensures $t - \tau_{ij}(t)$ have differential inverse functions, denoted by $\varphi_{ij}(t)$.

To proceed, we firstly give the following definition and lemma.

Definition 1

[9]: System (1) with (2) is said to be robust stable (globally robust stable) if its unique equilibrium $u^* = (u^*_1, u^*_2, \ldots, u^*_n) \in R^T$ is stable (or globally stable) for all

\[ A = \text{diag}(a_i)_{n \times n} \in A_f, \]
\[ W = \text{diag}(w_{ij})_{n \times n} \in W_f, \]
\[ W^T = \text{diag}(w_{ij})_{n \times n} \in W^T_f. \]

Lemma 1

[10-11]: Under assumptions (H1) and (H2), the neural network (1) with (2) has an equilibrium point.

Now, suppose that $u^* = (u^*_1, u^*_2, \ldots, u^*_n)^T \in R^n$ is an equilibrium point of (1) with (2), and let $x_i(t) = u_i(t) - u^*_i$, $i = 1, 2, \ldots, n$. Then, system (1) with (2) can be rewritten as

\[ \frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} w_{ij} g_j(x_j(t)) \]
\[ + \sum_{j=1}^{n} w_{ij}^T g_j(x_j(t - \tau_{ij}(t))), \]
\[ i = 1, 2, \ldots, n, \]

(3)

where $g_j(x_j(t)) = f_j(x_j(t) + x^*_j) - f_j(x^*_j)$, $j = 1, 2, \ldots, n$. According to the assumed characteristics of $f_j$, functions $g_i$ possess the following properties:

(H1) $0 \leq \frac{g_i(x_i)}{x_i} \leq L_i$, $0 < L_i \leq +\infty$ and $g_i(0) = 0$, $x_i g_i(x_i) > 0$, $i = 1, 2, \ldots, n$.

Obviously, an equilibrium $u^*$ of system (1) with (2) is globally robust stable if and only if the trivial solution of (3) with (2) is globally robust stable.

3. Robust Stability Criteria for IDHNNNTVD

In this section, we consider the case where the time delays $\tau_{ij}$ are variables of $t$.

Similar to the main theorem given in Ref. 9, we have the following result for the corresponding time-varying delays system:

Theorem 1

If there exist positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

\[ \alpha' \triangleq \max_{1 \leq i \leq n} \left\{ \frac{L_i}{\sum_{j=1}^{n} \lambda_j w_{ij}^T} \right\} < 1, \]

where

\[ w_{ij}^* = \max_{1 \leq i \leq n} \left\{ |w_{ij}| + |w_{ij}^T| \varphi_{ij}(t) \right\}, \]
\[ \left( |\tilde{w}_{ij}| + |\tilde{w}_{ij}^T| \varphi_{ij}(t) \right) \]

then system (1) with (2) has a unique and robust stable equilibrium $u^*$ for each constant input $I = (I_1, I_2, \ldots, I_n)^T \in R^n$.

Proof

A proof for the existence and uniqueness is obtained by a similar argument for Theorem 1 derived in Ref. 9. We only prove its robust stability here.

Construct a Lyapunov functional of the form

\[ V(x_1, x_2, \ldots, x_n)(t) \]
\[ = \sum_{i=1}^{n} \lambda_i \left\{ |x_i(t)| + \sum_{j=1}^{n} |w_{ij}^T| \int_{t}^{\varphi_{ij}(t)} \right. \]
\[ \times |g_j(x_j(s - \tau_{ij}(s))| \right\}. \]

We can calculate the upper right Dini derivative of $V(x_1, x_2, \ldots, x_n)(t)$, along the solution of Eq. (3),
as follows:

\[ D^+ V(x_1, x_2, \ldots, x_n)(t) = \sum_{i=1}^{n} \lambda_i \left[ -a_i x_i(t) + \sum_{j=1}^{n} w_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} w_{ij}^\tau g_j(x_j(t - \tau_{ij}(t))) \right] \text{sgn}(x_i(t)) \]

\[ + \sum_{j=1}^{n} |w_{ij}^\tau| \|g_j(x_j(\varphi_{ij}(t) - \tau_{ij}(\varphi_{ij}(t))))\varphi_{ij}'(t)\| - \|g_j(x_j(t - \tau_{ij}(t))))\| \right) \]

\[ \leq \sum_{i=1}^{n} \lambda_i \left[ -a_i |x_i(t)| + L_j \sum_{j=1}^{n} \lambda_j (|w_{ji}| + |w_{ji}^\tau| |\varphi_{ji}(t)|) \right] |x_i(t)| \]

\[ \leq \sum_{i=1}^{n} \alpha_i \lambda_i \left[ -1 + \frac{1}{\alpha_i \lambda_i} L_j \sum_{j=1}^{n} \lambda_j w_{ji}^* \right] |x_i(t)| \]

\[ \leq (-1 + \alpha') \sum_{i=1}^{n} \alpha_i \lambda_i |x_i(t)| < 0. \]

Following the derivation given in Ref. 9, we finally conclude that system (1) with (2) is globally robust stable. \( \square \)

**Corollary 1**

If \( A = A, W = W, W^\tau = W^\tau \) and if there exist positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that

\[ \alpha_i^* = \max_{1 \leq i \leq n} \left\{ \frac{L_i}{\alpha_i \lambda_i} \sum_{j=1}^{n} \lambda_j (|w_{ji}| + |w_{ji}^\tau| |\varphi_{ji}(t)|) \right\} < 1, \quad (7) \]

then system (1) is globally asymptotically stable.

**Remark 2**

In Ref. 8, the pure-delay model, i.e., with \( w_{ij} = 0 \), was studied. Hence, when \( w_{ij} = 0 \), the result of Eq. (7) reduces to that given in Ref. 8. A major difference is that in Ref. 8, the sigmoid-type activation function was required to be continuous, differentiable, and bounded. However, these properties are no longer required here.

By constructing another Lyapunov functional, we also obtain the following result.

**Theorem 2**

If there exist positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and \( r_1 \in [0, 1], r_2 \in [0, 1] \) such that

\[ \beta^* = \max_{1 \leq i \leq n} \left\{ \frac{1}{\alpha_i \lambda_i} \sum_{j=1}^{n} \lambda_j L_j^2(1-r_1)|w_{ij}^*| + \lambda_i L_j^2 r_2 |w_{ij}^*| \right\} < 2, \quad (8) \]

where

\[ \begin{cases} w_{ij}^* = \max_{1 \leq i, j \leq n} \{|w_{ij}|, |w_{ij}^\tau|\} \\ w_{ij}^{**} = \max_{1 \leq i, j \leq n} \{|w_{ij}^\tau|, |w_{ij}^{\tau\tau}|\} \end{cases} \quad (9) \]

then system (1) with (2) is globally robust stable.

**Proof**

The existence of the equilibrium has been given in Lemma 1. The proof of the uniqueness is obtained by an approach similar to that given in Ref. 11. We thus only prove its robust stability.

Consider the following Lyapunov functional:

\[ V(x_1, x_2, \ldots, x_n)(t) = \sum_{i=1}^{n} \lambda_i \left\{ x_i^2(t) + \sum_{j=1}^{n} |w_{ij}^\tau| L_j^2(1-r_2) \int_{t}^{\varphi(t)} \varphi_{ij}'(s) ds \right\} \times x_j^2(s - \tau_{ij}(s)) ds \quad (10) \]

We calculate the derivative of \( V(x_1, x_2, \ldots, x_n)(t) \), along the solution of Eq. (3), as follows:
\[ \dot{V}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i \left( -a_i x_i(t) + \sum_{j=1}^{n} w_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} w_{ij}^r g_j(x_j(t - \tau_{ij}(t))) \right) \]

\[ \quad + \sum_{j=1}^{n} |w_{ij}^r| L_j^2(1-r^2) [x_j^2(t) \phi_{ij}(t) - x_j^2(t - \tau_{ij}(t))] \]

\[ \leq \sum_{i=1}^{n} \lambda_i \left( -2a_i x_i^2(t) + \sum_{j=1}^{n} 2|w_{ij}|(L_{ij}^2 x_i(t))(L_{ij}^{1-r_1} x_j(t)) + \sum_{j=1}^{n} 2|w_{ij}^r|(L_{ij}^{1-r_2} x_i(t))(L_{ij}^{1-r_2} x_j(t - \tau_{ij}(t))) \right) \]

\[ \quad + \sum_{j=1}^{n} |w_{ij}^r| L_j^2(1-r_2) x_j^2(t - \tau_{ij}(t)) \]

\[ \leq \sum_{i=1}^{n} \lambda_i \left( -2a_i \lambda_i + \sum_{j=1}^{n} \lambda_j L_{ij}^2 |w_{ij}| + \sum_{j=1}^{n} \lambda_j L_{ij}^{2(1-r_1)} |w_{ji}| + \sum_{j=1}^{n} \lambda_j L_{ij}^{2r_2} |w_{ij}^r| \right) \]

\[ \quad + \sum_{j=1}^{n} \lambda_j L_{ij}^{2(1-r_2)} |w_{ij}^r| \phi_{ji}(t) \]

\[ \leq \sum_{i=1}^{n} \alpha_i \lambda_i \left( -2 + \frac{1}{\alpha_i \lambda_i} \sum_{j=1}^{n} (\lambda_j L_{ij}^{2r_1} w_{ij}^* + \lambda_j L_{ij}^{2(1-r_1)} w_{ji}^{**} + \lambda_j L_{ij}^{2r_2} w_{ij}^{***} + \lambda_j L_{ij}^{2(1-r_2)} w_{ji}^{****} \phi_{ji}(t)) \right) x_i^2(t) \]

\[ \leq (-2 + \beta^r) \sum_{i=1}^{n} \alpha_i \lambda_i x_i^2(t) < 0. \]

Now, by a standard Lyapunov-type theorem for functional differential equations,\(^{18}\) for \(\beta^r < 2\), the trivial solution of Eq. (3) with Eq. (2) is globally robust stable. Therefore, \(u^*\) is globally robust stable for system (1) with (2). \(\square\)

**Corollary 2**

If \(A = \hat{A}, W = \hat{W}, W^r = \hat{W}^r\) and if there exist positive numbers \(\lambda_1, \lambda_2, \ldots, \lambda_n\), and \(r_1 \in [0, 1], r_2 \in [0, 1]\), such that

\[ \beta_i^r = \max_{1 \leq i \leq n} \left\{ \frac{1}{\alpha_i \lambda_i} \sum_{j=1}^{n} (\lambda_j L_{ij}^{2r_1} |w_{ij}| + \lambda_j L_{ij}^{2(1-r_1)} |w_{ji}| \right. \]

\[ \quad + \lambda_j L_{ij}^{2r_2} |w_{ij}^r| + \lambda_j L_{ij}^{2(1-r_2)} |w_{ji}^r| \phi_{ji}(t) \left. \right) < 2, \]

(11)

then system (1) has the unique and globally asymptotically stable equilibrium \(u^*\) for \(I = (I_1, I_2, \ldots, I_n)^T \in \mathbb{R}^n\).

**Remark 3**

In Refs. 8 and 9, the proof of the existence and uniqueness for the equilibrium requires that the activation functions are continuous, monotonic, differentiable, and bounded. However, in the proof of Theorem 2 above, the activation functions only need to satisfy conditions (H1) or (H1'). Hence, our results given here are stronger and less restrictive than those given in Refs. 8 and 9.

**Remark 4**

If we set \(r_1 = r_2 = 0\), then condition (8) becomes

\[ \alpha^t \Delta \max_{1 \leq i \leq n} \left\{ \frac{1}{\alpha_i \lambda_i} \sum_{j=1}^{n} (\lambda_i (w_{ji}^* + w_{ij}^{**}) \right. \]

\[ \quad + \lambda_i L_i^2 (w_{ji}^* + w_{ij}^{***}) \phi_{ji}(t)) \left. \right) < 2, \]

(12)

which corresponds to the derivation of the following
Lyapunov functional:

\[
V(x_1, x_2, \ldots, x_n)(t) = \sum_{i=1}^{n} \lambda_i \left( x_i^2(t) + \sum_{j=1}^{n} |w_{ij}^w|^2 \int_t^{\varphi_{ij}(t)} x_j^2(s - \tau_{ij}(s)) \, ds \right) \times x_j^2(s - \tau_{ij}(s)) \, ds .
\]

(13)

However, this Lyapunov functional is similar to the one given in Ref. 4. The major difference is that the neural networks studied in Ref. 4 have constant delays but here they have time-varying time-delays. Hence, the result obtained here is more general. The following result is immediate.

**Theorem 3**

Assume that there exist positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that

\[
\gamma^t = \max_{1 \leq i \leq n} \left\{ \frac{L_i}{\lambda_i} \left| \sum_{j=1}^{n} (|\lambda_i w_{ij}^{**} + \lambda_j w_{ji}^{**}| + \lambda_i w_{ij}^{**} + \lambda_j w_{ji}^{**}) \right| \right\} < 2 ,
\]

(14)

where

\[
\left\{ \begin{array}{l}
 w_{ij}^{**} = \max_{1 \leq i,j \leq n} \{ |w_{ij}|, |w_{ij}^w| \} , \\
 w_{ij}^{**} = \max_{1 \leq i,j \leq n} \{ |w_{ij}^w|, |w_{ij}^w| \} ,
\end{array} \right.
\]

(15)

then system (1) with (2) has a unique and globally robust stable equilibrium \( u^* \) for each constant input \( I = (I_1, I_2, \ldots, I_n)^T \in \mathbb{R}^n \).

**Proof**

The existence of the equilibrium has been given in Lemma 1. The proof of the uniqueness is obtained by an approach similar to that given in Ref. 11. We therefore only prove the global robust stability of the equilibrium \( u^* \).

Define a Lyapunov functional as follows.

\[
\dot{V}(x_1, x_2, \ldots, x_n)(t) = \sum_{i=1}^{n} \lambda_i \left( g_i(x_i(t)) \left[ -a_i x_i(t) + \sum_{j=1}^{n} w_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} w_{ij}^w g_j(x_j(t - \tau_{ij}(t))) \right] \\
+ \frac{1}{2} \sum_{j=1}^{n} |w_{ij}^w| |g_j^2(x_j(t))| \varphi'_{ij}(t) - g_j^2(x_j(t - \tau_{ij}(t))) \right) \right\}
\]

\[
\leq \sum_{i=1}^{n} \left\{ -a_i \lambda_i x_i(t) g_i(x_i(t)) + \sum_{j=1}^{n} g_i(x_i(t)) \lambda_i w_{ij} g_j(x_j(t)) \\
+ \frac{\lambda_i}{2} \sum_{j=1}^{n} |w_{ij}^w|^2 (g_j^2(x_j(t - \tau_{ij}(t))) + g_j^2(x_j(t))) \right\}
\]

\[
\leq \sum_{i=1}^{n} \left\{ -a_i \lambda_i g_i(x_i(t)) + \frac{L_i}{2} \sum_{j=1}^{n} \lambda_i w_{ij} + \lambda_j w_{ji} g_j(x_j(t)) + \frac{1}{2} \sum_{j=1}^{n} \lambda_i |w_{ij}^w| + \lambda_j |w_{ji}^w| \varphi'_{ij}(t) \right\} g_j(x_j(t))
\]

\[
\leq \sum_{i=1}^{n} \left\{ -2 + \gamma^t \right\} \sum_{i=1}^{n} \frac{\lambda_i}{2L_i} g_i^2(x_i(t)).
\]
By a standard Lyapunov-type theorem for functional differential equations, for \( \gamma^t < 2 \), the trivial solution of Eq. (3) is globally robust stable. Therefore, \( u^* \) is globally robust stable for system (1) with (2). \( \square \)

**Corollary 3**

If \( A = \bar{A}, W = \bar{W}, W^T = \tilde{W}^T = \tilde{W}^\tau \) and there exist positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that

\[
\gamma_1 \triangleq \max_{1 \leq i \leq n} \left\{ \frac{L_i}{\mu_i \lambda_i} \sum_{j=1}^{n} (\lambda_i w_{ij} + \lambda_j w_{ji}) + \lambda_i |w_{ij}^{max} + \lambda_j |w_{ji}^*| |\varphi_j^*(t)| < 2, \right. \]

then system (1) has a unique and globally asymptotically stable equilibrium \( u^* \) for each constant input \( I = (I_1, I_2, \ldots, I_n)^T \in \mathbb{R}^n \).

**Remark 5**

The proof of Theorem 2 makes sufficient use of the condition \( x_i g_i(x_i) > 0 \). However, the proof of Theorem 1 does not use this property. This hints that in Theorems 1 and 2 the requirement on the activation functions can be somewhat released: they only need to satisfy

\[
\frac{f_i(x_j) - f_j(y_j)}{x_j - y_j} \leq L_j, \quad L_j > 0, x_j = y_j, \quad j = 1, 2, \ldots, n.
\]

If \( \tau_{ij} = \tau_j \), then we immediately obtain the following:

**Theorem 4**

If \( \tau_{ij} = \tau_j \) in Eq. (1) for all \( i = 1, 2, \ldots, n \), and if there exist positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that

\[
\mu^t \triangleq \max_{1 \leq i \leq n} \left\{ \frac{L_i}{\mu_i \lambda_i} \sum_{j=1}^{n} (\lambda_i w_{ij}^{*} + \lambda_j w_{ji}^{**}) + (1 + \varphi_j^*(t)) \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i w_{ij}^{**})^2 \right)^{1/2} \right\} < 1,
\]

where \( w_{ij}^{**} \) and \( w_{ij}^{***} \) are given in Eq. (15), then system (1) with (2) has a unique and globally asymptotically stable equilibrium \( u^* \) for each constant input \( I = (I_1, I_2, \ldots, I_n)^T \in \mathbb{R}^n \).

**Proof**

The existence of the equilibrium has been given in Lemma 1. The proof of the uniqueness is obtained by an approach similar to that given in Ref. 11.

We next only prove the global robust stability of the equilibrium \( u^* \).

Define a Lyapunov functional as follows:

\[
V(x_1, x_2, \ldots, x_n)(t) = \sum_{i=1}^{n} \lambda_i \int_{0}^{x_i(t)} g_i(s)ds + \frac{1}{2} \times \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i w_{ij}^{**})^2 \right)^{1/2} \sum_{j=1}^{n} \int_{t}^{x_j(t)} g_j^2(x_j(s - \tau_{ij}(s)))ds.
\]

By the Cauchy–Schwarz inequality, we can calculate the derivative of \( V(x_1, x_2, \ldots, x_n) \), along the solution of Eq. (3), as follows:
\[
+ \frac{1}{2} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i w_{ij})^2 \right) \frac{1}{2} \sum_{j=1}^{n} \left[ g_i^2(x_i(t)) + g_j^2(x_j(t - \tau_{ij})) \right]
\]

\[
+ \frac{1}{2} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i w_{ij})^2 \right) \frac{1}{2} \sum_{j=1}^{n} \left[ g_j^2(x_j(t)) \phi_{ij}(t) - g_j^2(x_j(t - \tau_{ij}(t))) \right]
\]

\[
\leq \sum_{i=1}^{n} \left\{ - \frac{a_i \lambda_i}{L_i} + \frac{1}{2} \sum_{j=1}^{n} |(\lambda_i w_{ij} + \lambda_j w_{ji})| + (1 + \phi_{ij}(t)) \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i w_{ij}^T)^2 \right) \right\} g_i^2(x_i(t))
\]

\[
\leq \sum_{i=1}^{n} \left\{ - \frac{a_i \lambda_i}{L_i} \left[ 1 + \frac{L_i}{a_i \lambda_i} \left( \frac{1}{2} \sum_{j=1}^{n} |\lambda_i w_{ij}^{**} + \lambda_j w_{ji}^{**}| + (1 + \phi_{ij}(t)) \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i w_{ij}^T)^2 \right) \right) \right] \right\} g_i^2(x_i(t))
\]

\[
\leq \left( -1 + \mu^{tt} \right) \sum_{i=1}^{n} \left\{ \frac{a_i \lambda_i}{L_i} \right\} g_i^2(x_i(t)).
\]

According to Ref. 18, for \( \mu^{tt} < 1 \) the trivial solution of Eq. (3) is globally robust stable. Therefore, \( u^* \) is globally robust stable for system (1) with (2). \( \square \)

**Corollary 4**

If \( A = A = \bar{A}, W = W = \bar{W}, W^T = W^T = \bar{W}^T \) and if there exist positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that

\[
\mu_i^1 = \max_{1 \leq i \leq n} \left\{ \frac{L_i}{a_i \lambda_i} \left( \frac{1}{2} \sum_{j=1}^{n} |\lambda_i w_{ij} + \lambda_j w_{ji}| + (1 + \phi_{ij}(t)) \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i w_{ij}^T)^2 \right) \right)^{1/2} \right\} < 1,
\]

then system (1) has a unique and globally asymptotically stable equilibrium \( u^* \) for each constant input \( I = (I_1, I_2, \ldots, I_n)^T \in \mathbb{R}^n \).

In the following theorem, we do not need the time-varying delays to be continuous and differentiable. They only need to satisfy the condition \( 0 \leq \tau_{ij}(t) \leq \bar{\tau} \).

**Theorem 5**

If there exist positive constants \( a_i' > 0, b_i' > 0 \) and \( \rho > 1 \) such that

\[
\delta^t = \max_{1 \leq i \leq n} \left( \frac{1}{a_i} \left( L_i \sum_{j=1}^{n} w_{ij}^+ \phi_i(x_i(t) + 0) \right) \right) < 1,
\]

where

\[
c_i = \max_{1 \leq i \leq n} \{ a_i', b_i' \},
\]

\[
d_i = \min_{1 \leq i \leq n} \{ a_i', b_i' \},
\]

\[
\phi_i(x_i) = \begin{cases} a_i', & x_i \geq 0 \\ -b_i', & x_i < 0 \end{cases}
\]

\[
w_{ij}^+ = \max_{1 \leq i \leq n} \{ W_{ij}, |\bar{w}_{ij}| \},
\]

\[
\bar{w}_{ij}^+ = \max_{1 \leq i, j \leq n} \{ \bar{w}_{ij}^T, |\bar{w}_{ij}^T| \},
\]

then the equilibrium \( u^* \) of system (1) with (2) is unique and is globally robust stable.

**Proof**

Lemma 1 has ensured the existence of the equilibrium. We next prove its uniqueness. Without loss of generality, assume that system (1) with (2) have two equilibrium points, \( u^* \) and \( u^{**} \). By Eq. (3) and the definition of \( g_t \), we have the following inequality:
which implies that

\[ 0 \leq \sum_{i=1}^{n} \phi_i(u_i^* - u_i^{**})(u_i^* - u_i^{**}) \left( -a_\omega + L_i \sum_{j=1}^{n} |w_{ij}| + w_{ij}^* \right) \left| \frac{\phi_j(u_j^* - u_j^{**})}{\phi_i(u_i^* - u_i^{**})} \right| \]

\[ \leq \sum_{i=1}^{n} \phi_i(u_i^* - u_i^{**})(u_i^* - u_i^{**}) \left( -a_\omega + L_i \sum_{j=1}^{n} w_{ij} \phi_j(u_j^* - u_j^{**}) \phi_i(u_i^* - u_i^{**}) \right) \]

\[ \leq \sum_{i=1}^{n} \phi_i(u_i^* - u_i^{**})(u_i^* - u_i^{**}) a_i \leq 0. \]

Hence, we have \( u_i^* = u_i^{**}, \) \( i = 1, 2, \ldots, n. \) The uniqueness of the equilibrium is thus proven.

Next, we prove the global robust stability. Define a Lyapunov functional as follows:

\[ V(x_1, x_2, \ldots, x_n)(t) = \sum_{s=1}^{n} \phi_s(x_s)x_s. \]  

(23)

We then calculate the upper right derivative of \( V(x_1, x_2, \ldots, x_n)(t), \) along the solution of Eq. (3), and obtain

\[ D^+ V(x_1, x_2, \ldots, x_n)(t) \mid_{t=\tau_s} \]

\[ = \sum_{s=1}^{n} \phi_s(x_s(t) \pm 0) \left( -a_s x_s(t) + \sum_{j=1}^{n} w_{sj} g_j(x_j(t)) + \sum_{j=1}^{n} w_{sj}^* g_j(x_j(t - \tau_{sj}(t))) \right) \]

\[ \leq \sum_{j=1}^{n} \phi_j(x_j(t)) x_j(t) \left( -a_j + L_j \sum_{s=1}^{n} w_{sj} \frac{\phi_s(x_s(t) \pm 0)}{\phi_j(x_j(t))} \right) + \sum_{s=1}^{n} \sum_{j=1}^{n} L_j \phi_s(x_s(t) \pm 0) w_{sj}^* |x_j(t - \tau_{sj}(t))|. \]

Choose \( \rho > 1 \) such that

\[ V(x_1(t + \theta), x_2(t + \theta), \ldots, x_n(t + \theta)) \leq \rho V(x_1(t), x_2(t), \ldots, x_n(t)), \quad \theta \in [-\tau, 0]. \]

Then, we have

\[ |x_j(t + \theta)| \leq \frac{\rho}{|\phi_j(x_j(t + \theta))|} \sum_{s=1}^{n} \phi_s(x_s(t))x_s(t), \]
so that

\[ D^+ V(x_1, x_2, \ldots, x_n)(t) \leq \sum_{j=1}^{n} \phi_j(x_j(t))x_j(t) \left( -a_j + L_j \sum_{s=1}^{n} \left| w_{sj}^+ \frac{\phi_s(x_s(t) \pm 0)}{\phi_j(x_j(t))} \right| \right) + \rho \sum_{i=1}^{n} \sum_{s=1}^{n} L_s \left| \sum_{k=1}^{n} \phi_k(x_k(t))x_k(t) \right| \]

This implies that the equilibrium \( u^* \) is globally robust stable. \( \square \)

In Theorem 5, let \( a_s = a^s, b_s = a^s, \) where \( a^s \) is the \( s \)th power of \( a. \)

**Corollary 5**

If there exist constants \( a > 0 \) and \( \rho > 1 \) such that

\[
\begin{align*}
(i) \quad & \frac{\alpha_j}{L_j} \frac{w_{sj}^+}{1 - w_{jj}^+} \leq \frac{a_{j-s}}{n}, & s \neq j, s, j = 1, 2, \ldots, n \\
(ii) \quad & \frac{(1 - w_{jj}^+)}{n \sum_{i=1}^{n} \sum_{j=1}^{n} a^{s-1} L_s w_{si}^+} \leq \rho > 1, & j = 1, 2, \ldots, n.
\end{align*}
\]

then the equilibrium \( u^* \) of system (1) with (2) is globally robust stable.

**Proof**

The proof for the existence and uniqueness of the equilibrium \( u^* \) is similar to that given in the proof of Theorem 5. Therefore, we only prove the global robust stability.

By Theorem 5, we have

\[ D^+ V(x_1, x_2, \ldots, x_n)(t) \]

\[
\leq \sum_{j=1}^{n} \phi_j(x_j(t))x_j(t) \left( -a_j + L_j \sum_{s=1}^{n} \left| w_{sj}^+ \frac{\phi_s(x_s(t) \pm 0)}{\phi_j(x_j(t))} \right| \right) + \rho \sum_{i=1}^{n} \sum_{s=1}^{n} L_s \left( \frac{c_s}{d_i} w_{si}^+ \right) \]

\[ = \sum_{j=1}^{n} \phi_j(x_j(t))x_j(t) \left( 1 - w_{jj}^+ \right) \left( -a_j + L_j \sum_{s=1, s \neq j}^{n} \left| w_{sj}^+ \frac{\phi_s(x_s(t) \pm 0)}{1 - w_{jj}^+} \right| \right) + \rho \sum_{i=1}^{n} \sum_{s=1}^{n} L_s \left( \frac{c_s}{d_i} w_{si}^+ \right) \]

\[
\leq \sum_{j=1}^{n} \phi_j(x_j(t))x_j(t) \left( 1 - w_{jj}^+ \right) \left( -1 + \frac{n-1}{n} + \frac{1}{n} \right) a_j = 0.
\]

This completes the proof of the theorem. \( \square \)
4. Discussions and Conclusions

It is well known that one of the most investigated problems in nonlinear dynamical system theory is that of the existence, uniqueness and global and robust stability of the equilibrium point, (see e.g. Refs. 2-6, 9-11 and 16-18). However, the property of global and robust stability, which means that the domain of attraction of the equilibrium point is the whole space, is of importance from a theoretical as well as application point of view in several field.\cite{2-11} In particular, in the neural network field, global and robust stable networks are known to be well suited for solving some classes of optimization problems. Actually, a global and robust stable neural network is guaranteed to compute the global optimal solution independently of the initial condition, which in turn implies that the network is devoid of spurious suboptimal responses. Such global and robust neural networks can also be useful for accomplishing other interesting cognitive or computational tasks.\cite{1}

In the section, some novel sufficient criteria for delay-dependent global and robust stability INHTVD have been derived. The main results were given in conditions (4), (7), (8), (14), (18) and (21), which have been generalized and improved the existing results given in Refs. 2-11. As pointed out in Refs. 9-11, conditions for ensuring global and robust stability of the neural networks (1) with (2) have been investigated for a long time. In the present context, it can be carried out in the following two different ways: (1) finding classes of interconnection weight matrices; (2) developing algorithms that provide a numerical answer to the problem for neural networks of realistic sizes or even higher-dimension neural networks.

It follows from the proof of Theorem 1 that one knows how to find parameters \( \lambda_i \) and \( L_i \) \((i = 1, 2, \ldots, n)\), such that they satisfy inequality (4). A practical algorithm is then suggested as follows:

(i) Select \( L_i \) \((i = 1, 2, \ldots, n)\) to satisfy (H1) and (H2). Note that the choice of \( L_i \) might depend on the properties of the activation functions used.

(ii) Select \( \lambda_i \) \((i = 1, 2, \ldots, n)\). Start with a small value of \( \lambda_i \) \((i = 1, 2, \ldots, n)\), and then increase this value gradually. Note that \( \lambda_i \) might indicate some sort of stability margins.

(iii) Compute \( \alpha^i \) on the left side of Eq. (4).

(iv) Check whether inequality (4) is satisfied; if not, return to Step (ii) and increase the values of \( \lambda_i \).

Remark 6

To implement those conditions (7), (8), (14), (18) and (21), an algorithm similar to the one outline above is needed and omitted here.

In this paper, we have derived some criteria for the global and robust stability of a class of interval as well as conventional Hopfield neural networks with time-varying delays. We have in effect extended some previously known results, obtained by Liao and Yu,\cite{9-11} to the time-varying delay case. These new results have been shown to be a generalization of some other existing results reported recently. Moreover, these results require weaker conditions on the activation functions of the networks. However, how to find appropriate positive constants \( \lambda_i (i = 1, 2, \ldots, \ldots) \) to achieve less conservative robust stability bounds still remains a question for future research.

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References


