Transition from Sinusoidal to Relaxation Oscillations
in Emitter-Coupled Multivibrators

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Abstract — Method of quasilinear approximation is applied to investigate the sinusoidal oscillations obtained in emitter-coupled multivibrators. The amplitude and frequency are found, and frequency decreasing occurring in development of oscillation from the starting point to steady state is explained. The transition from sinusoidal to relaxation oscillations is explained by the modification of the phase plane partitioning when the coupling capacitance is varying. The jumps of the describing point in the phase plane help to understand the difficulties occurring in simulation of multivibrators. Finally, the formula for calculation of frequency in relaxation oscillations is given. All results are verified in simulations.

Index term — Emitter-coupled multivibrators, sinusoidal oscillations, relaxation oscillations, van der Pol equations.

I. INTRODUCTION

Emitter-coupled multivibrators are used very frequently in the systems requiring voltage-controlled oscillators, the phase-locked loops can be a typical, but not exhaustive, representative example. Forcing these circuits to oscillate at the maximal frequency (this is achieved reducing the coupling capacitance value) results in the oscillations which are more resembling to sinusoidal and not to relaxation oscillations. The transition from relaxation to sinusoidal oscillations (in the paper we move in the opposite direction) is continuous. The investigation of this process is important for using these multivibrators in high speed applications.

It is known that classical Abraham-Bloch multivibrators at high frequencies may slip into sinusoidal regime. It was established experimentally long ago [1], and later on was explained analytically [2] using the method of energy cycles (we prefer to call it K.F. Theodorchik’s method). Yet, the analytical treatment of similar transition in emitter-coupled multivibrators was never tried. Here we apply the method of [2] for consideration of sinusoidal oscillations (in the paper we move in the opposite direction) is continuous. The investigation of this process is important for using these multivibrators in high speed applications.

II. SINUSOIDAL REGIME DIFFERENTIAL EQUATION

Fig. 1 shows the emitter-coupled multivibrator (Fig. 1, a) and its simplified small-signal circuit (Fig. 1, b). Assuming sinusoidal oscillations and symmetric circuit (i.e. $r_{n1} = r_{n2} = r_n$, $C_{c1} = C_{c2} = C_c$, and $g_{m1} = g_{m2} = g_m$) one can find that the input impedance, $z_{in}$, connected to the coupling capacitor, $C$, is equal to

$$z_{in} = \frac{2[r_n + R(1 - r_c g_m) + R C_c s]}{1 + r_c g_m + r_c C_c s}.$$  (1)

Substituting (1) in the oscillation condition

$$iz_{in} + \frac{1}{C} \int idt = 0,$$  (2)

and treating $s$ as the time-domain differentiation operator one can obtain the differential equation for the capacitor current, $i$, as following
\[
\frac{d^2 x}{dt^2} + \omega^2 x = F(t) \cos \omega t + f(t) \sin \omega t
\]  
(11)

Hence, for slowly changing \( F(t) \) the approximate solution of the equation

\[
\frac{d^2 x}{dt^2} + \omega^2 x = F(t) \cos \omega t + f(t) \sin \omega t
\]  
(11)

will be

\[
x = \frac{1}{2 \omega} \int_0^t F(\xi) d\xi - \frac{1}{2 \omega} \cos \omega t \int_0^t f(\xi) d\xi ,
\]

i.e.,

\[
x = a(t) \sin \omega t - b(t) \cos \omega t ,
\]

where (approximately)

\[
a(t) = \frac{1}{2 \omega} \int_0^t F(\xi) d\xi , \quad b(t) = \frac{1}{2 \omega} \int_0^t f(\xi) d\xi .
\]

Differentiating (14) one obtains

\[
\frac{da}{dt} = \frac{1}{\omega} F(t) ; \quad \frac{db}{dt} = \frac{1}{\omega} f(t)
\]

Let us now apply these results to obtain the solution of (7). We rewrite this equation as

\[
\frac{d^2 x}{dt^2} + \omega^2 x = a(t) \sin \omega t - b(t) \cos \omega t
\]

and substitute (17) in the right side of (16). Yet, in doing so we consider that the amplitudes in (17) are changing so slowly that we may neglect their derivatives (this is not an error, it simply means that the mathematical foundation of the proposed approach lies in the averaging methods [3]). This gives us

\[
\frac{d^2 x}{dt^2} + \omega^2 x = \frac{a(t) \sin \omega t - b(t) \cos \omega t}{2} + \frac{a(t)^2 x(1 - x^2)}{4}.
\]

III. AMPLITUDE TRANSIENT EQUATIONS

For the zero initial conditions \( x(0) = (dx/dt) \big|_{t=0} = 0 \), the full solution of the differential equation

\[
\frac{d^2 x}{dt^2} + \omega^2 x = \tilde{g}(t)
\]  
(8)

is equal to [2]

\[
x = \frac{1}{\omega} \sin \omega t \int_0^t \tilde{g}(\xi) \cos \omega \xi d\xi - \frac{1}{\omega} \cos \omega t \int_0^t \tilde{g}(\xi) \sin \omega \xi d\xi .
\]  
(9)

Assume now that \( \tilde{g}(t) = F(t) \cos \omega t \) where \( F(t) \) is a function of time. Then (9) gives us

\[
x = \frac{1}{\omega} \sin \omega t \int_0^t F(\xi) \cos 2 \omega \xi d\xi + \frac{1}{\omega} \int_0^t \sin \omega t F(\xi) \cos 2 \omega \xi d\xi
\]  
(10)

If \( F(t) = F_0 = \text{const} \) the second term of (10) becomes zero. When \( F(t) \) is not constant, this term will be much less than the first term, if \( F(t) \) changes much slower than \( \cos \omega t \).
IV. ANALYSIS OF TRANSIENT EQUATIONS

The steady-state solutions of (19) are obtained equating the derivatives to zero. This results in two mutually excluding solutions. The first one is \( B = 0 \), and the amplitude \( A \) and frequency \( \omega \) are obtained from the system of equations

\[
\begin{align*}
2\delta_0 \omega - 2\delta_0 \omega \frac{A^3}{4} &= 0 \\
\omega^2 A - \omega_0^2 A + \omega_0^2 \frac{3A^3}{4} &= 0
\end{align*}
\]

(20)

The second solution is \( A = 0 \), and amplitude and frequency that are obtained from the system that is the same as (20), and where \( A \) is replaced by \( B \), and the signs of all terms in the second equation are opposite. Hence, as soon as oscillation starts, one solution is developed (let it be the first one) and the other is suppressed. One finds from (20) that

\[
A = A_0 = 2 \frac{\delta_0}{\omega_0}, \quad \omega = \omega_0 \sqrt{1 - \frac{3 \delta_0}{\delta_0}}.
\]

(21)

Hence, the oscillation initial frequency is \( \omega_0 \), and when the oscillation arrives to steady state its frequency is reduced. The results (21) could be obtained by the harmonic balance method [3] as well.

Considering that the steady state solution is given by (21) and \( B = 0 \), we are left with equation

\[
\frac{dA}{dt} = \frac{1}{2\omega^2} \left[ 2\omega \delta_0 A - \frac{2\omega \delta_0 A^3}{4} \right].
\]

(22)

Substituting \( A = A_0 + \Delta A \) in (25) and doing linearization one finds that the perturbation, \( \Delta A \), of the limit cycle amplitude is described by the differential equation

\[
\frac{d\Delta A}{dt} = -2\delta_0 \Delta A.
\]

(23)

The characteristic equation of (23) has one negative root \( p = -2\delta_0 \). Hence, the oscillation is stable, and its stability increases when \( \delta_0 \) is increasing, i.e. when the multivibrator moves from sinusoidal to relaxation oscillations.

Strictly speaking there is no pure sinusoidal regime in the multivibrators (as in any other oscillator as well), including the considered one. When the circuit moves to the relaxation regime, the oscillation becomes reach in harmonics [4]. The oscillation, first, starts to include the third harmonic. Using the method of [5] one can find that the development of the third harmonic increases the stability of the first harmonic.

V. RELAXATION OSCILLATIONS

When the ratio \( C_z/(2C) \rightarrow 0 \), the oscillator moves from nearly sinusoidal to relaxation oscillations. This transition can be traced using the phase plane. Using (6), and substituting the normalized variable \( x = i/I_0 \), one introduces \( y = \frac{dx}{dt} \). Then (6) may be rewritten as

\[
\frac{dy}{dx} = - \left[ \frac{1 + C_z}{2C} \frac{RI_x}{V_T} \frac{1 - x^2}{(1 - x^2)} \right] y + \frac{I_0}{2CV_T} x(1 - x^2) + \frac{yRC_T}{201}.
\]

(24)

One can see that the point \( x = 0; y = 0 \) is an unstable focus. The borders of increment and decrement regions can be obtained from the equation

\[
1 + \frac{C_z}{2C} \frac{RI_0}{V_T} (1 - x^2) = 0.
\]

(25)

This gives

\[
x_b = \pm \sqrt{1 - \frac{1 + \frac{C_z}{2C}}{I_0R}}.
\]

(26)

The isoline of horizontal tangents can be obtained from the equation

\[
y_z = - \frac{I_0}{2CV_T} \frac{x(1 - x^2)}{1 + \frac{C_z}{2C} \frac{RI_0}{V_T} (1 - x^2)}.
\]

(27)

This gives three branches. The results may be summarized the following way (Fig. 2).

![Fig. 2 Phase-plane partitioning](image-url)
addition, this point is obliged now to move very close (nearly along) to the isoclines of horizontal tangents, or along the line \( y = 0 \). The phase plane now includes a wide incremental zone (Fig. 2b). In the steady state oscillation, the describing point most of the first semi-period is sitting in, say, \( x = 1 \) (point A), then it starts to move to the border of incremental zone, in the vicinity of the horizontal axis. As soon as it enters the incremental zone it will be in a very short time on the isocline of horizontal tangents, and arrives to the point B. Here it makes a jump on another branch of the isocline of horizontal tangents, to the point C, then quickly arrives to A. The first semi-period is finished.

Another semi-period is similar: i.e. for most of the time the describing point is sitting in the point \( A_1 \), then moves to the border of incremental zone, enters it, quickly arrives to \( B_1 \), has a jump on another branch of the isocline of horizontal tangents, to the point \( C_1 \), and then quickly arrives to A.

This behaviour explains very well: a) the difficulty of some simulators to simulate the multivibrators when \( C_e \) is removed from the transistor model, and b) the appearance of very sharp and fast spikes in the current (seen at the collector resistances) because the point \( C_1 \) is further from the \( y \)-axis than A (or C is further from \( y \)-axis than \( A_1 \)). In reality, the isocline of horizontal tangents is a continuous line going from a local maximum to the minimum (or vice versa) at the borders of incremental zone, and the jump from \( B_1 \) to \( C_1 \) or from \( B \) to \( C \) will be at the points of such local maximums or minimums.

Hence, the movement of the describing point from A (or from \( A_1 \)) to the incremental zone is the only time when noise or other interference signal can interact with the multivibrator. It is very difficult to say when the describing point moved sufficiently off the line \( x = 0 \) so that the accelerating properties of the phase plane will bring it to another point of rest. Yet, this time is important for calculation of the oscillation frequency. A good result gives the following approach.

When one transistor is ON and another is just start to operate the resistor seen at the capacitor terminals is

\[
z_m(0) = \frac{1}{g_m} + \frac{1}{g_m} - 2R = \frac{V_t}{2I_c} - \frac{V_t}{2I_c} - 2R.
\]

Let us consider that the describing point is sufficiently moved from the horizontal axis when this value becomes equal to \(-R\). This gives us

\[
\Delta t = I_c(1-a),
\]

where \( a = \sqrt{1 - \frac{2V_t}{I_cR}} \). The voltage at the capacitor at this instant is equal to

\[
V_c = 2I_cR \left[ a - \frac{V_t}{2I_cR} \ln \left( \frac{1+a}{1-a} \right) \right].
\]

Assuming now that the capacitor is recharging from \( V_c \) to \(-V_c\) by the current \( I_c \) one finds the oscillation frequency as

\[
f_o = \frac{1}{8RC} \left[ a - \frac{V_t}{2I_cR} \ln \left( \frac{1+a}{1-a} \right) \right]
\]

This formula gives better results than calculating the frequency considering the capacitor as the short circuit, finding the loop transfer function, and equating it to 1 [6].

VI. SIMULATION RESULTS

The essential simplification of the transistor model allows one to analyze the sinusoidal and relaxation regimes of the emitter-coupled multivibrator. Yet, it results in some difference between simulation results and the results given in the paper. The oscillation frequency \( \omega \) in simulations is lower by approximately 20%. A similar error occurs for the final frequency of \( \omega \). The results are improved, if the capacitance due to Miller effect of \( C_e \), the capacitor between collector and base, is added to \( C_e \).

VII. CONCLUSION

The theory of energy cycles (we called it here “method of quasilinear approximation”) developed by K.F. Theodorchik allows one to solve rather complicated problems of the oscillations theory in applications to microelectronics. We demonstrated it by investigating the transition from sinusoidal to relaxation oscillation in the emitter-coupled multivibrator. The results obtained show that the sinusoidal regime may be used in pushing the circuit to the maximal available speed, even though this results in a less stable limit cycle.

The method can be also expanded on the system of coupled oscillators. This expansion is now under development.

REFERENCES