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GEOMETRIC STRUCTURE IN THE PRINCIPAL SERIES OF THE p-ADIC GROUP G_2

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Abstract. In the representation theory of reductive p-adic groups G, the issue of reducibility of induced representations is an issue of great intricacy. It is our contention, expressed as a conjecture in (2007), that there exists a simple geometric structure underlying this intricate theory.

We will illustrate here the conjecture with some detailed computations in the principal series of G_2.

A feature of this article is the role played by cocharacters $h_{\mathbf{c}}$ attached to two-sided cells $\mathbf{c}$ in certain extended affine Weyl groups.

The quotient varieties which occur in the Bernstein programme are replaced by extended quotients. We form the disjoint union $\mathfrak{A}(G)$ of all these extended quotient varieties. We conjecture that, after a simple algebraic deformation, the space $\mathfrak{A}(G)$ is a model of the smooth dual $\mathrm{Irr}(G)$. In this respect, our programme is a conjectural refinement of the Bernstein programme.

The algebraic deformation is controlled by the cocharacters $h_{\mathbf{c}}$. The cocharacters themselves appear to be closely related to Langlands parameters.

1. Introduction

In the representation theory of reductive p-adic groups, the issue of reducibility of induced representations is an issue of great intricacy; see, for example, the classic article by Bernstein and Zelevinsky [6] on $\mathrm{GL}(n)$ and the more recent article by Muić [21] on G_2. It is our contention, expressed as a conjecture in [3], that there exists a simple geometric structure underlying this intricate theory. We will illustrate here the conjecture with some detailed computations in the principal series of G_2.

Let F be a local nonarchimedean field, let G be the group of F-rational points in a connected reductive algebraic group defined over F, and let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible smooth representations of G.

Our programme is a conjectural refinement of the Bernstein programme, as we now explain. Denote by $\mathfrak{Z}(G)$ the centre of the category of smooth G-modules. According to Bernstein [5], the centre $\mathfrak{Z}(G)$ is isomorphic to the product of finitely generated subalgebras, each of which is the coordinate algebra of a certain irreducible algebraic variety, the quotient D/Γ of an algebraic variety D by a finite group Γ. Let $\Omega(G)$ denote the disjoint union of all these quotient varieties. The *infinitesimal character inf.ch.* is a finite-to-one map

$$inf.ch. : \mathrm{Irr}(G) \to \Omega(G).$$

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Our basic idea is simple: we replace each quotient variety D/Γ by the extended quotient $D/\!/\Gamma$, and form the disjoint union $\mathfrak{A}(G)$ of all these extended quotient varieties. We conjecture that, after a simple algebraic deformation, the space $\mathfrak{A}(G)$ is a model of the smooth dual $\mathrm{Irr}(G)$.

The algebraic deformation is controlled by finitely many cocharacters $h_{\mathbf{c}}$, one for each two-sided cell $\mathbf{c}$ in the extended affine Weyl group corresponding to (D, Γ). In fact, the cells $\mathbf{c}$ determine a decomposition of each extended quotient $D/\!/\Gamma$. The cocharacters themselves appear to be closely related to Langlands parameters.

In this article, we verify the conjecture in [3] for the principal series of the p-adic group G_2. We have chosen this example as a challenging test case.

Hence let $G = \mathrm{G}_2(F)$ be the group of F-points of a reductive algebraic group of type G_2. Let T denote a maximal torus in G_2, and let $T^\vee$ denote the dual torus in the Langlands dual $\mathrm{G}_2^\vee$:

$$T^\vee \subset \mathrm{G}_2^\vee = \mathrm{G}_2(\mathbb{C}).$$

Since we are working with the principal series of G_2, the algebraic variety D has the structure of the complex torus $T^\vee$.

Let $X(T^\vee)$ denote the group of characters of $T^\vee$ and let $X_*(T)$ denote the group of cocharacters of T. By duality, these two groups are identified: $X_*(T) = X(T^\vee)$. Let $\Psi(T)$ denote the group of unramified characters of T. We have an isomorphism

$$T^\vee \cong \Psi(T), \qquad t \mapsto \chi_t$$

where

$$\chi_t(\phi(\varpi_F)) = \phi(t)$$

for all $t \in T^\vee$, $\phi \in X_*(T) = X(T^\vee)$, and ϖ_F is a uniformizer in F.

We consider pairs (T, λ) consisting of a maximal torus T of G and a smooth quasicharacter λ of T. Two such pairs (T_i, λ_i) are *inertially equivalent* if there exists $g \in G$ and $\psi \in \Psi(T_2)$ such that

$$T_2 = T_1^g \qquad \text{and} \qquad \lambda_1^g = \lambda_2 \otimes \psi.$$

Here, $T_1^g = g^{-1}T_1 g$ and $\lambda_1^g : x \mapsto \lambda_1(gxg^{-1})$ for $x \in T_1^g$. We write $[T, \lambda]_G$ for the inertial equivalence class of the pair (T, λ) and $\mathfrak{T}(G)$ for the set of all inertial equivalence classes of the form $[T, \lambda]_G$.

We will choose a point $\mathfrak{s} \in \mathfrak{T}(G)$. Let $(T, \lambda) \in \mathfrak{s}$. We will write

$$D^{\mathfrak{s}} := \{\lambda \otimes \psi : \psi \in \Psi(T)\}$$

for the $\Psi(T)$-orbit of λ in $\mathrm{Irr}(T)$. Let $W(T)$ be the Weyl group $\mathrm{N}_G(T)/T$. We set

$$W^{\mathfrak{s}} := \{w \in W(T) : w \cdot \lambda \in D^{\mathfrak{s}}\}. \tag{1}$$

We have the standard projection

$$\pi^{\mathfrak{s}} \colon D^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to D^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

Section 1: This leads up to our main result; see Theorem 1.4.

Section 2: We explain the strategy of our proof.

Section 3: This contains background material on G_2.

Sections 4–8: These sections are devoted to our proof, which requires 20 lemmas. The lemmas are arranged in a logical fashion: Lemma x.y is a proof of part y of the conjecture for the character λ of T which appears in section x. These lemmas involve some detailed representation theory, and some calculations of the ideals $J_{\mathbf{c}}$ in the asymptotic algebra J of Lusztig corresponding to the complex Lie group

$\mathrm{SO}(4,\mathbb{C})$. The computation of the ideal $J_{\mathbf{e}_0}$ in Section 8 is intricate. Our result here is especially interesting. We establish a geometric equivalence (see the definition below)

$$J_{\mathbf{e}_o} \asymp \mathcal{O}(T^\vee/W^\mathfrak{s}) \oplus \mathbb{C} \tag{2}$$

where $\mathbf{e}_0$ is the lowest two-sided cell and $\lambda = \chi \otimes \chi$ with χ is a ramified quadratic character of $F^\times$. This geometric equivalence has the effect of *separating* the two constituents of an L-packet in the principal series of G_2.

In [2, §4] we introduced a *geometrical equivalence* $\asymp$ between finite type k-algebras, which is generated by elementary steps of the following three types: Morita equivalences, morphisms which are spectrum-preserving with respect to filtrations, and deformations of central characters. *The assertion that $A \asymp B$ will mean that a definite geometrical equivalence has been constructed between A and B.* We would like to emphasize the fact that in (2) no deformation of central character is used.

Let $\mathrm{Irr}(G)^\mathfrak{s}$ denote the $\mathfrak{s}$-component of $\mathrm{Irr}(G)$ in the Bernstein decomposition of $\mathrm{Irr}(G)$. We will give the quotient variety $T^\vee/W^\mathfrak{s}$ the Zariski topology, and $\mathrm{Irr}(G)^\mathfrak{s}$ the Jacobson topology. We note that irreducibility is an *open* condition, and so the set $\mathfrak{R}^\mathfrak{s}$ of reducible points in $T^\vee/W^\mathfrak{s}$, i.e., those $(M, \psi \otimes \lambda)$ such that when parabolically induced to G, $\psi \otimes \lambda$ becomes reducible, is a sub-variety of $T^\vee/W^\mathfrak{s}$. The reduced scheme associated to a scheme $\mathfrak{X}^\mathfrak{s}$ will be denoted $\mathfrak{X}^\mathfrak{s}_{\mathrm{red}}$ as in [11, p. 25]. In the present context, a *cocharacter* will mean a homomorphism of algebraic groups $\mathbb{C}^\times \to T^\vee$.

Let $\mathcal{H}^\mathfrak{s}(G)$ be the Bernstein ideal of the Hecke algebra of G determined by $\mathfrak{s} \in \mathfrak{T}(G)$. The point $\mathfrak{s} \in \mathfrak{T}(G)$ and the two-sided ideal $\mathcal{H}^\mathfrak{s}(G)$ are said to be *toral*. Then $\mathfrak{s}$ is toral if and only if $D^\mathfrak{s}/W^\mathfrak{s}$ has maximal dimension (that is, 2 here) in $\Omega(G)$.

Let $G^\vee = \mathrm{G}_2(\mathbb{C})$ be the complex reductive Lie group dual to G, and let $T^\vee \subset \mathrm{G}_2(\mathbb{C})$. We define

$$\widetilde{W}^\mathfrak{s}_{\mathrm{a}} := W^\mathfrak{s} \ltimes X(T^\vee). \tag{3}$$

The group $W^\mathfrak{s}$ is a *finite Weyl group* and $\widetilde{W}^\mathfrak{s}_{\mathrm{a}}$ is an *extended affine Weyl group* (that is, the semidirect product of an affine Weyl group by a finite abelian group), see Section 2. Let $\Phi^{\mathfrak{s}\vee}$ denote the coroot system of $W^\mathfrak{s}$, and let $Y(T^\vee)$ denote the group of cocharacters of $T^\vee$. Then let $H^\mathfrak{s}$ be the complex Lie group with root datum $(X(T^\vee), \Phi^\mathfrak{s}, Y(T^\vee), \Phi^{\mathfrak{s}\vee})$. We will see that the possible groups $H^\mathfrak{s}$ are $\mathrm{G}_2(\mathbb{C})$, $\mathrm{GL}(2,\mathbb{C})$, $\mathrm{SL}(3,\mathbb{C})$ and $\mathrm{SO}(4,\mathbb{C})$. We will consider these cases in sections 4, 6, 7 and 8, respectively.

For simplicity, we shall assume in the Introduction that $H^\mathfrak{s}$ has a simply connected derived group. This will be the case in Sections 4, 6 and 7 (not in Section 8).

As any extended affine Weyl group, the group $\widetilde{W}^\mathfrak{s}_{\mathrm{a}}$ is partitioned into *two-sided cells*. This partition arises (together with Kazhdan-Lusztig polynomials) from comparison of the Kazhdan-Lusztig basis for the Iwahori-Hecke algebra of $\widetilde{W}^\mathfrak{s}_{\mathrm{a}}$ with the standard basis. Let $\mathrm{Cell}(\widetilde{W}^\mathfrak{s}_{\mathrm{a}})$ be the set of two-sided cells in $\widetilde{W}^\mathfrak{s}_{\mathrm{a}}$. The definition of cells yields a natural partial ordering on $\mathrm{Cell}(\widetilde{W}^\mathfrak{s}_{\mathrm{a}})$. We will denote by $\mathbf{c}_0$ the *lowest* two-sided cell in $\widetilde{W}^\mathfrak{s}_{\mathrm{a}}$.

Let $J^\mathfrak{s}$ denote the Lusztig asymptotic algebra of the group $\widetilde{W}^\mathfrak{s}_{\mathrm{a}}$ defined in [17, §1.3]; this is a $\mathbb{C}$-algebra whose structure constants are integers and which may be regarded as an asymptotic version of the Iwahori-Hecke algebras $\mathcal{H}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}},\tau)$ of $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$, where $\tau\in\mathbb{C}^{\times}$. Moreover, $J^{\mathfrak{s}}$ admits a canonical decomposition into finitely many two-sided ideals $J^{\mathfrak{s}}=\bigoplus J_{\mathbf{c}}^{\mathfrak{s}}$, labelled by the the two-sided cells in $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$.

Proposition 1.1. *There exists a partition of $T^{\vee}//W^{\mathfrak{s}}$ indexed by the two-sided cells in $\widetilde{W}_{a}^{\mathfrak{s}}$:*

$$T^{\vee}//W^{\mathfrak{s}}=\bigcup_{\mathbf{c}\in\mathrm{Cell}(\widetilde{W}_{a}^{\mathfrak{s}})}(T^{\vee}//W^{\mathfrak{s}})_{\mathbf{c}}.$$

The partition can be chosen so that the following property holds:

$$T^{\vee}/W^{\mathfrak{s}}\subset(T^{\vee}//W^{\mathfrak{s}})_{\mathbf{c}_0}. \tag{4}$$

Remark 1.2. The cell decomposition in Proposition 1.1 is inherited from the canonical decomposition of the asymptotic algebra $J^{\mathfrak{s}}$ into two-sided ideals $J_{\mathbf{c}}^{\mathfrak{s}}$; see (32), (46), (50), (54), and Lemmas 4.1, 6.1, 7.1, 8.1. We will also see there that the inclusion in (4) can be strict.

We choose a partition

$$T^{\vee}//W^{\mathfrak{s}}=\bigcup_{\mathbf{c}\in\mathrm{Cell}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}})}(T^{\vee}//W^{\mathfrak{s}})_{\mathbf{c}},$$

so that (4) holds. We will call $(T^{\vee}//W^{\mathfrak{s}})_{\mathbf{c}}$ the $\mathbf{c}$-*component* of $T^{\vee}//W^{\mathfrak{s}}$.

We will denote by k the coordinate algebra $\mathcal{O}(T^{\vee}/W^{\mathfrak{s}})$ of the ordinary quotient $T^{\vee}/W^{\mathfrak{s}}$. Then k is isomorphic to the centre of $\mathcal{H}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}},\tau)$, [15, §8.1]. Let

$$\phi_{\tau}\colon\mathcal{H}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}},\tau)\to J^{\mathfrak{s}} \tag{5}$$

be the $\mathbb{C}$-algebra homomorphism that Lusztig defined in [17, §1.4]. The centre of $J^{\mathfrak{s}}$ contains $\phi_{\tau}(k)$; see [17, Prop. 1.6]. This provides $J^{\mathfrak{s}}$ (and also each $J_{\mathbf{c}}^{\mathfrak{s}}$) with a structure of a k-module algebra (this structure depends on the choice of τ). Both $\mathcal{H}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}},\tau)$ and $J^{\mathfrak{s}}$ are finite type k-algebras.

We will assume that $p\neq2,3,5$ in order to be able to apply the results of Roche in [24]. By combining [24, Theorem 6.3] and [2, Theorem 1], we obtain that the ideal $\mathcal{H}^{\mathfrak{s}}(G)$ is a k-algebra Morita equivalent to the k-algebra $\mathcal{H}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}},q)$, where q is the order of the residue field of F. On the other hand, the morphism $\phi_q\colon\mathcal{H}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}},q)\to J^{\mathfrak{s}}$ is spectrum-preserving with respect to filtrations; see [4, Theorem 9].

It follows that the Bernstein ideal $\mathcal{H}^{\mathfrak{s}}(G)$ is *geometrically equivalent* to $J^{\mathfrak{s}}$ (which is equipped with the structure of a k-algebra induced by ϕ_q):

$$\mathcal{H}^{\mathfrak{s}}(G)\asymp J^{\mathfrak{s}}. \tag{6}$$

Remark 1.3. We observe that similar arguments show that the geometrical equivalence in (6) is true for each toral Bernstein ideal $\mathcal{H}^{\mathfrak{s}}(G)$ of any p-adic group G (with the same restrictions on p as in [24, §4]), which is the group of F-points of an F-split connected reductive algebraic group $\mathbf{G}$ such the centre of $\mathbf{G}$ is connected.

Remark 1.4. We note that geometric equivalence respects direct sums. Suppose that we are given geometric equivalences

$$\alpha_1:A_1\asymp B_1,\quad \alpha_2:A_2\asymp B_2.$$

We then have

$$\alpha_1\oplus1:A_1\oplus A_2\asymp B_1\oplus A_2,\qquad 1\oplus\alpha_2:B_1\oplus A_2\asymp B_1\oplus B_2,$$

so that we have a definite geometrical equivalence

$$(1 \oplus \alpha_2)(\alpha_1 \oplus 1) : A_1 \oplus A_2 \asymp B_1 \oplus B_2.$$

By a case-by-case analysis for $G = \mathrm{G}_2(F)$, we will prove that the k-algebra $J^{\mathfrak{s}}$ (equipped here with the structure of a k-algebra induced by ϕ_1) is itself geometrically equivalent to the coordinate algebra $\mathcal{O}(T^\vee /\!/ W^{\mathfrak{s}})$ of the extended quotient $T^\vee /\!/ W^{\mathfrak{s}}$:

$$J^{\mathfrak{s}} \asymp \mathcal{O}(T^\vee /\!/ W^{\mathfrak{s}}). \tag{7}$$

The geometric equivalence (7) comes from geometric equivalences

$$J^{\mathfrak{s}}_{\mathbf{c}} \asymp \mathcal{O}((T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{c}}), \qquad \text{for each } \mathbf{c} \in \mathrm{Cell}(\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}) \tag{8}$$

that will be constructed in Lemmas 4.1, 6.1, 7.1, 8.1.

Let $\mathbf{c}$ be a two-sided cell of $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$. Then (8) induces a bijection

$$\eta^{\mathfrak{s}}_{\mathbf{c}} \colon (T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{c}} \to \mathrm{Irr}(J^{\mathfrak{s}}_{\mathbf{c}}). \tag{9}$$

We will denote by $\eta^{\mathfrak{s}} \colon T^\vee /\!/ W^{\mathfrak{s}} \to \mathrm{Irr}(J^{\mathfrak{s}})$ the bijection defined by

$$\eta^{\mathfrak{s}}(t) = \eta^{\mathfrak{s}}_{\mathbf{c}}(t), \quad \text{for } t \in (T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{c}}. \tag{10}$$

On the other hand, let $\phi^{\mathfrak{s}}_{q,\mathbf{c}} \colon \mathcal{H}(\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}, q) \to J^{\mathfrak{s}}_{\mathbf{c}}$ denote the composition of the map $\phi^{\mathfrak{s}}_q$ and of the projection of J onto $J^{\mathfrak{s}}_{\mathbf{c}}$. Let E be a simple $J^{\mathfrak{s}}_{\mathbf{c}}$-module, through the homomorphism $\phi^{\mathfrak{s}}_{q,\mathbf{c}}$, it is endowed with an $\mathcal{H}(\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}, q)$-module structure. We denote the $\mathcal{H}(\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}, q)$-module by $(\phi^{\mathfrak{s}}_{q,\mathbf{c}})^*(E)$. Lusztig showed in [17] (see also [27, §5.13]) that the set

$$\left\{ (\phi^{\mathfrak{s}}_{q,\mathbf{c}})^*(E) \,:\, \begin{matrix} \mathbf{c} & \text{a two-sided cell of } \widetilde{W}^{\mathfrak{s}}_{\mathrm{a}} \\ E & \text{a simple } J^{\mathfrak{s}}_{\mathbf{c}}\text{-module} \end{matrix} \quad \text{(up to isomorphisms)} \right\} \tag{11}$$

is a complete set of simple $\mathcal{H}(\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}, q)$-modules.

Hence, we obtain a bijection

$$\tilde{\mu}^{\mathfrak{s}} \colon T^\vee /\!/ W^{\mathfrak{s}} \to \mathrm{Irr}(\mathcal{H}(\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}, q)) \tag{12}$$

by setting

$$\tilde{\mu}^{\mathfrak{s}}(t) = (\phi^{\mathfrak{s}}_{q,\mathbf{c}})^* \left(\eta^{\mathfrak{s}}_{\mathbf{c}}(t)\right) \qquad \text{for } t \in (T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{c}}. \tag{13}$$

Let

$$\mu^{\mathfrak{s}} \colon T^\vee /\!/ W^{\mathfrak{s}} \to \mathrm{Irr}(G)^{\mathfrak{s}} \tag{14}$$

denote the composition of $\tilde{\mu}^{\mathfrak{s}}$ with the bijection

$$\theta^{\mathfrak{s}} \colon \mathrm{Irr}(\mathcal{H}(\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}, q)) \to \mathrm{Irr}(G)^{\mathfrak{s}}$$

defined by Roche [24].

We have

$$\mu^{\mathfrak{s}} := \theta^{\mathfrak{s}} \circ (\phi^{\mathfrak{s}}_q)^* \circ \eta^{\mathfrak{s}}. \tag{15}$$

We should emphasize that the map $\mu^{\mathfrak{s}}$ is not canonical: it depends on a choice of geometrical equivalence inducing $\eta^{\mathfrak{s}}$.

Main result. Here is our main result, which is a consequence of 20 lemmas: Lemmas 4.1, 4.2, 4.3, 4.4, 4.5, 6.1, 6.2, 6.3, ..., 8.1, 8.2, 8.3, 8.4, 8.5.

Theorem 1.5. *Let $G = G_2(F)$ with $p \neq 2, 3, 5$. Let $\mathfrak{s} = [T, \lambda]_G$ with λ a smooth character of T. Then we have*

(1) *The Bernstein ideal $\mathcal{H}^{\mathfrak{s}}(G)$ is geometrically equivalent to the coordinate algebra $\mathcal{O}(T^\vee /\!/ W^{\mathfrak{s}})$ of the affine variety $T^\vee /\!/ W^{\mathfrak{s}}$ and this determines a bijection*

$$\mu^{\mathfrak{s}} : T^\vee /\!/ W^{\mathfrak{s}} \to \mathit{Irr}(G)^{\mathfrak{s}}.$$

(2) *There is a flat family $\mathfrak{X}^{\mathfrak{s}}_\tau$ of subschemes of $T^\vee / W^{\mathfrak{s}}$, with $\tau \in \mathbb{C}^\times$, such that*

$$\mathfrak{X}^{\mathfrak{s}}_1 = \pi^{\mathfrak{s}}(T^\vee /\!/ W^{\mathfrak{s}} - T^\vee / W^{\mathfrak{s}}), \qquad \mathfrak{X}^{\mathfrak{s}}_{\sqrt{q}} = \mathfrak{R}^{\mathfrak{s}}.$$

The schemes $\mathfrak{X}^{\mathfrak{s}}_1$, $\mathfrak{X}^{\mathfrak{s}}_{\sqrt{q}}$ are reduced.

(3) *There exists a correcting system of cocharacters for the bijection*

$$\mu^{\mathfrak{s}} : T^\vee /\!/ W^{\mathfrak{s}} \to \mathit{Irr}(G)^{\mathfrak{s}}.$$

(4) *The geometrical equivalence in (1) can be chosen so that*

$$(inf.ch.) \circ \mu^{\mathfrak{s}} = \pi^{\mathfrak{s}}_{\sqrt{q}}.$$

(5) *Let $E^{\mathfrak{s}}$ denote the maximal compact subgroup of $T^\vee$. Then we have*

$$\mu^{\mathfrak{s}}(E^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}) = \mathit{Irr}^t(G)^{\mathfrak{s}}.$$

Correcting system of cocharacters. Recall (see [2]) that, by definition, $T^\vee /\!/ W^{\mathfrak{s}}$ is the quotient $\widetilde{T^\vee} / W^{\mathfrak{s}}$. Here

$$\widetilde{T^\vee} := \{(w, t) \in W^{\mathfrak{s}} \times T^\vee : wt = t\}$$

and

$$T^\vee /\!/ W^{\mathfrak{s}} := \widetilde{T^\vee} / W^{\mathfrak{s}}$$

where $W^{\mathfrak{s}}$ acts on $\widetilde{T^\vee}$ by

$$\alpha(w, t) = (\alpha w \alpha^{-1}, \alpha t)$$

with $\alpha \in W^{\mathfrak{s}}$, $(w, t) \in \widetilde{T^\vee}$. Then $\rho^{\mathfrak{s}}$ denotes the quotient map $\widetilde{T^\vee} \to T^\vee /\!/ W^{\mathfrak{s}}$ and $\pi^{\mathfrak{s}}$ denotes the evident projection

$$T^\vee /\!/ W^{\mathfrak{s}} \to T^\vee / W^{\mathfrak{s}}, \qquad (w, t) \mapsto t.$$

The extended quotient $T^\vee /\!/ W^{\mathfrak{s}}$ is a complex affine variety and thus is the union of its irreducible components $Z_1, Z_2, \ldots, Z_r$. A *correcting system of cocharacters* for the bijection

$$\mu^{\mathfrak{s}} : T^\vee /\!/ W^{\mathfrak{s}} \to \mathrm{Irr}(G)^{\mathfrak{s}}$$

is an assignment to each Z_j of a cocharacter h_j of $T^\vee$. Each h_j is a morphism of algebraic groups

$$h_j \colon \mathbb{C}^\times \to T^\vee.$$

If Z_j and Z_k are labelled by the same two-sided cell $\mathbf{c}$ of $\widetilde{W^{\mathfrak{s}}_{\mathrm{a}}}$, then $h_j = h_k$ and we will write $h_j = h_k = h_{\mathbf{c}}$. We have $h_{\mathbf{c}_0} = 1$. We require that once the finite sequence $h_1, h_2, \ldots, h_r$ has been fixed, it is possible to choose irreducible components $X_1, X_2, \ldots, X_r$ of the affine variety $\widetilde{T^\vee}$ such that

- $\rho^{\mathfrak{s}}(X_j) = Z_j, \qquad j = 1, 2, \ldots, r.$

- For each $\tau \in \mathbb{C}^\times$, the map $X_j \to T^\vee/W^{\mathfrak{s}}$ which is the composition
$$X_j \to T^\vee \to T^\vee/W^{\mathfrak{s}},$$
$$(w,t) \mapsto h_j(\tau)t \mapsto p^{\mathfrak{s}}(h_j(\tau)t)$$
factors through $\rho^{\mathfrak{s}} \colon X_j \to Z_j$ and so gives a well-defined map
$$\pi^{\mathfrak{s}}_\tau \colon Z_j \to T^\vee/W^{\mathfrak{s}}$$
with commutativity in the diagram
$$\begin{tikzcd} X_j \arrow[r] \arrow[d, "\rho^{\mathfrak{s}}"'] & T^\vee/W^{\mathfrak{s}} \arrow[d, "id"] \\ Z_j \arrow[r, "\pi^{\mathfrak{s}}_\tau"'] & T^\vee/W^{\mathfrak{s}} \end{tikzcd}$$
Note that $h_j(\tau)t$ is the product in the algebraic group $T^\vee$ of $h_j(\tau)$ and t, and $p^{\mathfrak{s}} \colon T^\vee \to T^\vee/W^{\mathfrak{s}}$ is the standard quotient map.
- Since $T^\vee/\!/W^{\mathfrak{s}}$ is the union of the Z_j, we have a morphism of affine varieties
$$\pi^{\mathfrak{s}}_\tau \colon T^\vee/\!/W^{\mathfrak{s}} \to T^\vee/W^{\mathfrak{s}}$$
which we require to be a finite morphism with
$$\pi^{\mathfrak{s}}_\tau(T^\vee/\!/W^{\mathfrak{s}} - T^\vee/W^{\mathfrak{s}}) = (\mathfrak{X}^{\mathfrak{s}}_\tau)_{\rm red}$$
and with
$$(inf.ch.) \circ \mu^{\mathfrak{s}} = \pi^{\mathfrak{s}}_{\sqrt{q}}\,.$$

Remark 1.6. Theorem 1.5 shows, in particular, that Conjecture 3.1 in [3] and part (1) of Conjecture 1 in [2] are both true for the principal series of $\mathrm{G}_2(F)$. Moreover, we observe that the statement (3) in Theorem 1.5 is stronger than the statement (2) in Conjecture 3.1 of [3] in the sense that cocharacters are dependent only on the two-sided cells, and not on the irreducible components of $T^\vee/\!/W^{\mathfrak{s}}$ (in general, $(T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{c}}$ contains more than one irreducible component; see Lemmas 4.1, 6.1 and 7.1). Also, the bijection $\mu^{\mathfrak{s}}$ is not a homeomorphism in general (see the Note after the proof of Lemma 8.1).

2. The strategy of the proof

Let $G = \mathrm{G}_2(F)$ and let $\mathfrak{s} \in \mathfrak{T}(G)$. In this section we will both explain the strategy of the proof of Theorem 1 and recall some needed results from [13], [18] and [23].

2.1. The extended affine Weyl group $\widetilde{W}^{\mathfrak{s}}_{\mathbf{a}}$. It will follow from equations (28), (44), (45), (49) and (52), that the possible groups $W^{\mathfrak{s}}$ are the dihedral group of order 12, $\mathbb{Z}/2\mathbb{Z}$ (the cyclic group of order 2), the symmetric group S_3, and the direct product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In particular, $W^{\mathfrak{s}}$ is a finite Weyl group. Let $\Phi^{\mathfrak{s}}$ denote its root system, and let $Q^{\mathfrak{s}} = \mathbb{Z}\Phi^{\mathfrak{s}}$ be the corresponding root lattice.

The group $W^{\mathfrak{s}}_{\rm a} := W^{\mathfrak{s}} \ltimes Q^{\mathfrak{s}}$ is an *affine Weyl group*. Let $S^{\mathfrak{s}}$ be a set of simple reflections of $W^{\mathfrak{s}}_{\rm a}$ such that $S^{\mathfrak{s}} \cap W^{\mathfrak{s}}$ generates $W^{\mathfrak{s}}$ and is a set of simple reflections of $W^{\mathfrak{s}}$. Then one can find an abelian subgroup $C^{\mathfrak{s}}$ of $\widetilde{W}^{\mathfrak{s}}_{\rm a}$ such that $cS^{\mathfrak{s}} = S^{\mathfrak{s}}c$ for any $c \in C^{\mathfrak{s}}$ and we have $\widetilde{W}^{\mathfrak{s}}_{\rm a} = W^{\mathfrak{s}}_{\rm a} \ltimes C^{\mathfrak{s}}$. This shows that $\widetilde{W}^{\mathfrak{s}}_{\rm a}$ is an *extended affine Weyl group*. Let ℓ denote the length function on $W^{\mathfrak{s}}_{\rm a}$. We extend ℓ to $\widetilde{W}^{\mathfrak{s}}_{\rm a}$ by $\ell(wc) = \ell(w)$, if $w \in W^{\mathfrak{s}}_{\rm a}$ and $c \in C$.

2.2. **The Iwahori-Hecke algebra $\mathcal{H}(\widetilde{W}_{\mathbf{a}}^{\mathfrak{s}}, \tau)$.** Let $\boldsymbol{\tau}$ be an indeterminate and let $\mathcal{A} = \mathbb{C}[\boldsymbol{\tau}, \boldsymbol{\tau}^{-1}]$. Let $\mathcal{W} \in \{W_{\mathrm{a}}^{\mathfrak{s}}, \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}\}$. We denote by $\mathcal{H}(\mathcal{W}, \boldsymbol{\tau}^2)$ the *generic Iwahori-Hecke algebra* of $\mathcal{W}$, that is, the free $\mathcal{A}$-module with basis $(T_w)_{w\in\mathcal{W}}$ and multiplication defined by the relations

$$T_w T_{w'} = T_{ww'}, \quad \text{if } \ell(ww') = \ell(w) + \ell(w'), \tag{16}$$

$$(T_s - \boldsymbol{\tau})(T_s + \boldsymbol{\tau}^{-1}) = 0, \quad \text{if } s \in S^{\mathfrak{s}}. \tag{17}$$

The Iwahori-Hecke algebra $\mathcal{H}(\mathcal{W}, \tau^2)$ associated to $(\mathcal{W}, \tau)$ with $\tau \in \mathbb{C}^\times$ is obtained from $\mathcal{H}(\mathcal{W}, \boldsymbol{\tau}^2)$ by specializing $\boldsymbol{\tau}$ to τ, that is, it is the algebra generated by T_w, $w \in \mathcal{W}$, with relations (16) and the analog of (17) in which $\boldsymbol{\tau}$ has been replaced by τ.

We observe that the works of Reeder [23] and Roche [24] reduce the study of $\mathrm{Irr}^{\mathfrak{s}}(G)$ to those of the simple modules of $\mathcal{H}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}, q)$. A classification of these simple modules by *indexing triples* (t, u, ρ) is provided by [13] and [23]. We will recall some features of this classification in the next subsection.

2.3. **The indexing triples.** Let I_F be the inertia subgroup of the Weil group W_F, let $\mathrm{Frob}_F \in W_F$ be a geometric Frobenius (a generator of $W_F/I_F = \mathbb{Z}$), and let Φ be a Langlands parameter (that is, the equivalence class modulo inner automorphisms by elements of $H^{\mathfrak{s}}$ of a homomorphism from $W_F \times \mathrm{SL}(2,\mathbb{C})$ to $H^{\mathfrak{s}}$ which is admissible in the sense of [7, § 8.2]). We assume that Φ is *unramified*, that is, that Φ is trivial on I_F. We will still denote by Φ the restriction of Φ to $\mathrm{SL}(2,\mathbb{C})$.

Let u be the unipotent element of $H^{\mathfrak{s}}$ defined by

$$u = \Phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

We set $t = \Phi(\mathrm{Frob}_F)$. Then t is a semisimple element in $H^{\mathfrak{s}}$ which commutes with u. Let $\mathrm{Z}_{H^{\mathfrak{s}}}(t)$ denote the centralizer of t in $H^{\mathfrak{s}}$ and let $\mathrm{Z}^\circ_{H^{\mathfrak{s}}}(t)$ be the identity connected component of $\mathrm{Z}_{H^{\mathfrak{s}}}(t)$. We observe that if $H^{\mathfrak{s}}$ is one of the groups $\mathrm{G}_2(\mathbb{C})$, $\mathrm{GL}(2,\mathbb{C})$, $\mathrm{SL}(3,\mathbb{C})$, then $\mathrm{Z}_{H^{\mathfrak{s}}}(t)$ is always connected.

For each $\tau \in \mathbb{C}^\times$, we set

$$t(\tau) = \Phi \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} \in \mathrm{Z}^\circ_{H^{\mathfrak{s}}}(t).$$

Lusztig constructed in [18, Theorem 4.8] a bijection $\mathcal{U} \mapsto \mathbf{c}(\mathcal{U})$ between the set of unipotent classes in $H^{\mathfrak{s}}$ and the set of two-sided cells of $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$. Let $\mathbf{c}$ be the two-sided cell of $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$ which corresponds by this bijection to the unipotent class to which u belongs and then let the L-parameter Φ be such that (18) holds. We will denote by $F_{\mathbf{c}}$ the centralizer in $H^{\mathfrak{s}}$ of $\Phi(\mathrm{SL}(2,\mathbb{C}))$; then $F_{\mathbf{c}}$ is a maximal reductive subgroup of $\mathrm{Z}_{H^{\mathfrak{s}}}(u)$.

For any $\tau \in \mathbb{C}^\times$, we set

$$h_{\mathbf{c}}(\tau) := \Phi \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}.$$

Since Φ is only determined up to conjugation, we can always assume that $h_{\mathbf{c}}(\tau)$ belongs to $T^\vee$. Then it defines a cocharacter:

$$h_{\mathbf{c}} \colon \mathbb{C}^\times \to T^\vee.$$

The element

$$\sigma := h_{\mathbf{c}}(\sqrt{q}) \cdot t \tag{19}$$

satisfies the equation

$$\sigma u \sigma^{-1} = u^q. \tag{20}$$

For σ a semisimple element in $H^{\mathfrak{s}}$ and u a unipotent element in $H^{\mathfrak{s}}$ such that (20) holds, let $\mathcal{B}^{\mathfrak{s}}_{\sigma,u}$ be the variety of Borel subgroups of $H^{\mathfrak{s}}$ containing σ and u, and let $A_{\sigma,u}$ be the component group of the simultaneous centralizer of σ and u in $H^{\mathfrak{s}}$. Let $\mathcal{T}(H^{\mathfrak{s}})$ denote the set of triples (σ, u, ρ) such that σ is a semisimple element in $H^{\mathfrak{s}}$, u is a unipotent element in $H^{\mathfrak{s}}$ which satisfy (20), and ρ is an irreducible representation of $A_{\sigma,u}$ such that ρ appears in the natural representation of $A_{\sigma,u}$ on $H^*(\mathcal{B}^{\mathfrak{s}}_{\sigma,u}, \mathbb{C})$.

Reeder proved in [23], using the construction of Roche [24], that the set $\mathrm{Irr}^{\mathfrak{s}}(G)$ is in bijection with the $H^{\mathfrak{s}}$-conjugacy classes of triples $(\sigma, u, \rho) \in \mathcal{T}(H^{\mathfrak{s}})$. The irreducible G-module corresponding to the $H^{\mathfrak{s}}$-conjugacy class of (σ, u, ρ) will be denoted $\mathcal{V}^{\mathfrak{s}}_{\sigma,u,\rho}$ and we will refer to the triples (σ, u, ρ) as *indexing triples* for $\mathrm{Irr}^{\mathfrak{s}}(G)$.

2.4. **The L-parameters.** Let W_F^a be the topological abelianization of W_F and let I_F^a be the image in W_F^a of the inertia subgroup I_F. We denote by

$$r_F \colon W_F^a \to F^\times$$

the reciprocity isomomorphism of abelian class-field theory, and set $\varpi_F := r_F(\mathrm{Frob}_F)$ a prime element in F. Then the map $x \mapsto x(\varpi_F)$ defines an embedding of $X(T^\vee) = Y(T)$ in the p-adic torus T. This embedding gives a splitting $T = T(\mathfrak{o}_F) \times X(T^\vee)$.

We assume given (t, u) as above, i.e., $t = \Phi(\mathrm{Frob}_F)$ and u satisfies (18).

Let $\mathfrak{o}_F$ denote the ring of integers of F, let λ° be an irreducible character of $T(\mathfrak{o}_F)$, and let λ be an extension of λ^0 to T. Let

$$\hat{\lambda} \colon I_F^a \to T^\vee$$

be the unique homomorphism satisfying

$$x \circ \hat{\lambda} = \lambda^\circ \circ x \circ r_F, \quad \text{for } x \in X(T^\vee), \tag{21}$$

where x is viewed as in $X(T^\vee)$ on the left side of (21) and as an element of $Y(T)$ (a cocharacter of T) on the right side.

The choice of Frobenius Frob_F determines a splitting

$$W_F^a = I_F^a \times \langle \mathrm{Frob}_F \rangle,$$

so we can extend $\hat{\lambda}$ to a homomorphism $\hat{\lambda}_t \colon W_F^a \to G^\vee$ by setting

$$\hat{\lambda}_t(\mathrm{Frob}_F) := t.$$

Then we define (see [23, § 4.2]):

$$\tilde{\Phi} \colon W_F^a \times \mathrm{SL}(2, \mathbb{C}) \to G^\vee \quad (w, m) \mapsto \hat{\lambda}_t(w) \cdot \Phi(m).$$

2.5. **The asymptotic Hecke algebra $J^{\mathfrak{s}}$.** Let $^{-}\colon \mathcal{A} \to \mathcal{A}$ be the ring involution which takes $\boldsymbol{\tau}^n$ to $\boldsymbol{\tau}^{-n}$ and let $h \mapsto \bar{h}$ be the unique endomorphism of $\mathcal{H}(W_{\mathrm{a}}^{\mathfrak{s}}, \boldsymbol{\tau})$ which is $\mathcal{A}$-semilinear with respect to $^{-}\colon \mathcal{A} \to \mathcal{A}$ and satisfies $\bar{T}_s = T_s^{-1}$ for any $s \in S^{\mathfrak{s}}$. Let $w \in W_{\mathrm{a}}^{\mathfrak{s}}$. There is a unique

$$C_w \in \bigoplus_{w \in W_{\mathrm{a}}^{\mathfrak{s}}} \mathbb{Z}[\boldsymbol{\tau}^{-1}] T_w \quad \text{such that}$$

$$\bar{C}_w = C_w \quad \text{and} \quad C_w = T_w \quad (\mathrm{mod} \bigoplus_{y \in W_{\mathrm{a}}^{\mathfrak{s}}} (\bigoplus_{m<0} \mathbb{Z}\boldsymbol{\tau}^m) T_y)$$

(see for instance [19, Theorem 5.2 (a)]). We write

$$C_w = \sum_{y \in W_{\mathrm{a}}^{\mathfrak{s}}} P_{y,w}\, T_y, \quad \text{where } P_{y,w} \in \mathbb{Z}[\boldsymbol{\tau}^{-1}].$$

For $y \in W_{\mathrm{a}}^{\mathfrak{s}}$, $c, c' \in C^{\mathfrak{s}}$, we define $P_{yc,wc'}$ as $P_{y,w}$ if $c = c'$ and as 0 otherwise. Then for $w \in \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$, we set $C_w = \sum_{y \in \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}} P_{y,w}\, T_y$. It follows from [19, Theorem 5.2 (b)] that $(C_w)_{w \in \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}}$ is an $\mathcal{A}$-basis of $\mathcal{H}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}, \boldsymbol{\tau})$. For x, y, w in W, let $h_{x,y,w} \in \mathcal{A}$ be defined by

$$C_x \cdot C_y = \sum_{w \in \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}} h_{x,y,w}\, C_w.$$

For any $w \in \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$, there exists a nonnegative integer $a(w)$ such that

$$\begin{aligned} h_{x,y,w} &\in \quad \boldsymbol{\tau}^{a(w)}\, \mathbb{Z}[\boldsymbol{\tau}^{-1}] \quad \text{for all } x, y \in \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}, \\ h_{x,y,w} &\notin \quad \boldsymbol{\tau}^{a(w)-1}\, \mathbb{Z}[\boldsymbol{\tau}^{-1}] \quad \text{for some } x, y \in \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}. \end{aligned}$$

Let $\gamma_{x,y,w^{-1}}$ be the coefficient of $\boldsymbol{\tau}^{a(w)}$ in $h_{x,y,w}$.

Let $\underline{J}^{\mathfrak{s}}$ denote the free Abelian group with basis $(t_w)_{w \in \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}}$. Lusztig has defined an associative ring structure on $\underline{J}^{\mathfrak{s}}$ by setting

$$t_x \cdot t_y := \sum_{w \in \widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}} \gamma_{x,y,w^{-1}}\, t_w. \quad \text{(This is a finite sum.)}$$

The ring $\underline{J}^{\mathfrak{s}}$ is called the *based ring* of $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$. It has a unit element. The $\mathbb{C}$-algebra

$$J^{\mathfrak{s}} := \underline{J}^{\mathfrak{s}} \otimes_{\mathbb{Z}} \mathbb{C} \tag{22}$$

is called the *asymptotic Hecke algebra* of $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$.

For each two-sided cell $\mathbf{c}$ in $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$, the subspace $\underline{J}_{\mathbf{c}}^{\mathfrak{s}}$ spanned by the t_w, $w \in \mathbf{c}$, is a two-sided ideal of $\underline{J}^{\mathfrak{s}}$. The ideal $\underline{J}_{\mathbf{c}}^{\mathfrak{s}}$ is an associative ring, with a unit, which is called the based ring of the two-sided cell $\mathbf{c}$ and

$$\underline{J}^{\mathfrak{s}} = \bigoplus_{\mathbf{c} \in \mathrm{Cell}(\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}})} \underline{J}_{\mathbf{c}}^{\mathfrak{s}} \tag{23}$$

is a direct sum decomposition of $\underline{J}^{\mathfrak{s}}$ as a ring. We set $J_{\mathbf{c}}^{\mathfrak{s}} := \underline{J}_{\mathbf{c}}^{\mathfrak{s}} \otimes_{\mathbb{Z}} \mathbb{C}$.

3. Background on the group G_2

Let $\mathbf{G} = \mathrm{G}_2$ be a group of type G_2 over a commutative field $\mathbb{F}$, and let $\mathrm{G}_2(\mathbb{F})$ denote its group of $\mathbb{F}$-points.

TABLE 1. Values of a and b.

$a(\alpha) = -\alpha$	$a(\beta) = 3\alpha + \beta$
$a(\alpha + \beta) = 2\alpha + \beta$	$a(2\alpha + \beta) = \alpha + \beta$
$\alpha(3\alpha + \beta) = \beta$	$a(3\alpha + 2\beta) = 3\alpha + 2\beta$
$b(\alpha) = \alpha + \beta$	$b(\beta) = -\beta$
$b(\alpha + \beta) = \alpha$	$b(2\alpha + \beta) = 2\alpha + \beta$
$b(3\alpha + \beta) = 3\alpha + 2\beta$	$b(3\alpha + 2\beta) = 3\alpha + \beta$

3.1. **Roots and fundamental reflexions.** Denote by $\mathbf{T}$ a maximal split torus in $\mathbf{G}$, and by Φ the set of roots of $\mathbf{G}$ with respect to $\mathbf{T}$. Let $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be the canonical basis of $\mathbb{R}^3$, equipped with the scalar product $(\,|\,)$ for which this basis is orthonormal. Then $\alpha := \varepsilon_1 - \varepsilon_2$, $\beta := -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ defines a basis of Φ and

$$\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$$

is a subset of positive roots in Φ (see [8, Planche IX]).

We have

$$(\alpha|\alpha) = 2, \quad (\beta|\beta) = 6 \quad \text{and} \quad (\alpha|\beta) = -3. \tag{24}$$

Hence, α is short root, while β is a long root.

We set

$$n(\gamma', \gamma) := \langle \gamma', \gamma^\vee \rangle = \frac{2(\gamma'|\gamma)}{(\gamma|\gamma)}, \tag{25}$$

(see [8, Chap. VI, §1.1 (7)]). We will denote by s_γ the reflection in W which corresponds to γ, i.e., $s_\gamma(x) := x - \langle x, \gamma^\vee \rangle \gamma$. We set $a := s_\alpha$, $b := s_\beta$ and $r := ba$.

The Cartan matrix for $\mathrm{G}_2(\mathbb{F})$ is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, and the values of a and b on the elements of Φ^+ are given in Table 1.

We write $B = TU$ for the corresponding Borel subgroup in $\mathrm{G}_2(\mathbb{F})$ and $\bar{B} = T\bar{U}$ for the opposite Borel subgroup. Denote by $X(T)$ the group of rational characters of T. We have

$$X(T) = \mathbb{Z}(2\alpha + \beta) + \mathbb{Z}(\alpha + \beta). \tag{26}$$

We identify $T \cong \mathbb{F}^\times \times \mathbb{F}^\times$ by

$$\xi_\alpha \colon t \longmapsto ((2\alpha + \beta)(t), (\alpha + \beta)(t)). \tag{27}$$

In this realization we have

$$\begin{cases} \alpha(t_1, t_2) = t_1 t_2^{-1}, & \beta(t_1, t_2) = t_1^{-1} t_2^2, \\ a(t_1, t_2) = (t_2, t_1), & b(t_1, t_2) = (t_1, t_1 t_2^{-1}). \end{cases}$$

The Weyl group $W = \mathrm{N}_{\mathrm{G}_2(\mathbb{F})}(T)/T$ has 12 elements. They are described along with the action on the character $\chi_1 \otimes \chi_2$ of $T \cong \mathbb{F}^\times \times \mathbb{F}^\times$:

(28)

	w	$w(\chi_1 \otimes \chi_2)$,
1.	e	$\chi_1 \otimes \chi_2$,
2.	a	$\chi_2 \otimes \chi_1$,
3.	b	$\chi_1\chi_2 \otimes \chi_2^{-1}$,
4.	ab	$\chi_2^{-1} \otimes \chi_1\chi_2$,
5.	ba	$\chi_1\chi_2 \otimes \chi_1^{-1}$,
6.	aba	$\chi_1^{-1} \otimes \chi_1\chi_2$, .
7.	bab	$\chi_1 \otimes \chi_1^{-1}\chi_2^{-1}$,
8.	$abab$	$\chi_1^{-1}\chi_2^{-1} \otimes \chi_1$,
9.	$baba$	$\chi_2 \otimes \chi_1^{-1}\chi_2^{-1}$,
10.	$ababa$	$\chi_1^{-1}\chi_2^{-1} \otimes \chi_2$,
11.	$babab$	$\chi_2^{-1} \otimes \chi_1^{-1}$,
12.	$bababa$	$\chi_1^{-1} \otimes \chi_2^{-1}$.

3.2. **Affine Weyl group, two-sided cells and unipotent orbits.** Let $W_{\mathrm{a}} := W \ltimes X(T^\vee)$ denote the affine Weyl group of the p-adic group $G = \mathrm{G}_2(F)$. Denote by $\{a, b, d\}$ the set of simple reflections in W_{a}, with $W = \langle a, b\rangle$ and $(ab)^6 = (da)^2 = (db)^3 = \mathrm{e}$.

As in the case of an arbitrary Coxeter group, the group W_{a} is partitioned into *two-sided cells*. The definition of cells yields a natural partial ordering on the set $\mathrm{Cell}(W_{\mathrm{a}})$ of two-sided cells in W_{a}. The *highest* cell $\mathbf{c}_e$ in this ordering contains just the identity element of W_{a}. Lusztig defined in [16] an *a-invariant* for each two-sided cell. The a-invariant respects (inversely) the partial ordering on $\mathrm{Cell}(W_{\mathrm{a}})$.

The group W_{a} has five two-sided cells $\mathbf{c}_{\mathrm{e}}$, $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ and $\mathbf{c}_0$ (see for instance [27, §11.1]) and the ordering occurs to be total: $\mathbf{c}_0 \le \mathbf{c}_3 \le \mathbf{c}_2 \le \mathbf{c}_1 \le \mathbf{c}_e$, with

$$\begin{aligned}
\mathbf{c}_{\mathrm{e}} &= \{w \in W_{\mathrm{a}} \,:\, a(w) = 0\} = \{\mathrm{e}\},\\
\mathbf{c}_1 &= \{w \in W_{\mathrm{a}} \,:\, a(w) = 1\},\\
\mathbf{c}_2 &= \{w \in W_{\mathrm{a}} \,:\, a(w) = 2\},\\
\mathbf{c}_3 &= \{w \in W_{\mathrm{a}} \,:\, a(w) = 3\},\\
\mathbf{c}_0 &= \{w \in W_{\mathrm{a}} \,:\, a(w) = 6\} \quad \text{(the \textit{lowest} two-sided cell)}.
\end{aligned}$$

For a visual realization of the two-sided cells see [12, p. 529].

Let $\mathcal{U}$ denote the unipotent variety in the Langlands dual $G^\vee = \mathrm{G}_2(\mathbb{C})$ of G. For $\mathcal{C}, \mathcal{C}'$ two unipotent classes in $G^\vee$, we will write $\mathcal{C}' \le \mathcal{C}$ if $\mathcal{C}'$ is contained in the Zariski closure of $\mathcal{C}$. The relation $\le$ defines a partial ordering on $\mathcal{U}$. In the Bala-Carter classification, the unipotent classes in $G^\vee$ are $1 \le \mathrm{A}_1 \le \widetilde{\mathrm{A}}_1 \le \mathrm{G}_2(a_1) \le \mathrm{G}_2$ (see for instance [10, p. 439]). The dimensions of these varieties are 0, 6, 8, 10, 12. The component groups are trivial except for $\mathrm{G}_2(a_1)$ in which case the component group is the symmetric group S_3. The group S_3 admits 3 irreducible representations; two of these, the trivial and the 2-dimensional representations, namely ρ_1, ρ_2, appear in our construction. In [22], Ram refers to 1, A_1, $\widetilde{\mathrm{A}}_1$, $\mathrm{G}_2(a_1)$ and G_2 as the *trivial*, *minimal*, *subminimal*, *subregular* and *regular* orbit, respectively.

The bijection between $\mathrm{Cell}(W_{\mathrm{a}})$ and $\mathcal{U}$ that Lusztig has constructed in [18] is order-preserving. Under this bijection, $\mathbf{c}_e$ corresponds to the *regular* unipotent class and $\mathbf{c}_0$ corresponds to the *trivial* class. If the two-sided cell $\mathbf{c}$ corresponds to the orbit of some unipotent element $u \in G^\vee$, then $a(\mathbf{c}) = \dim \mathcal{B}_u$, where $\mathcal{B}_u$ denotes the Springer fibre of u (that is, the set of Borel subgroups in $G^\vee$ containing u). Lusztig's bijection is described as follows:

$$\mathbf{c}_{\mathrm{e}} \leftrightarrow \mathrm{G}_2 \quad \mathbf{c}_1 \leftrightarrow \mathrm{G}_2(a_1) \quad \mathbf{c}_2 \leftrightarrow \widetilde{\mathrm{A}}_1 \quad \mathbf{c}_3 \leftrightarrow \mathrm{A}_1 \quad \mathbf{c}_0 \leftrightarrow 1.$$

3.3. **Representations.** Let $R(\mathrm{G}_2(F))$ denote the Grothendieck group of admissible representations of finite length of $\mathrm{G}_2(F)$. With $\lambda \in \Psi(T)$, we will write $I(\lambda) := i^G_{T\subset B}(\lambda)$ for the induced representation (normalized induction). We will denote by $\ell(i^G_{T\subset B}(\lambda))$ for the length of this representation, by $|i^G_{T\subset B}(\lambda)|$ the number of *inequivalent* constituents.

It is a result of Keys that the unitary principal series $I(\chi_1 \otimes \chi_2)$ is reducible if and only if χ_1 and χ_2 are different characters of order 2 [14, Theorem G_2]. When reducible, it is of multiplicity one and length two.

Let ν denote the normalized absolute value of F. Using [21, Prop. 3.1] we have the following result: $I(\psi_1\chi_1 \otimes \psi_2\chi_2)$, with ψ_i, χ_i $(i = 1, 2)$ smooth characters of $F^\times$, ψ_1, ψ_2 unramified, and $\psi_1\chi_1$, $\psi_2\chi_2$ nonunitary, is reducible if and only if at least one of the following holds:

$$\begin{array}{cccc} \psi_1\chi_1 = \nu & \psi_1^2\psi_2\chi_1^2\chi_2 = \nu & \psi_1^2\psi_2\chi_1^2\chi_2 = \nu^{-1} & \psi_1\psi_2\chi_1\chi_2 = \nu^{-1} \\ \psi_2\chi_2 = \nu & \psi_1\chi_1 = \nu^{-1} & \psi_1\psi_2^2\chi_1\chi_2^2 = \nu & \psi_1\psi_2^2\chi_1\chi_2^2 = \nu^{-1} \\ \psi_2\chi_2 = \nu^{-1} & \psi_1\psi_2\chi_1\chi_2 = \nu & \psi_1\psi_2^{-1}\chi_1\chi_2^{-1} = \nu & \psi_1\psi_2^{-1}\chi_1\chi_2^{-1} = \nu^{-1}, \end{array} \tag{29}$$

that is, if and only if there exists a root $\gamma \in \Phi$ such that

$$(\chi_1 \otimes \chi_2) \circ \gamma^\vee = \nu^{\pm 1}. \tag{30}$$

From now on we will assume that $\mathbb{F} = F$, a local non-Archimedean field. Let $G = \mathrm{G}_2$ and let $\mathfrak{s} = [T, \chi_1 \otimes \chi_2]_G$. Let $\Psi(F^\times)$ denote the group of unramified quasicharacters of $F^\times$. We have

$$D^{\mathfrak{s}} = \left\{\psi_1\chi_1 \otimes \psi_2\chi_2 \,:\, \psi_1, \psi_2 \in \Psi(F^\times)\right\} \cong \left\{(z_1, z_2) \,:\, z_1, z_2 \in \mathbb{C}^\times\right\} \cong T^\vee,$$

the Langlands dual of T, a complex torus of dimension 2.

Let $\Psi^{\mathrm{t}}(F^\times)$ denote the group of unramified unitary quasicharacters of $F^\times$ and let $E = E^{\mathfrak{s}}$ be the maximal compact subgroup of $D^{\mathfrak{s}}$. We have

$$E^{\mathfrak{s}} = \left\{\psi_1\chi_1 \otimes \psi_2\chi_2 : \psi_1, \psi_2 \in \Psi^{\mathrm{t}}(F^\times)\right\}.$$

Let $w \in W(T) = W$. Then we have

$$\mathfrak{s} = [T, \chi_1 \otimes \chi_2]_G = [T, w \cdot (\chi_1 \otimes \chi_2)]_G. \tag{31}$$

We are also free to give χ an unramified twist; this will not affect the inertial support.

4. The Iwahori point in $\mathfrak{T}(\mathrm{G}_2)$

We will assume in this section that $\mathfrak{s} = \mathfrak{i} = [T, 1]_G$. We have $W^{\mathfrak{i}} = W$ and $W^{\mathfrak{i}}_{\mathrm{a}} = \widetilde{W}_{\mathrm{a}}$. The group W is a finite Coxeter group of order 12:

$$W = \langle a, b, a^2 = b^2 = (ab)^6 = e\rangle.$$

Let $r = ab$. Then representatives of W-conjugacy classes are:

$$\{e, r, r^2, r^3, a, b\}.$$

Definition. We define the following partition of $T^\vee//W$:

$$T^\vee//W = \bigsqcup_{\mathbf{c}_i \in \mathrm{Cell}(W_\mathrm{a})} (T^\vee//W)_{\mathbf{c}_i}, \tag{32}$$

where

$$\begin{aligned}
(T^\vee//W)_{\mathbf{c}_e} &:= (T^\vee)^r/\mathrm{Z}(r),\\
(T^\vee//W)_{\mathbf{c}_1} &:= (T^\vee)^{r^3}/\mathrm{Z}(r^3) \sqcup (T^\vee)^{r^2}/\mathrm{Z}(r^2),\\
(T^\vee//W)_{\mathbf{c}_2} &:= (T^\vee)^a/\mathrm{Z}(a),\\
(T^\vee//W)_{\mathbf{c}_3} &:= (T^\vee)^b/\mathrm{Z}(b),\\
(T^\vee//W)_{\mathbf{c}_0} &:= T^\vee/W.
\end{aligned}$$

Note 1. In the Definition above we had the freedom to permute a and b (that is, we could as well have attached (b) to $\mathbf{c}_2$ and then (a) to $\mathbf{c}_3$).

Note 2. The Springer correspondence for the group G_2 (see [10, p. 427]) is as follows:

$$\begin{array}{ccc}
\phi_{1,0} & \leftrightarrow & \mathbf{c}_e\\
\phi_{2,1} & \leftrightarrow & (\mathbf{c}_1, \rho_1)\\
\phi'_{1,3} & \leftrightarrow & (\mathbf{c}_1, \rho_2)\\
\phi_{2,2} & \leftrightarrow & \mathbf{c}_2\\
\phi''_{1,3} & \leftrightarrow & \mathbf{c}_3\\
\phi_{1,6} & \leftrightarrow & \mathbf{c}_0.
\end{array}$$

Each of the following two bijections:

$$\begin{array}{ccc}
(r) & \leftrightarrow & \phi_{1,0}\\
(r^3) & \leftrightarrow & \phi_{2,1}\\
(r^2) & \leftrightarrow & \phi'_{1,3}\\
(a) & \leftrightarrow & \phi_{2,2}\\
(b) & \leftrightarrow & \phi''_{1,3}\\
(e) & \leftrightarrow & \phi_{1,6}
\end{array}
\qquad\qquad
\begin{array}{ccc}
(r) & \leftrightarrow & \phi_{1,0}\\
(r^3) & \leftrightarrow & \phi'_{1,3}\\
(r^2) & \leftrightarrow & \phi_{2,1}\\
(a) & \leftrightarrow & \phi_{2,2}\\
(b) & \leftrightarrow & \phi''_{1,3}\\
(e) & \leftrightarrow & \phi_{1,6}
\end{array}$$

by composing with the Springer correspondence, sends

- (r) to the unipotent class in $G^\vee$ corresponding to $\mathbf{c}_e$,
- (r^2) and (r^3) to the unipotent class in $G^\vee$ corresponding to $\mathbf{c}_1$,
- (a) to the unipotent class in $G^\vee$ corresponding to $\mathbf{c}_2$,
- (b) to the unipotent class in $G^\vee$ corresponding to $\mathbf{c}_3$,
- (e) to the unipotent class in $G^\vee$ corresponding to $\mathbf{c}_0$.

This is compatible with (32). Thanks to Jim Humphreys for a helpful comment at this point.

Note 3. The correspondence between conjugacy classes in W and unipotent classes in $G^\vee$ in Note 2 can be interpreted as a *partition* of the set $\underline{W}$ of conjugacy classes in W indexed by unipotent classes in $G^\vee$:

$$\underline{W} = \bigsqcup_u \mathbf{s}_u, \tag{33}$$

where u runs over the set of unipotent classes in $G^\vee$. Moreover, each $\mathbf{s}_u$ is a union of n_u conjugacy classes in W, where n_u is the number of isomorphism classes of irreducible representations of the component group $\mathrm{Z}_{G^\vee}(u)/\mathrm{Z}_{G^\vee}(u)^0$ which appear in the Springer correspondence for $G^\vee$. In [20, §§8-9], Lusztig defined similar kinds of partitions in a more general setting and more canonical way for adjoint algebraic reductive groups over an algebraic closure of a finite field. The partition (33) coincides with those obtained by Lusztig for a group of type G_2 on the top of page 7 of [20].

Let $J = J^{\mathrm{i}}$ be the asymptotic Iwahori-Hecke algebra of W, let $\mathbb{A}^1$ denote the affine complex line, let $\mathbb{I}$ denote the unit interval, and let $\asymp$ be the geometrical equivalence defined in [2, §4].

Lemma 4.1. *We have the following isomorphisms of algebraic varieties:*

$$(T^\vee/\!/W)_{\mathbf{c}_e} \xrightarrow{\sim} pt_* \qquad (T^\vee/\!/W)_{\mathbf{c}_1} \xrightarrow{\sim} pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$$
$$(T^\vee/\!/W)_{\mathbf{c}_2} \xrightarrow{\sim} \mathbb{A}^1 \qquad (T^\vee/\!/W)_{\mathbf{c}_3} \xrightarrow{\sim} \mathbb{A}^1,$$

$$E/\!/W \xrightarrow{\sim} pt_* \sqcup (pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4) \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup E/W,$$

and $J \asymp \mathcal{O}(T^\vee/\!/W)$*, where* $J_{\mathbf{c}_1} \asymp \mathcal{O}((T^\vee/\!/W)_{\mathbf{c}_1})$*, and*

$$J_{\mathbf{c}_e} \sim_{\text{morita}} \mathcal{O}((T^\vee/\!/W)_{\mathbf{c}_e}) \qquad J_{\mathbf{c}_2} \sim_{\text{morita}} \mathcal{O}((T^\vee/\!/W)_{\mathbf{c}_2}),$$
$$J_{\mathbf{c}_3} \sim_{\text{morita}} \mathcal{O}((T^\vee/\!/W)_{\mathbf{c}_3}) \qquad J_{\mathbf{c}_0} \sim_{\text{morita}} \mathcal{O}((T^\vee/\!/W)_{\mathbf{c}_0}).$$

Proof. The centralizers in W are:

$$\mathrm{Z}(e) = W, \quad \mathrm{Z}(r) = \langle r\rangle, \quad \mathrm{Z}(r^2) = \langle r\rangle, \quad \mathrm{Z}(r^3) = W,$$
$$\mathrm{Z}(a) = \langle r^3, a\rangle, \quad \mathrm{Z}(b) = \langle r^3, b\rangle.$$

Case-by-case analysis. We will write $X = T^\vee$.

- $\mathbf{c} = \mathbf{c}_0$, $c = 1$. $X^c/\mathrm{Z}(c) = X/W$.
- $\mathbf{c} = \mathbf{c}_e$, $c = r$. $X^c = (1,1)$, $X^c/\mathrm{Z}(c) = pt_*$.
- $\mathbf{c} = \mathbf{c}_1$, $c = r^3$. $X^c = \{(1,1), (-1,1), (1,-1), (-1,-1)\}$. Note that $(1,1)$ is fixed by W and
$$(-1,-1) \sim_{bab} (-1,1) \sim_a (1,-1)$$
are in a single W-orbit. $X^c/\mathrm{Z}(c) = pt_1 \sqcup pt_2$. Also, attached to this cell, $c = r^2$. $X^c = \{(1,1), (j,j), (j^2,j^2)\}$, where $j = \exp(2\pi i/3)$. Now
$$(j,j) \sim_{ba} (j^2, j^2)$$
are in the same $\mathrm{Z}(c)$-orbit. $X^c/\mathrm{Z}(c) = pt_3 \sqcup pt_4$.
- $\mathbf{c} = \mathbf{c}_2$, $c = a$. $X^c = \{(z,z) : z \in \mathbb{C}^\times\}$.
$$X^c/\mathrm{Z}(c) = \{\{(z,z), (z^{-1}, z^{-1})\} : z \in \mathbb{C}^\times\} \cong \mathbb{A}^1.$$
- $\mathbf{c} = \mathbf{c}_3$, $c = b$. $X^c = \{(z,1) : z \in \mathbb{C}^\times\}$.
$$X^c/\mathrm{Z}(c) = \{\{(z,1), (z^{-1}, 1)\} : z \in \mathbb{C}^\times\} \cong \mathbb{A}^1.$$

Let $F_{\mathbf{c}}$ denote the maximal reductive subgroup of the centralizer in $G^\vee$ of the unipotent class corresponding to $\mathbf{c}$ and let $R_{F_{\mathbf{c}}}$ denote the complexified representation ring of $F_{\mathbf{c}}$.

• We have $F_{\mathbf{c}_0} = G^\vee$. Since the group G_2 is F-split adjoint, we have (see [2, Theorem 4]):

$$J_{\mathbf{c}_0} \simeq M_{|W|}(\mathcal{O}(T^\vee/W)).$$

This isomorphism induces a homeomorphism of primitive ideal spectra:

$$\eta_{\mathbf{c}_0} \colon T^\vee/W \to \mathrm{Irr}(J_{\mathbf{c}_0}).$$

• The reductive group $F_{\mathbf{c}_{\mathrm{e}}}$ is the centre of $G^\vee$ and $J_{\mathbf{c}_{\mathrm{e}}} = \mathbb{C}$. Let $\eta_{\mathbf{c}_{\mathrm{e}}}(pt_*)$ be the simple module of $J_{\mathbf{c}_{\mathrm{e}}}$.

• Let $i \in \{2, 3\}$. We have $F_{\mathbf{c}_i} \simeq \mathrm{SL}(2, \mathbb{C})$. In proving the Lusztig conjecture [18, Conjecture 10.5] on the structure of the asymptotic Hecke algebra, Xi constructed in [27, §11.2] an isomorphism

$$J_{\mathbf{c}_i} \simeq \mathrm{M}_6(R_{F_{\mathbf{c}_i}}).$$

Let $\mathbb{C}[t, t^{-1}]$ denote the algebra of Laurent polynomials in one indeterminate t. Let α denote the generator of $\mathbb{Z}/2\mathbb{Z}$. The group $\mathbb{Z}/2\mathbb{Z}$ acts as automorphisms of $\mathbb{C}[t, t^{-1}]$, with $\alpha(t) = t^{-1}$. The fixed algebra is isomorphic to the coordinate algebra of the affine line $\mathbb{A}^1$:

$$\mathbb{C}[t, t^{-1}]^{\mathbb{Z}/2\mathbb{Z}} \simeq \mathcal{O}(\mathbb{A}^1), \qquad t + t^{-1} \mapsto t.$$

We then have

$$R_{\mathrm{SL}(2,\mathbb{C})} \simeq \mathbb{C}[t, t^{-1}]^{\mathbb{Z}/2\mathbb{Z}} \simeq \mathcal{O}(\mathbb{A}^1).$$

The isomorphism

$$J_{\mathbf{c}_i} \simeq M_6(\mathcal{O}(\mathbb{A}^1))$$

induces a homeomorphism of primitive ideal spectra:

$$\mathbb{A}^1 \simeq \mathrm{Irr}(J_{\mathbf{c}_i}).$$

This in turn determines a homeomorphism

$$\eta_{\mathbf{c}_i} : (T^\vee/\!/W)_{\mathbf{c}_i} \simeq \mathrm{Irr}(J_{\mathbf{c}_i}) \tag{34}$$

for $i = 2, 3$.

• According to [27, §11.2], we have $F_{\mathbf{c}_1} = S_3$, the symmetric group on $\{1, 2, 3\}$. We write $S_3 = \{e, (12), (13), (123), (132), (2, 3)\}$. Let

$$\overline{e} = \{e\}, \quad \overline{(12)} = \{(12), (13), (23)\} \quad \text{and} \quad \overline{(123)} = \{(123), (132)\}$$

denote the three conjugacy classes in S_3. According to [27, §11.3, §12], the based ring $J_{\mathbf{c}_1}$ has four simple modules: (E_1, π_1), (E_2, π_2), (E_3, π_3), (E_4, π_4) with $\dim E_1 = \dim E_2 = 3$, $\dim E_3 = 2$, $\dim E_4 = 1$, where

$$E_1 = E_{\overline{e},\rho_1}, \quad E_2 = E_{\overline{(12)}}, \quad E_3 = E_{\overline{(123)}}, \quad E_4 = E_{\overline{e},\rho_2}, \tag{35}$$

using the notation in [27, §5.4].

Consider the map $\delta_{\mathbf{c}_1} : J_{\mathbf{c}_1} \longrightarrow \mathrm{M}_3(\mathbb{C}) \oplus \mathrm{M}_3(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C}) \oplus \mathbb{C}$, defined by

$$\delta_{\mathbf{c}_1}(x) = (\pi_1(x), \pi_2(x), \pi_3(x), \pi_4(x)), \text{ for } x \in J_{\mathbf{c}_1}.$$

The map $\delta_{\mathbf{c}_1}$ is spectrum-preserving. For the primitive ideal space of $J_{\mathbf{c}_1}$ is the discrete space $\{E_1, E_2, E_3, E_4\}$ and the primitive ideal space of $\mathrm{M}_3(\mathbb{C}) \oplus \mathrm{M}_3(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C}) \oplus \mathbb{C}$ is $\{pt \sqcup pt \sqcup pt \sqcup pt\}$.

Then we get $J_{\mathbf{c}_1} \asymp \mathbb{C}^4 \asymp \mathcal{O}((T^\vee/\!/W)_{\mathbf{c}_1})$. Moreover, we can choose the geometrical equivalence $J_{\mathbf{c}_1} \asymp \mathcal{O}((T^\vee/\!/W)_{\mathbf{c}_1})$ in order that the induced bijection $\eta_{\mathbf{c}_1} : (T^\vee/\!/W)_{\mathbf{c}_1} \to \mathrm{Irr}(J_{\mathbf{c}_1})$ satisfies

$$\eta_{\mathbf{c}_1}(pt_i) = E_i \quad \text{for } 1 \leq i \leq 4. \tag{36}$$

□

Lemma 4.2. *The flat family is given by*

$$\mathfrak{X}_\tau : (1 - \tau^2 y)(x - \tau^2 y) = 0.$$

Proof. Since there is only one quadratic unramified character, the unitary principal series $I(\psi_1 \otimes \psi_2)$ is always irreducible.

Then we write down all the nonunitary quasicharacters of T which obey the reducibility conditions (29):

$$\nu \otimes \psi_2,\ \nu^{-1} \otimes \psi_2,\ \psi_1 \otimes \nu,\ \psi_1 \otimes \nu^{-1},\ \psi \otimes \psi^{-2}\nu,\ \psi \otimes \psi^{-2}\nu^{-1},$$
$$\psi^{-2}\nu \otimes \psi,\ \psi^{-2}\nu^{-1} \otimes \psi,\ \psi \otimes \psi^{-1}\nu,\ \psi \otimes \psi^{-1}\nu^{-1},\ \psi \otimes \psi\nu^{-1},\ \psi \otimes \psi\nu$$

with ψ an unramified quasicharacter of $F^\times$. These quasicharacters of T fall into two W-orbits:

$$\begin{cases} 1. & W \cdot (\nu\psi \otimes \psi), \\ 2. & W \cdot (\nu \otimes \psi). \end{cases}$$

For the first W-orbit, we obtain the curve

$$\mathfrak{C}_2 = \{W \cdot (\nu\psi \otimes \psi) \,:\, \psi \in \Psi(F^\times)\} \cong \{W \cdot (z, q^{-1}z) \,:\, z \in \mathbb{C}^\times\}.$$

The second W-orbit creates the curve

$$\mathfrak{C}_3 = \{W \cdot (\nu \otimes \psi) \,:\, \psi \in \psi(F^\times)\} \cong \{W \cdot (z, q^{-1}) \,:\, z \in \mathbb{C}^\times\}.$$

We obtain

$$\mathfrak{R} = \mathfrak{C}_2 \cup \mathfrak{C}_3.$$

Let ϵ be the unique unramified quadratic character of $F^\times$, and let ω denote an unramified cubic character of $F^\times$. In the article of Ram [22] there is a list $t_a, \ldots, t_j$ of central characters, their calibration graphs, Langlands parameters and indexing triples. After computing the calibration graphs, we are now able to identify these central characters with points in the complex torus $T^\vee \cong \Psi(T)$:

$$t_a = (q^{-1}, q^{-2}) = \nu \otimes \nu^2,\ t_b = (z, q^{-1}z) = \psi \otimes \nu\psi,\ t_c = (j, q^{-1}j) = \omega \otimes \nu\omega,$$
$$t_d = (-1, q^{-1}) = \nu \otimes \epsilon,\ t_e = (1, q^{-1}) = \nu \otimes 1,\ t_f = (q^{2/3}, q^{-1/3}) = \nu^{-2/3} \otimes \nu^{1/3},$$
$$t_g = (q^{1/2}, q^{-1/2}) = \nu^{-1/2} \otimes \nu^{1/2},\ t_h = (q^{-1}, z) = \nu \otimes \psi,$$
$$t_i = (q^{-1}, q^{-1}) = \nu \otimes \nu,\ t_j = (q^{-1}, q^{1/2}) = \nu \otimes \nu^{-1/2}$$

with $z = \psi(\varpi_F)$. We have

$$\mathfrak{C}_2 \cap \mathfrak{C}_3 = \{t_a, t_d, t_e\}.$$

The flat family

$$\mathfrak{X}_\tau : (1 - \tau^2 y)(x - \tau^2 y) = 0$$

has the property that $\mathfrak{X}_{\sqrt{q}} = \mathfrak{R}$. □

Lemma 4.3. *For each two-sided cell* $\mathbf{c}$ *of* $W^{\mathfrak{i}}$, *the cocharacters* $h_{\mathbf{c}}$ *are as follows:*

$$h_{\mathbf{c}}(\tau) = (1, \tau^{-2}) \text{ if } \mathbf{c} = \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3,$$
$$h_{\mathbf{c}_0} = 1,\ h_{\mathbf{c}_e}(\tau) = (\tau^{-2}, \tau^{-4}).$$

Now define

$$\pi_\tau(x) = \pi(h_{\mathbf{c}}(\tau) \cdot x)$$

for all x *in the* $\mathbf{c}$*-component. Then, for all* $t \in T^\vee/W$ *we have*

$$|\pi^{-1}_{\sqrt{q}}(t)| = |i^G_{T \subset B}(t)|.$$

Proof. We compare the description of the irreducible components of $I(1\otimes\nu)$ given by Muić [21] with those which occur in Ram's table [25, p. 20]. Then it follows from [21, p. 476 and Prop. 4.3] that

$$I(1\otimes\nu)=\pi(1)+\pi(1)'+J_\alpha(1/2,\delta(1))+2J_\beta(1/2,\delta(1))+J_\beta(1,\pi(1,1))$$

so that

$$\ell(I(1\otimes\nu))=6,\qquad |I(1\otimes\nu)|=5.$$

When we collate the data in the table of Ram [25, p. 20], we find that

$$\begin{gathered}|i^G_{T\subset B}(t_e)|=5,\\ |i^G_{T\subset B}(t)|=4\quad\text{if}\quad t=t_a,\ t_c,\ t_d,\\ |i^G_{T\subset B}(t)|=2\quad\text{if}\quad t=t_b,t_f,t_g,t_h,t_i,t_j.\end{gathered}$$

We will now compute a correcting system of cocharacters (see the paragraph after Theorem 1.5). The extended quotient $T^\vee/\!/W$ is a disjoint union of 8 irreducible components $Z_1, Z_2, \ldots, Z_8$. Our notation is such that $Z_1=pt_1$, $Z_2=pt_2$, $Z_3=pt_3$, $Z_4=pt_4$, $Z_5=pt_*$, $Z_6\simeq\mathbb{A}^1$, $Z_7\simeq\mathbb{A}^1$, $Z_8=T^\vee/W$.

In the following table, the first column comprises 8 irreducible components $X_1,\ldots,X_8$ of $\widetilde{T^\vee}$ for which $\rho^{\mathrm{i}}(X_j)=Z_j,\ j=1,\ldots,8$. Let $[z_1,z_2]$ denote the image of (z_1,z_2) via the standard quotient map $p^{\mathrm{i}}\colon T^\vee\to T^\vee/W$, so that $[z_1,z_2]=W\cdot(z_1,z_2)$. When the first column itemizes the pairs $(w,t)\in\widetilde{T^\vee}$, the second column itemizes $p^{\mathrm{i}}(h_j(\tau)t)$. The third column itemizes the corresponding correcting cocharacters.

$X_1=(r^2,(j,j))$	$[j,\tau^{-2}j]$	$h_1(\tau)=(1,\tau^{-2})$
$X_2=(r^2,(1,1))$	$[1,\tau^{-2}]$	$h_2(\tau)=(1,\tau^{-2})$
$X_3=(r^3,(1,1))$	$[1,\tau^{-2}]$	$h_3(\tau)=(1,\tau^{-2})$
$X_4=(r^3,(-1,1))$	$[-1,\tau^{-2}]$	$h_4(\tau)=(1,\tau^{-2})$
$X_5=(r,(1,1))$	$[\tau^{-2},\tau^{-4}]$	$h_5(\tau)=(\tau^{-2},\tau^{-4})$
$X_6=\{(a,(z,z)):z\in\mathbb{C}^\times\}$	$\{[z,\tau^{-2}z]:z\in\mathbb{C}^\times\}$	$h_6(\tau)=(1,\tau^{-2})$
$X_7=\{(b,(z,1)):z\in\mathbb{C}^\times\}$	$\{[z,\tau^{-2}]:z\in\mathbb{C}^\times\}$	$h_7(\tau)=(1,\tau^{-2})$
$X_8=\{(e,(z_1,z_2)):z_1,z_2\in\mathbb{C}^\times\}$	$\{[z_1,z_2]:z_1,z_2\in\mathbb{C}^\times\}$	$h_8(\tau)=1$

It is now clear that cocharacters can be assigned to two-sided cells as follows: $h_{\mathbf{c}}(\tau)=(1,\tau^{-2})$ if $\mathbf{c}=\mathbf{c}_1,\mathbf{c}_2,\mathbf{c}_3$, $h_{\mathbf{c}_0}=1$, $h_{\mathbf{c}_e}(\tau)=(\tau^{-2},\tau^{-4})$.

We are now in a position to write down explicitly the fibres $\pi^{-1}_{\sqrt{q}}(t)$ of the points of reducibility in the quotient variety $T^\vee/W$. We recall that $t_e=(1,q^{-1})$, $t_a=(q^{-1},q^{-2})$, $t_d=(q^{-1},-1)$ and $t_c=(j,q^{-1}j)$. We have, for example,

$$\begin{gathered}\pi^{-1}_{\sqrt{q}}(t_e)=\{\rho^{\mathrm{i}}(e,(1,q^{-1})),\rho^{\mathrm{i}}(r^2,(1,1)),\rho^{\mathrm{i}}(r^3,(1,1)),\rho^{\mathrm{i}}(a,(1,1)),\rho^{\mathrm{i}}(b,(1,1))\},\\ \pi^{-1}_{\sqrt{q}}(t_a)=\{\rho^{\mathrm{i}}(e,(q^{-1},q^{-2})),\rho^{\mathrm{i}}(r,(1,1)),\rho^{\mathrm{i}}(a,(q^{-1},q^{-1})),\rho^{\mathrm{i}}(b,(q^{-1},1))\},\\ \pi^{-1}_{\sqrt{q}}(t_d)=\{\rho^{\mathrm{i}}(e,(q^{-1},-1)),\rho^{\mathrm{i}}(r^3,(-1,1)),\rho^{\mathrm{i}}(a,(-1,-1)),\rho^{\mathrm{i}}(b,(-1,1))\},\\ \pi^{-1}_{\sqrt{q}}(t_c)=\{\rho^{\mathrm{i}}(e,(j,j/q)),\rho^{\mathrm{i}}(r^2,(j,j)),\rho^{\mathrm{i}}(a,(j,j)),\rho^{\mathrm{i}}(a,(j^2,j^2))\}.\end{gathered}$$

The two points $(a,(j,j))$ and $(a,(j^2,j^2))$ are especially interesting. The map $\pi_{\sqrt{q}}$ sends these two points to the one point $t_c\in T^\vee/W$ since $(j^2,j^2/q)$ and $(j,j/q)$ are in the same W-orbit: $(j,j/q)\sim_{aba}(j^2,j^2/q)$. In the map

$$\pi_{\sqrt{q}}:T^\vee/\!/W\to T^\vee/W$$

the image of the affine line $(T^\vee)^a/Z(a)$ has a *self-intersection point*, namely t_c.

We have

$$\begin{aligned} |\pi^{-1}_{\sqrt{q}}(t_e)| &= 5, \\ |\pi^{-1}_{\sqrt{q}}(t)| &= 4 \quad \text{if} \quad t = t_a,\ t_c,\ t_d, \\ |\pi^{-1}_{\sqrt{q}}(t)| &= 2 \quad \text{if} \quad t = t_b, t_f, t_g, t_h, t_i, t_j. \end{aligned}$$

□

Lemma 4.4. *Part (4) of Theorem* 1.5 *is true for* $\mathfrak{i} \in \mathfrak{B}(G_2)$.

Proof. We will denote elements in the five unipotent classes of $\mathrm{G}_2(\mathbb{C})$ by 1, u_3, u_2, u_1, u_e (trivial, minimal, subminimal, subregular, regular).

We recall that the irreducible G-module in $\mathrm{Irr}^{\mathfrak{i}}(G)$ corresponding to the Kazhdan-Lusztig triple (σ, u, ρ) is denoted $\mathcal{V}^{\mathfrak{i}}_{\sigma,u,\rho}$. According to [22, Table 6.1], we have $\dim \mathcal{V}^{\mathfrak{i}}_{t_e,u_1,\rho_2} = 1$, $\dim \mathcal{V}^{\mathfrak{i}}_{t_e,u_1,\rho_1} = \dim \mathcal{V}^{\mathfrak{i}}_{t_d,u_1,1} = 3$, and $\dim \mathcal{V}^{\mathfrak{i}}_{t_c,u_1,1} = 2$. Hence we have

$$\mathcal{V}^{\mathfrak{i}}_{t_e,u_1,\rho_2} = E_{4,q}, \quad \mathcal{V}^{\mathfrak{i}}_{t_e,u_1,\rho_1} = E_{1,q}, \quad \mathcal{V}^{\mathfrak{i}}_{t_d,u_1,1} = E_{2,q}, \quad \mathcal{V}^{\mathfrak{i}}_{t_c,u_1,1} = E_{3,q},$$

where $E_{i,q} := \phi^*_{q,\mathbf{c}_1}(E_i)$, with E_i as (35).

The semisimple elements t, σ below are always related as in equation (19), that is,

$$\sigma = h_{\mathbf{c}}(\sqrt{q}) \cdot t = \pi_{\sqrt{q}}(t).$$

Then the definition (13) of $\mu^{\mathfrak{i}}$ gives (using the maps $\eta_{\mathbf{c}}$ defined in the proof of Lemma 4.1):

$$\mu^{\mathfrak{i}}(t) = \begin{cases} \mathcal{V}^{\mathfrak{i}}_{\sigma,1,1}\,, & \text{if } t \in T^\vee/W, \\ \mathcal{V}^{\mathfrak{i}}_{\sigma,u_2,1}\,, & \text{if } t \in \mathbb{A}^1 \text{ (attached to } \mathbf{c}_2), \\ \mathcal{V}^{\mathfrak{i}}_{\sigma,u_3,1}\,, & \text{if } t \in \mathbb{A}^1 \text{ (attached to } \mathbf{c}_3); \end{cases}$$

two of the isolated points are sent to the L-indistinguishable elements in the discrete series which admit nonzero Iwahori fixed vectors:

$$\mu^{\mathfrak{i}}(pt_1) = \mathcal{V}^{\mathfrak{i}}_{t_e,u_1,\rho_1}, \quad \mu^{\mathfrak{i}}(pt_4) = \mathcal{V}^{\mathfrak{i}}_{t_e,u_1,\rho_2}$$

and

$$\begin{aligned} \mu^{\mathfrak{i}}(pt_2) &= \mathcal{V}^{\mathfrak{i}}_{t_d,u_1,1}, \\ \mu^{\mathfrak{i}}(pt_3) &= \mathcal{V}^{\mathfrak{i}}_{t_c,u_1,1}, \\ \mu^{\mathfrak{i}}(pt_*) &= \mathcal{V}^{\mathfrak{i}}_{t_a,u_e,1}. \end{aligned}$$

Now the infinitesimal character of $\mathcal{V}^{\mathfrak{i}}_{\sigma,u,\rho}$ is σ, therefore the map $\mu^{\mathfrak{i}}$ satisfies

$$inf.ch. \circ \mu^{\mathfrak{i}} = \pi_{\sqrt{q}}.$$

The map $\mu^{\mathfrak{i}}$ is compatible with the cell-partitions

$$\mu^{\mathfrak{i}}((T^\vee/\!/W)_{\mathbf{c}}) \subset \mathrm{Irr}^{\mathfrak{i}}(G)_{\mathbf{c}}.$$

□

Lemma 4.5. *Part (5) of Theorem* 1.5 *is true for* $\mathfrak{i} \in \mathfrak{B}(G_2)$.

Proof. As for the compact extended quotient, this is accounted for as follows: The compact quotient E/W is the unitary principal series, one unit interval is one intermediate unitary series, the other unit interval is the other intermediate unitary series, and the five isolated points are the remaining tempered representations itemized in [22, 21]. □

Note. Among the tempered representations of G which admit nonzero Iwahori fixed vectors, those which have real central character are in bijection with the conjugacy classes in W. For G of type G_2, they are (see [22, Fig. 6.1, Tab. 6.3]):

$$\mathcal{V}^{\mathrm{i}}_{t_0,1,1},\ \mathcal{V}^{\mathrm{i}}_{t_g,u_3,1},\ \mathcal{V}^{\mathrm{i}}_{t_j,u_2,1},\ \mathcal{V}^{\mathrm{i}}_{t_e,u_1,\rho_1},\ \mathcal{V}^{\mathrm{i}}_{t_e,u_1,\rho_2},\ \mathcal{V}^{\mathrm{i}}_{t_a,u_e,1}.$$

These representations correspond (via the inverse map of μ^{i}) to points in $T^\vee/W$, $(T^\vee/\!/W)_{\mathbf{c}_3}$, $(T^\vee/\!/W)_{\mathbf{c}_2}$, $pt_1 \in (T^\vee/\!/W)_{\mathbf{c}_1}$, $pt_4 \in (T^\vee/\!/W)_{\mathbf{c}_1}$, and $pt_* \in (T^\vee/\!/W)_{\mathbf{c}_e}$, respectively.

Hence the correspondence with conjugacy classes in W that we obtained is the following:

$$\begin{array}{ccc}
\mathcal{V}^{\mathrm{i}}_{t_0,1,1} & \leftrightarrow & (e) \\
\mathcal{V}^{\mathrm{i}}_{t_g,u_3,1} & \leftrightarrow & (b) \\
\mathcal{V}^{\mathrm{i}}_{t_j,u_2,1} & \leftrightarrow & (a) \\
\mathcal{V}^{\mathrm{i}}_{t_e,u_1,\rho_1} & \leftrightarrow & (r^3) \\
\mathcal{V}^{\mathrm{i}}_{t_e,u_1,\rho_2} & \leftrightarrow & (r^2) \\
\mathcal{V}^{\mathrm{i}}_{t_a,u_e,1} & \leftrightarrow & (r)
\end{array}$$

5. Some preparatory results

5.1. The group $W^{\mathfrak{s}}$.

When $W^{\mathfrak{s}} = \{e\}$, our conjecture is easily verified.

Lemma 5.1. *We have $W^{\mathfrak{s}} \neq \{e\}$ if and only if $\mathfrak{s} = [T, \chi \otimes \chi]_G$ or $\mathfrak{s} = [T, \chi \otimes 1]_G$ with χ an irreducible character of $F^\times$.*

Proof. From (1), we have

$$W^{\mathfrak{s}} = \{w \in W \ :\ w \cdot (\chi_1 \otimes \chi_2) = \psi(\chi_1 \otimes \chi_2) \ \text{ for some } \psi \in \Psi(T)\}\,.$$

Let $\sigma^\circ := \chi_1|_{\mathfrak{o}_F^\times} \otimes \chi_2|_{\mathfrak{o}_F^\times}$. Then we get

$$W^{\mathfrak{s}} = \{w \in W \ :\ w \cdot \sigma^\circ = \sigma^\circ\}. \tag{37}$$

Let $\chi_i^\circ := \chi_i|_{\mathfrak{o}_F^\times}$. From (28), it follows that we have $W^{\mathfrak{s}} = \{e\}$ if and only if

$$\chi_1^\circ \neq 1,\ \ \chi_2^\circ \neq 1,\ \ \chi_1^\circ\chi_2^\circ \neq 1,\ \ \chi_1^\circ \neq \chi_2^\circ,\ \ (\chi_1^\circ)^2\chi_2^\circ \neq 1,\ \ \chi_1^\circ(\chi_2^\circ)^2 \neq 1. \tag{38}$$

Hence we have $W^{\mathfrak{s}} \neq \{e\}$ if and only if we are in one of the following cases:

(1) We have $\chi_1^\circ = \chi_2^\circ$. We may and do assume that $\chi_1 = \chi_2 = \chi$.
(2) We have $\chi_2^\circ = 1$. We may and do assume that $\chi_1 = \chi$ and $\chi_2 = 1$. □

Remark 5.2. We observe that the condition (38) is equivalent to the condition

$$((\chi_1 \otimes \chi_2) \circ \gamma^\vee)|_{\mathfrak{o}_F^\times} \neq 1, \ \text{ for all } \gamma \in \Phi.$$

Note that this condition is closely related to the condition (30).

Remark 5.3. The group $W^{\mathfrak{s}}$ admits the following description (which is compatible with [24, Lemma 6.2]):

$$W^{\mathfrak{s}} = \langle s_\gamma \ :\ \gamma \in \Phi \text{ such that } ((\chi_1 \otimes \chi_2) \circ \gamma^\vee)|_{\mathfrak{o}_F^\times} = 1\rangle.$$

In particular, this shows that $W^{\mathfrak{s}}$ is a finite Weyl group.

5.2. **The list of cases to be studied.**

5.2.1. *W-orbits.* 1. The orbit $W\cdot(\chi\otimes\chi)$ consists of the following characters:

$$\chi\otimes\chi,\ \chi^{-1}\otimes\chi^{-1},\ \chi^2\otimes\chi^{-1},\ \chi^{-1}\otimes\chi^2,\ \chi\otimes\chi^{-2},\ \chi^{-2}\otimes\chi. \tag{39}$$

It follows that

$$W\cdot(\chi\otimes\chi)=\begin{cases}\chi\otimes\chi,\ \chi\otimes 1,\ 1\otimes\chi & \text{if } \chi \text{ is quadratic,}\\ \chi\otimes\chi,\ \chi^{-1}\otimes\chi^{-1} & \text{if } \chi \text{ is cubic.}\end{cases}$$

We have

$$|W\cdot(\chi\otimes\chi)|=\begin{cases}1 & \text{if } \chi \text{ is trivial,}\\ 3 & \text{if } \chi \text{ is quadratic,}\\ 2 & \text{if } \chi \text{ is cubic,}\\ 6 & \text{otherwise.}\end{cases} \tag{40}$$

2. The orbit $W\cdot(\chi\otimes 1)$ consists of the following characters:

$$\chi\otimes 1,\ 1\otimes\chi,\ \chi\otimes\chi^{-1},\ \chi^{-1}\otimes 1,\ 1\otimes\chi^{-1},\ \chi^{-1}\otimes\chi. \tag{41}$$

If χ is quadratic, then we have

$$W\cdot(\chi\otimes 1)=\{\chi\otimes\chi,\ \chi\otimes 1,\ 1\otimes\chi\}.$$

We have

$$|W\cdot(\chi\otimes 1)|=\begin{cases}1 & \text{if } \chi \text{ is trivial,}\\ 3 & \text{if } \chi \text{ is quadratic,}\\ 6 & \text{otherwise.}\end{cases} \tag{42}$$

5.2.2. *The cases.* From now on we will assume that $W^{\mathfrak{s}}\neq\{e\}$. Then the above discussion leads to the following cases:

(1) $\mathfrak{s}=\mathfrak{i}=[T,1]_G$. Here $W^{\mathfrak{s}}=W$. Already studied in Section 4.
(2) $\mathfrak{s}=[T,\chi\otimes 1]_G$ with χ ramified nonquadratic; see Section 6.
(3) $\mathfrak{s}=[T,\chi\otimes\chi]_G$ with χ ramified, neither quadratic nor cubic; see Section 6.
(4) $\mathfrak{s}=[T,\chi\otimes\chi]_G$ with χ ramified cubic; see Section 7.
(5) $\mathfrak{s}=[T,\chi\otimes\chi]_G$ with χ ramified quadratic; see Section 8.

5.3. **Lengths of the induced representations.** We fix homomorphisms $x_\gamma\colon F\to G$ and $\zeta_\gamma\colon \mathrm{SL}(2,F)\to G$ $\gamma\in\Phi$ such that (see [9, (6.1.3) (b)]):

$$x_\gamma(u)=\zeta_\gamma\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix},\quad x_{-\gamma}(u)=\zeta_{-\gamma}\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}\quad\text{and}\quad \gamma^\vee(t)=\zeta_\gamma\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}.$$

For $\gamma\in\{\alpha,\beta\}$, let P_γ be the maximal standard parabolic subgroup of G corresponding to γ, and M_γ be the centralizer of the image of $(\gamma')^\vee$ in G, where γ' is the unique positive root orthogonal to γ, that is,

$$\gamma'=\begin{cases}3\alpha+2\beta & \text{if } \gamma=\alpha,\\ 2\alpha+\beta & \text{if } \gamma=\beta.\end{cases}$$

Then M_γ is a Levi factor for P_γ.

We extend $\zeta_\gamma\colon \mathrm{SL}(2,F)\to M_\gamma$ to an isomorphism $\zeta_\gamma\colon \mathrm{GL}(2,F)\to M_\gamma$ by

$$\zeta_\gamma\left(\begin{pmatrix}t & 0\\ 0 & t\end{pmatrix}\right):=\zeta_{\gamma'}\left(\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right),\quad \text{for } t\in F^\times.$$

Then the restriction to T of the inverse map of ζ_γ coincides with the isomorphism $\xi_\gamma\colon T \xrightarrow{\sim} F^\times \times F^\times$, where ξ_α has been defined in (27), while

$$\xi_\beta\colon t \mapsto ((\alpha+\beta)(t), \alpha(t)).$$

For $\gamma \in \{\alpha,\beta\}$, and σ a smooth irreducible representation of GL(2), let $I_\gamma(\sigma)$ denote the parabolically induced representation of G:

$$\mathrm{Ind}_{M_\gamma \subset P_\gamma}^{G}(\sigma \circ \zeta_\gamma^{-1}). \tag{43}$$

Let δ be the Steinberg representation of GL(2) and let $\delta(\chi)$ denote the twist of δ by the one-dimensional representation $\chi \circ \det$. Then $\delta(\chi)$ is the unique irreducible subrepresentation of $\mathrm{Ind}_B^{\mathrm{GL}(2)}(\nu^{1/2}\chi \otimes \nu^{-1/2}\chi)$. It is square integrable if χ is unitary. The representation δ has torsion number 1, and so all the twists $\{\delta(\chi) : \chi \in \Psi(F^\times)\}$ are distinct.

The inertial support of the representation $I_\gamma(\delta(\chi))$ is $[T,(\chi\otimes\chi)\circ\xi_\alpha]_G$ if $\gamma=\alpha$ and $[T,(\chi\otimes\chi)\circ\xi_\beta]_G = [T,(\chi\otimes 1)\circ\xi_\alpha]_G$ if $\gamma=\beta$. We observe the following consequence (which will be crucial in the sequel of the paper):

Proposition 5.4. *The representations $I_\alpha(\delta(\chi))$ and $I_\beta(\delta(\chi))$ have same inertial support when $\chi^2 = 1$ and have distinct inertial supports otherwise.*

Proof. It follows from the orbit computation done in §5.2.1. □

Lemma 5.5. *Let χ, ψ be two characters of $F^\times$, with ψ unramified and χ ramified. Let ϵ be the unique unramified quadratic character of $F^\times$, and let ω denote an unramified cubic character of $F^\times$. We set*

$$\begin{aligned}
\mathcal{P}_2 &= \left\{(\nu^{\pm 1/2},\chi),(\nu^{\pm 1/2}\epsilon,\chi) \;:\; \chi \text{ is quadratic}\right\},\\
\mathcal{P}_3 &= \left\{(\nu^{\pm 1/2},\chi),(\nu^{\pm 1/2}\omega,\chi),(\nu^{\pm 1/2}\omega^2,\chi) \;:\; \chi \text{ is cubic}\right\},\\
\mathcal{P} &= \mathcal{P}_2 \cup \mathcal{P}_3.
\end{aligned}$$

Then we have

$$\begin{aligned}
\ell\left(I(\nu^{-1/2}\psi\chi \otimes \nu^{1/2}\psi\chi)\right) &= |I(\nu^{-1/2}\psi\chi \otimes \nu^{1/2}\psi\chi)| = \begin{cases} 4 & \text{if } (\psi,\chi) \in \mathcal{P},\\ 2 & \text{otherwise,}\end{cases}\\
\ell\left(I(\nu^{-1/2}\psi\chi \otimes \nu)\right) &= |I(\nu^{-1/2}\psi\chi \otimes \nu| = \begin{cases} 4 & \text{if } (\psi,\chi) \in \mathcal{P}_2,\\ 2 & \text{otherwise.}\end{cases}
\end{aligned}$$

Proof. In $R(M_\alpha)$, we have (see for instance [21, Proposition 1.1(ii)]):

$$\mathrm{Ind}_{T(U\cap M_\alpha)}^{M_\alpha}(\nu^{-1/2}\psi\chi \otimes \nu^{1/2}\psi\chi) = \delta(\psi\chi) \oplus (\psi\chi \circ \det).$$

Similarly, in $R(M_\beta)$ (using now [21, Proposition 1.1(iii)]), we get

$$\mathrm{Ind}_{T(U\cap M_\beta)}^{M_\beta}(\nu^{-1/2}\psi\chi \otimes \nu) = \delta(\psi\chi) \oplus (\psi\chi \circ \det).$$

Then, by transitivity of parabolic induction, we obtain

$$\begin{aligned}
I(\nu^{-1/2}\psi\chi \otimes \nu^{1/2}\psi\chi)) &= I_\alpha(\delta(\psi\chi)) + I_\alpha(\psi\chi \circ \det),\\
I(\nu^{-1/2}\psi\chi \otimes \nu) &= I_\beta(\delta(\psi\chi)) + I_\beta(\psi\chi \circ \det).
\end{aligned}$$

Applying the involution D_G defined in [1], it follows from [1, Th. 1.7] that, for $\gamma \in \{\alpha,\beta\}$, the induced representations $I_\gamma(\delta(\psi\chi))$ and $I_\gamma(\psi\chi \circ \det)$ have the same length.

To describe the length of $I_\alpha(\delta(\psi\chi))$, we write $\psi = \nu^s$, $s \in \mathbb{C}$. Now, in a different notation, we write

$$I_\alpha(\mathrm{Re}(s), \delta(\nu^{\sqrt{-1}\mathrm{Im}(s)}\chi)) = I_\alpha(\delta(\psi\chi)).$$

Then [21, Theorem 3.1 (i)] implies the following conclusion:

1. If χ is neither quadratic nor cubic, then $I_\alpha(\delta(\psi\chi))$ is irreducible. Hence, $\ell\left(I(\nu^{-1/2}\psi\chi \otimes \nu^{1/2}\psi\chi)\right) = 2$.

2. If χ is ramified quadratic, then $I_\alpha(\delta(\psi\chi))$ reduces if and only if $\nu^{\sqrt{-1}\mathrm{Im}(s)} \in \{1, \epsilon\}$ and $\mathrm{Re}(s) = \pm 1/2$. Hence:

- If $\psi \notin \{\nu^{\pm 1/2}, \nu^{\pm 1/2}\epsilon\}$, then $\ell\left(I(\nu^{-1/2}\psi\chi \otimes \nu^{1/2}\psi\chi)\right) = 2$.
- Otherwise, Rodier's result [25, Corollary on p. 419] (see [21, Prop. 4.1]) implies that $I(\nu^{-1/2}\psi\chi \otimes \nu^{1/2}\psi\chi)$ has length 4 and multiplicity 1.

3. If χ is cubic ramified, then $I_\alpha(\delta(\psi\chi))$ reduces if and only if $\nu^{\sqrt{-1}\mathrm{Im}(s)} \in \{1, \omega, \omega^2\}$ and $\mathrm{Re}(s) = \pm 1/2$. Hence:

- If $\psi \notin \{\nu^{\pm 1/2}, \nu^{\pm 1/2}\omega, \nu^{\pm 1/2}\omega^2\}$, then $\ell\left(I(\nu^{-1/2}\psi\chi \otimes \nu^{1/2}\psi\chi)\right) = 2$.
- Otherwise, it follows from *loc.cit.* that $I(\nu^{-1/2}\psi\chi \otimes \nu^{1/2}\psi\chi)$ has length 4 and multiplicity 1.

If χ is (ramified) not quadratic, then $I_\beta(\delta(\psi\chi))$ is irreducible. Then

$$\ell\left(I(\nu^{-1/2}\psi\chi \otimes \nu)\right) = 2.$$

We assume from now on that χ is quadratic ramified. To describe the length of $I_\beta(\delta(\psi\chi))$, we write $\psi = \nu^s\psi_0$, where $s \in \mathbb{R}$, ψ_0 is unitary. Then $I_\beta(\delta(\psi\chi))$ reduces if and only if $s = \pm 1/2$ and $\psi_0^2 = 1$. Therefore, the length of $I(\nu^{-1/2}\psi\chi \otimes \nu)$ is two unless $\psi = \nu^{\pm 1/2}, \nu^{\pm 1/2}\epsilon$. □

5.4. **Two lemmas.** The next two lemmas will be needed in Section 8 in the proof of Lemma 8.1.

5.4.1. *Crossed product algebras.* Let A be a unital $\mathbb{C}$-algebra and let Γ be a finite group acting as automorphisms of the unital $\mathbb{C}$-algebra A. Let

$$A^\Gamma := \{a \in A \,:\, \gamma \cdot a = a, \quad \forall \gamma \in \Gamma\}.$$

Let $A \rtimes \Gamma$ denote the crossed product algebra for the action of Γ on A: The elements of $A \rtimes \Gamma$ are formal sums $\sum_{\gamma\in\Gamma} a_\gamma[\gamma]$, where:

- the addition is $(\sum_{\gamma\in\Gamma} a_\gamma[\gamma]) + (\sum_{\gamma\in\Gamma} b_\gamma[\gamma]) = \sum_{\gamma\in\Gamma}(a_\gamma + b_\gamma)[\gamma]$,
- the multiplication is determined by $(a_\gamma[\gamma])(b_\alpha[\alpha]) = a_\gamma(\gamma \cdot b_\alpha)[\gamma\alpha]$,
- the multiplication by $\lambda \in \mathbb{C}$ is given by $\lambda(\sum_{\gamma\in\Gamma} a_\gamma[\gamma]) = \sum_{\gamma\in\Gamma}(\lambda a_\gamma)[\gamma]$.

Let

$$e_\Gamma := |\Gamma|^{-1}\sum_{\gamma\in\Gamma}[\gamma].$$

Then e_Γ is an idempotent (i.e., $e_\Gamma^2 = e_\Gamma$).

Lemma 5.6. *The unital* $\mathbb{C}$*-algebras* A^Γ *and* $(A \rtimes \Gamma)e_\Gamma(A \rtimes \Gamma)$ *are Morita equivalent.*

Proof. See [26, Proposition p. 25]. □

5.4.2. *Ring homomorphisms.*

Lemma 5.7. *Let A be a ring with unit and let B be a ring (which is not required to have a unit). Let $\mathcal{J} \subset B$ be a two-sided ideal. Then any surjective homomorphism of rings $\varphi\colon \mathcal{J} \twoheadrightarrow A$ extends* uniquely *to a ring homomorphism $\tilde{\varphi}\colon B \to A$.*

Proof. Choose $\theta_0 \in \mathcal{J}$ such that $\varphi(\theta_0) = 1_A$ (the unit in A). Then, given $b \in B$, we define $\tilde{\varphi}(b)$ by $\tilde{\varphi}(b) := \varphi(\theta_0 b)$.

(1) We will check first that $\tilde{\varphi}$ is well defined, i.e., that the definition does not depend on the choice of θ_0. Indeed, for every $\theta \in \mathcal{J}$ such that $\varphi(\theta) = 1_A$, then we have, on one hand,
$$\varphi(\theta b \theta_0) = \varphi(\theta b)\varphi(\theta_0) = \varphi(\theta b);$$
and on the other hand,
$$\varphi(\theta b \theta_0) = \varphi(\theta)\varphi(b\theta_0) = \varphi(b\theta_0).$$
Hence, $\varphi(\theta b) = \varphi(b\theta_0)$. In particular, we have $\varphi(\theta_0 b) = \varphi(b\theta_0)$. Thus $\varphi(\theta b) = \varphi(\theta_0 b)$.

(2) Let $\tilde{\varphi}$ be any extension of φ. We have
$$\tilde{\varphi}(b) = 1_A\tilde{\varphi}(b) = \varphi(\theta_0)\tilde{\varphi}(b) = \tilde{\varphi}(\theta_0 b) = \varphi(\theta_0 b),$$
since $\theta_0 b \in \mathcal{J}$.

(3) Finally, we check that $\tilde{\varphi}$ is a ring homomorphism. Indeed,
$$\begin{aligned}\tilde{\varphi}(b_1 + b_2) &= \varphi(\theta_0(b_1 + b_2)) = \varphi(\theta_0 b_1 + \theta_0 b_2) = \tilde{\varphi}(b_1) + \tilde{\varphi}(b_2),\\ \tilde{\varphi}(b_1 b_2) &= \varphi(\theta_0 b_1 b_2) = \varphi(\theta_0 b_1 b_2 \theta_0) = \varphi(\theta_0 b_1)\varphi(b_2\theta_0) = \tilde{\varphi}(b_1)\tilde{\varphi}(b_2).\end{aligned}$$ □

6. The two cases for which $H^{\mathfrak{s}} = \mathrm{GL}(2,\mathbb{C})$

In this section, we will consider the following two cases.

Case 1: We assume here that $\chi_2 = 1$ and $\chi_1 = \chi$ with χ a ramified nonquadratic character. Then from (41) we obtain
$$\begin{array}{rcccccc}\mathfrak{s} & = & [T, \chi\otimes 1]_G & = & [T, 1\otimes\chi]_G & = & [T, \chi\otimes\chi^{-1}]_G,\\ & = & [T, \chi^{-1}\otimes 1]_G & = & [T, 1\otimes\chi^{-1}]_G & = & [T, \chi^{-1}\otimes\chi]_G.\end{array}$$
It follows from (28) that
$$W^{\mathfrak{s}} = \{e, b\} \cong S_2. \tag{44}$$

Case 2: We assume that $\chi_1 = \chi_2 = \chi$ with χ a ramified character which is neither quadratic nor cubic. From (39) we obtain
$$\begin{array}{rcccccc}\mathfrak{s} & = & [T, \chi\otimes\chi]_G & = & [T, \chi^{-1}\otimes\chi^{-1}]_G & = & [T, \chi^2\otimes\chi^{-1}]_G,\\ & = & [T, \chi^{-1}\otimes\chi^2]_G & = & [T, \chi\otimes\chi^{-2}]_G & = & [T, \chi^{-2}\otimes\chi]_G.\end{array}$$
It follows from (28) that
$$W^{\mathfrak{s}} = \{e, a\} \cong S_2. \tag{45}$$

In both Case 1 and Case 2, we have $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}} = S_2 \ltimes X(T^\vee)$. Hence, $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$ is the extended affine Weyl group of the p-adic group $\mathrm{GL}(2,F)$. There are 2 two-sided cells, say $\mathbf{b}_e$ and $\mathbf{b}_0$, in $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$; they correspond to the regular unipotent class $\mathcal{U}_e$ and to the trivial unipotent class of $\mathrm{GL}(2,\mathbb{C})$, respectively. Hence, $\mathbf{b}_e$ and $\mathbf{b}_0$ correspond to the partitions (2) and $(1,1)$ of 2, respectively. We have $\mathbf{b}_0 \leq \mathbf{b}_e$.

Definition. We define the following partition of $T^\vee//W^{\mathfrak{s}}$:

$$T^\vee//W^{\mathfrak{s}} = (T^\vee//W^{\mathfrak{s}})_{\mathbf{b}_e} \sqcup (T^\vee//W^{\mathfrak{s}})_{\mathbf{b}_0}, \tag{46}$$

where $(T^\vee//W^{\mathfrak{s}})_{\mathbf{b}_e} := (T^\vee)^c/\mathrm{Z}(c)$, where c is the nontrivial element in $W^{\mathfrak{s}}$, and $(T^\vee//W^{\mathfrak{s}})_{\mathbf{b}_0} := T^\vee/W^{\mathfrak{s}}$.

We will denote by $J^{\mathfrak{s}}$ the based ring of the extended affine Weyl group $\widetilde{W}^{\mathfrak{s}}_{\mathbf{a}}$ defined in (3) and set

$$U(1) := \{z \in \mathbb{C} : |z| = 1\}. \tag{47}$$

Lemma 6.1. *We have the following diffeomorphisms*

$$(T^\vee//W^{\mathfrak{s}})_{\mathbf{b}_e} \to \mathbb{C}^\times, \qquad E^{\mathfrak{s}}//W^{\mathfrak{s}} \to U(1) \sqcup E^{\mathfrak{s}}/W^{\mathfrak{s}},$$

and we have

$$J^{\mathfrak{s}} = J^{\mathfrak{s}}_{\mathbf{b}_e} + J^{\mathfrak{s}}_{\mathbf{b}_0} \sim_{\text{morita}} \mathcal{O}(T^\vee//W^{\mathfrak{s}}),$$

where $J^{\mathfrak{s}}_{\mathbf{b}_e} \sim_{\text{morita}} \mathcal{O}((T^\vee//W^{\mathfrak{s}})_{\mathbf{b}_e})$ *and* $J^{\mathfrak{s}}_{\mathbf{b}_0} \sim_{\text{morita}} \mathcal{O}((T^\vee//W^{\mathfrak{s}})_{\mathbf{b}_0})$.

Proof. Let $D = D^{\mathfrak{s}}$ and $E := E^{\mathfrak{s}}$. We give the case-by-case analysis.

- $c = 1$. $D^c/\mathrm{Z}(c) = D/W^{\mathfrak{s}}$ and $E^c/\mathrm{Z}(c) = E/W^{\mathfrak{s}}$.
- $c \neq 1$.

Case 1: $c = b$:

$$D^b/\mathrm{Z}(b) = D^b = \{(t,1) : t \in \mathbb{C}^\times\}. \quad E^b = \{(t,1) : t \in U(1)\}.$$

Case 2: $c = a$:

$$D^a/\mathrm{Z}(a) = D^a = \{(t,t) : t \in \mathbb{C}^\times\}. \quad E^b = \{(t,t) : t \in U(1)\}.$$

We have $J^{\mathfrak{s}} = J_{\mathbf{b}_e} + J_{\mathbf{b}_0}$ and (see [2, proof of Theorem 3]):

$$J_{\mathbf{b}_e} \sim_{\text{morita}} \mathcal{O}(\mathbb{C}^\times), \quad J_{\mathbf{b}_0} \sim_{\text{morita}} \mathcal{O}((\mathbb{C}^\times)^2/S_2) \cong \mathcal{O}(D^{\mathfrak{s}}/W^{\mathfrak{s}}).$$

It gives

$$J_{\mathbf{b}_i} \sim_{\text{morita}} \mathcal{O}(T^\vee//W^{\mathfrak{s}})_{\mathbf{b}_i}, \quad \text{for } i \in \{e, 0\}. \tag{48}$$

□

Lemma 6.2. *The flat family is given by*

$$\mathfrak{X}_\tau : 1 - \tau y = 0, \quad \textit{in Case 1;}$$
$$\mathfrak{X}_\tau : x - \tau^2 y = 0, \quad \textit{in Case 2.}$$

Proof. We will considerate the two cases separately.

Case 1: The curve of reducibility $\mathfrak{C}_1 = \mathfrak{X}_{\sqrt{q}} = \mathfrak{R}$ is given by

$$\mathfrak{C}_1 = \left\{\psi\chi\nu^{-1/2} \otimes \nu^{1/2} \,:\, \psi \in \Psi(F^\times)\right\} \cong \left\{(z\sqrt{q}, 1/\sqrt{q}) \,:\, z \in \mathbb{C}^\times\right\}.$$

We write down all the **nonunitary** quasicharacters $(\psi_1\chi \otimes \psi_2)$, with $\psi_1, \psi_2 \in \Psi(F^\times)$, which obey the reducibility conditions (29):

$$\psi\chi \otimes \nu^{-1}, \quad \psi\chi \otimes \nu, \quad \text{with } \psi \in \Psi(F^\times).$$

We get only one $W^{\mathfrak{s}}$-orbit of characters.

Case 2: The curve of reducibility $\mathfrak{C}_2 = \mathfrak{X}_{\sqrt{q}} = \mathfrak{R}$ is given by

$$\mathfrak{C}_2 = \left\{ \psi\chi\nu^{-1/2} \otimes \psi\chi\nu^{1/2} \,:\, \psi \in \Psi(F^\times) \right\} \cong \left\{ (z\sqrt{q}, z/\sqrt{q}) \,:\, z \in \mathbb{C}^\times \right\}.$$

We write down all the **nonunitary** quasicharacters of T which obey the reducibility conditions (29):

$$\psi\chi \otimes \psi\nu^{-1}\chi, \quad \psi\chi \otimes \psi\nu\chi, \qquad \text{with } \psi \in \Psi(F^\times).$$

We get only one $W^{\mathfrak{s}}$-orbit of characters. Indeed,

- the family of characters $\{\psi\chi \otimes \psi\nu\chi : \psi \in \Psi(F^\times)\}$, with the change of variable $\phi := \psi\nu^{1/2}$ is $\{\phi\nu^{-1/2}\chi \otimes \phi\nu^{1/2}\chi : \phi \in \Psi(F^\times)\}$;
- the family of characters $\{\psi\chi \otimes \psi\nu^{-1}\chi : \psi \in \Psi(F^\times)\}$, with the change of variable $\phi := \psi\nu^{-1/2}$ is $\{\phi\nu^{1/2}\chi \otimes \phi\nu^{-1/2}\chi \,:\, \phi \in \Psi(F^\times)\}$; by applying a, we then get $\{\phi\nu^{-1/2}\chi \otimes \phi\nu^{1/2}\chi : \phi \in \Psi(F^\times)\}$. □

Lemma 6.3. *The cocharacters are as follows:*

$$h_{\mathbf{b}_0} = 1, \;\; h_{\mathbf{b}_e}(\tau) = (\tau, \tau^{-1})$$

which leads to

$$\pi_\tau(v) = \pi(h_{\mathbf{b}_i}(\tau) \cdot v)$$

for all v in the $\mathbf{b}_i$-component, $i \in \{0, e\}$.

Proof. We apply Lemma 6.1. For all $v \in D^{\mathfrak{s}}/W^{\mathfrak{s}}$ we have

$$|\pi_{\sqrt{q}}^{-1}(v)| = |i_{T\subset B}^G(v)|.$$

If $v \notin \mathfrak{C} \cup \mathfrak{C}'$, we have $|i_{T\subset B}^G(v)| = 1 = |\pi_{\sqrt{q}}^{-1}(v)|$. On the other hand, for each $v \in \mathfrak{C} \cup \mathfrak{C}'$, from Lemma 5.5 we have

$$\ell(i_{T\subset B}^G(v)) = |i_{T\subset B}^G(v)| = 2 = |\pi_{\sqrt{q}}^{-1}(v)|, \qquad \square$$

due to Lemma 6.2.

Lemma 6.4. *Part (4) of Theorem* 1.5 *is true for the points $\mathfrak{s} = [T, \chi \otimes 1]_G$, with χ ramified nonquadratic and $\mathfrak{s} = [T, \chi \otimes \chi]_G$, with χ ramified neither quadratic nor cubic.*

Proof. The semisimple elements v, σ are always related as follows $\sigma = \pi_{\sqrt{q}}(v)$. Let $\eta^{\mathfrak{s}} : (T^\vee//W^{\mathfrak{s}}) \to \mathrm{Irr}(J^{\mathfrak{s}})$ be the bijection which is induced by the Morita equivalences in (48). Then the definition (13) of $\mu^{\mathfrak{s}} : (T^\vee//W^{\mathfrak{s}}) \to \mathrm{Irr}(G)^{\mathfrak{s}}$ gives:

$$\mu^{\mathfrak{s}}(v) = \begin{cases} \mathcal{V}^{\mathfrak{s}}_{\sigma,1,1}\,, & \text{if } v \in T^\vee/W^{\mathfrak{s}}, \\ \mathcal{V}^{\mathfrak{s}}_{\sigma,u_e,1}\,, & \text{if } v \in (T^\vee//W^{\mathfrak{s}})_{\mathbf{b}_e}. \end{cases}$$

Now the infinitesimal character of $\mathcal{V}^{\mathfrak{s}}_{\sigma,u,\rho}$ is σ, therefore the map $\mu^{\mathfrak{s}}$ satisfies

$$inf.ch. \circ \mu^{\mathfrak{s}} = \pi_{\sqrt{q}}. \qquad \square$$

Lemma 6.5. *Part (5) of Theorem* 1.5 *is true.*

Proof. As for the compact extended quotient, this is accounted for as follows: The compact quotient E/W is sent to the unitary principal series

$$\{I(\psi_1\chi \otimes \psi_2\chi) \,:\, \psi_1, \psi_2 \in \Psi(F^\times)\}/W^{\mathfrak{s}}$$

and the other component $U(1)$ to the intermediate unitary series

$$\{I_\alpha(\delta(\psi\chi)) : \psi \in \Psi^{\mathrm{t}}(F^\times)\}. \qquad \square$$

7. The case $H^{\mathfrak{s}} = \mathrm{SL}(3,\mathbb{C})$

We assume in this section that $\chi_1 = \chi_2 = \chi$, with χ a ramified character of order 3. We have

$$\mathfrak{s} = [T, \chi \otimes \chi]_G = [T, \chi^{-1} \otimes \chi^{-1}]_G = [T, \chi \otimes \chi^{-1}]_G = [T, \chi^{-1} \otimes \chi]_G.$$

It follows from (28) that

$$W^{\mathfrak{s}} = \{e, a, bab, abab, baba, ababa\} \cong S_3. \tag{49}$$

We have $a = s_\alpha$ and $bab = s_{\alpha+\beta}$. We observe that the root lattice $\mathbb{Z}\alpha \oplus \mathbb{Z}(\alpha+\beta)$ equals $X(T)$. It follows that $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$ (as defined in (3)) is the extended affine Weyl group of the p-adic group $G_{\mathfrak{s}} = \mathrm{PGL}(3, F)$.

There are 3 two-sided cells $\mathbf{d}_0$, $\mathbf{d}_1$, $\mathbf{d}_e$ in $\widetilde{W}_{\mathrm{a}}^{\mathfrak{s}}$, they are in bijection with the 3 unipotent classes of $\mathrm{SL}(3,\mathbb{C})$. The two-sided cell $\mathbf{d}_0$ corresponds to the trivial unipotent class, $\mathbf{d}_1$ corresponds to the subregular unipotent class, and $\mathbf{d}_e$ corresponds to the regular unipotent one. Hence we have $\mathbf{d}_0 \le \mathbf{d}_1 \le \mathbf{d}_e$ and $\mathbf{d}_e$, $\mathbf{d}_1$, $\mathbf{d}_0$ correspond, respectively, to the partitions (3), $(2,1)$ and $(1,1,1)$ of 3. We will denote elements in the three unipotent classes by 1, u_1, u_e (trivial, subregular, regular). The group $W^{\mathfrak{s}}$ admits 3 conjugacy classes: $\{e\}$, $\{a, bab, ababa\}$ and $\{abab, baba\}$. We recall that $r = ba$.

Definition. We define the following partition of $T^\vee /\!/ W^{\mathfrak{s}}$:

$$T^\vee /\!/ W^{\mathfrak{s}} = (T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_e} \sqcup (T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_1} \sqcup (T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_0}, \tag{50}$$

where

$$\begin{aligned}
(T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_e} &:= (T^\vee)^{r^2}/\mathrm{Z}(r^2),\\
(T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_1} &:= (T^\vee)^{a}/\mathrm{Z}(a),\\
(T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_0} &:= T^\vee / W^{\mathfrak{s}}.
\end{aligned}$$

Lemma 7.1. *We have*

$$\begin{aligned}
(T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_e} &= pt_1 \sqcup pt_2 \sqcup pt_3,\\
(T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_1} &= \mathbb{C}^\times,\\
E^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} &= (pt_1 \sqcup pt_2 \sqcup pt_3) \sqcup U(1) \sqcup E^{\mathfrak{s}}/W^{\mathfrak{s}},
\end{aligned}$$

and $J^{\mathfrak{s}} \sim_{\mathrm{morita}} \mathcal{O}(T^\vee /\!/ W^{\mathfrak{s}})$*, where*

$$\begin{aligned}
J_{\mathbf{d}_e} &\sim_{\mathrm{morita}} \mathcal{O}((T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_e}),\\
J_{\mathbf{d}_1} &\sim_{\mathrm{morita}} \mathcal{O}((T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_1}),\\
J_{\mathbf{d}_0} &\sim_{\mathrm{morita}} \mathcal{O}((T^\vee /\!/ W^{\mathfrak{s}})_{\mathbf{d}_0}).
\end{aligned}$$

Proof. We have $\mathrm{Z}_{W^{\mathfrak{s}}}(a) = \{e, a\}$ and $\mathrm{Z}_{W^{\mathfrak{s}}}(abab) = \{e, abab, baba\}$. Let $D := D^{\mathfrak{s}}$ and $E := E^{\mathfrak{s}}$. We obtain

$$D^a = \left\{(t,t) \,:\, t \in \mathbb{C}^\times\right\} \quad \text{and} \quad D^{abab} = \left\{(t, t^{-1}) \,:\, t \in \mathbb{C}^\times\right\}.$$

Case-by-case analysis.

- $\mathbf{d} = \mathbf{d}_e$, $c = abab$. $X^c/\mathrm{Z}_{W^{\mathfrak{s}}}(c) = \{(1,1), (j, j^2), (j^2, j)\} = E^c/\mathrm{Z}_{W^{\mathfrak{s}}}(c)$, where j is a primitive third root of unity. The points $(1,1)$, (j,j^2), (j^2,j) belong to 3 different $\mathrm{Z}_{W^{\mathfrak{s}}}(c)$-orbits. Therefore,

$$D^c/\mathrm{Z}_{W^{\mathfrak{s}}}(c) = E^c/\mathrm{Z}_{W^{\mathfrak{s}}}(c) = pt_1 \sqcup pt_2 \sqcup pt_3.$$

- $\mathbf{d} = \mathbf{d}_1$, $c = a$. $D^c/\mathrm{Z}_{W^\mathfrak{s}}(c) = D^c \cong \mathbb{C}^\times$.
$$E^c/\mathrm{Z}_{W^\mathfrak{s}}(c) = E^c = \{(t,t) : t \in U(1)\}.$$
- $\mathbf{d} = \mathbf{d}_0$, $c = 1$. $D^c/\mathrm{Z}_{W^\mathfrak{s}}(c) = D/W^\mathfrak{s}$. $E^c/\mathrm{Z}_{W^\mathfrak{s}}(c) = E/W^\mathfrak{s}$.

We have $J^\mathfrak{s} = J_{\mathbf{d}_e} + J_{\mathbf{d}_1} + J_{\mathbf{d}_0}$ and (see [2, proof of Theorem 4]):
$$J_{\mathbf{d}_e} \sim_{\text{morita}} \mathbb{C}^3, \quad J_{\mathbf{d}_1} \sim_{\text{morita}} \mathcal{O}(\mathbb{C}^\times), \quad J_{\mathbf{d}_0} \sim_{\text{morita}} \mathcal{O}(D^\mathfrak{s}/W^\mathfrak{s}).$$
It gives
$$J_{\mathbf{d}_i} \sim_{\text{morita}} \mathcal{O}((T^\vee/\!/W^\mathfrak{s})_{\mathbf{d}_i}), \quad \text{for } i \in \{e,1,0\}. \tag{51}$$
□

Lemma 7.2. *The flat family is given by*
$$\mathfrak{X}_\tau : (x - \tau^2 y = 0) \cup (j, \tau^{-2} j^2) \cup (j^2, \tau^{-2} j).$$

Proof. We write down all the quasicharacters of T which obey the reducibility conditions (29):
$$\begin{cases} 1. \ \ \psi\chi \otimes \psi^{-2}\nu\chi, \\ 2. \ \ \psi\chi \otimes \psi^{-2}\nu^{-1}\chi, \\ 3. \ \ \psi^{-2}\nu\chi \otimes \psi\chi, \\ 4. \ \ \psi^{-2}\nu^{-1}\chi \otimes \psi\chi, \\ 5. \ \ \psi\chi \otimes \psi\nu^{-1}\chi, \\ 6. \ \ \psi\chi \otimes \psi\nu\chi, \end{cases} \qquad \text{with } \psi \in \Psi(F^\times).$$
We get only one W-orbit of characters. Indeed,

6: the family of characters $\{\psi\chi \otimes \psi\nu\chi : \psi \in \Psi(F^\times)\}$, with the change of variable $\phi := \psi\nu^{1/2}$ is $\{\phi\nu^{-1/2}\chi \otimes \phi\nu^{1/2}\chi : \phi \in \Psi(F^\times)\}$;
5: the family of characters $\{\psi\chi \otimes \psi\nu^{-1}\chi : \psi \in \Psi(F^\times)\}$, with the change of variable $\phi := \psi\nu^{-1/2}$ is $\{\phi\nu^{1/2}\chi \otimes \phi\nu^{-1/2}\chi : \phi \in \Psi(F^\times)\}$; by applying a, we then get $\{\phi\nu^{-1/2}\chi \otimes \phi\nu^{1/2}\chi : \phi \in \Psi(F^\times)\}$;
4: we have $baba(\psi^{-2}\nu^{-1}\chi \otimes \psi\chi) = \psi\chi \otimes \psi\nu\chi$, which belongs to the family of characters in 6;
3: we have $baba(\psi^{-2}\nu\chi \otimes \psi\chi) = \psi\chi \otimes \psi\nu^{-1}\chi$, which belongs to the family of characters in 5;
2: we have $bab(\psi\chi \otimes \psi^{-2}\nu^{-1}\chi) = \psi\chi \otimes \psi\nu\chi$, which belongs to the family of characters in 6;
1: we have $bab(\psi\chi \otimes \psi^{-2}\nu\chi) = \psi\chi \otimes \psi\nu^{-1}\chi$, which belongs to the family of characters in 5.

The induced representations $\{I(\phi\nu^{-1/2}\chi \otimes \phi\nu^{1/2}\chi) : \phi \in \Psi(F^\times)\}$ are parametrized by $\{(zq^{1/2}, zq^{-1/2}) : z \in \mathbb{C}^\times\} = \{(z, zq^{-1}) : z \in \mathbb{C}^\times\}$. □

We set
$$y_a = (1, q^{-1}), \quad y'_a = (j, q^{-1} j^2), \quad y''_a = (j^2, q^{-1} j),$$
$$y_b = (z, q^{-1} z), \quad \text{where } z \notin \{1, j, j^2\}.$$

Lemma 7.3. *Define, for each two-sided cell* $\mathbf{d}$ *of* $W^\mathfrak{s}_a$, *cocharacters* $h_\mathbf{d}$ *as follows:*
$$h_{\mathbf{d}_0} = 1, \quad h_{\mathbf{d}_i}(\tau) = (1, \tau^{-2}) \quad \textit{for } i = 1, e.$$
Then, for all $y \in D^\mathfrak{s}/W^\mathfrak{s}$ *we have*
$$|\pi^{-1}_{\sqrt{q}}(y)| = |i^G_{T \subset B}(y)|.$$

Proof. If $y \notin \{y_a, y'_a, y''_a, y_b\}$, then we have $|i^G_{T\subset B}(y)| = 1 = |\pi^{-1}_{\sqrt{q}}(y)|$. On the other hand, Lemma 5.5 gives

$$|i^G_{T\subset B}(y_b)| = 2 = |\pi^{-1}_{\sqrt{q}}(y_b)|.$$

This leads to the following points of length 4:

$$\chi\otimes\nu\chi,\quad \omega\chi\otimes\nu\omega\chi,\quad \omega^2\chi\otimes\nu\omega^2\chi,\quad \nu^{-1}\chi\otimes\chi,\quad \nu^{-1}\omega\chi\otimes\omega\chi,\quad \nu^{-1}\omega^2\chi\otimes\omega^2\chi,$$

where ω denotes an unramified cubic character of $F^\times$. Since

$$\begin{aligned} baba(\nu^{-1}\chi\otimes\chi) &= \chi\otimes\nu\chi,\\ baba(\nu^{-1}\omega\chi\otimes\omega\chi) &= \omega\chi\otimes\nu\omega\chi,\\ baba(\nu^{-1}\omega^2\chi\otimes\omega^2\chi) &= \omega^2\chi\otimes\nu\omega^2\chi,\end{aligned}$$

this leads to exactly 3 points in the Bernstein variety $\Omega^{\mathfrak{s}}(G)$ which parametrize representations of length 4, namely

$$[T,\chi\otimes\nu\chi]_G,\quad [T,\omega\chi\otimes\nu\omega\chi]_G,\quad [T,\omega^2\chi\otimes\nu\omega^2\chi]_G.$$

The coordinates of these points in the algebraic surface $\Omega^{\mathfrak{s}}(G)$ are y_a, y'_a, y''_a, respectively.

We will now compute a correcting system of cocharacters. The extended quotient $T^\vee/\!/W^{\mathfrak{s}}$ is a disjoint union of 5 irreducible components Z_1, Z_2, …, Z_5. Our notation is such that $Z_1 = pt_1$, $Z_2 = pt_2$, $Z_3 = pt_3$, $Z_4 \simeq \mathbb{A}^1$, $Z_5 = T^\vee/W^{\mathfrak{s}}$.

In the following table, the first column comprises 5 irreducible components X_1, …, X_5 of $\widetilde{T^\vee}$ for which $\rho^{\mathfrak{s}}(X_j) = Z_j$, $j = 1,\dots,5$. Let $[z_1,z_2]$ denote the image of (z_1,z_2) via the standard quotient map $p^{\mathfrak{s}}\colon T^\vee \to T^\vee/W^{\mathfrak{s}}$, so that $[z_1,z_2] = W^{\mathfrak{s}}\cdot(z_1,z_2)$. When the first column itemizes the pairs $(w,t)\in\widetilde{T^\vee}$, the second column itemizes $p^{\mathfrak{s}}(h_j(\tau)t)$. The third column itemizes the corresponding correcting cocharacters.

$X_1 = ((ab)^2,(1,1))$	$[1,\tau^{-2}1]$	$h_1(\tau) = (1,\tau^{-2})$
$X_2 = ((ab)^2,(j,j^2))$	$[j,\tau^{-2}j^2]$	$h_2(\tau) = (1,\tau^{-2})$
$X_3 = ((ab)^2,(j^2,j))$	$[j^2,\tau^{-2}j]$	$h_3(\tau) = (1,\tau^{-2})$
$X_4 = \{(a,(z,z)) : z\in\mathbb{C}^\times\}$	$\{[z,\tau^{-2}z] : z\in\mathbb{C}^\times\}$	$h_4(\tau) = (1,\tau^{-2})$
$X_5 = \{(e,(z_1,z_2)) : z_1,z_2\in\mathbb{C}^\times\}$	$\{[z_1,z_2] : z_1,z_2\in\mathbb{C}^\times\}$	$h_5(\tau) = 1$

It is now clear that cocharacters can assigned to two-sided cells as follows: $h_{\mathbf{c}}(\tau) = (1,\tau^{-2})$ if $\mathbf{d} = \mathbf{d}_1, \mathbf{d}_e$, and $h_{\mathbf{d}_0} = 1$.

The map $\pi_{\sqrt{q}}$ sends the two distinct points $(1/\sqrt{q}, 1/\sqrt{q})$ and $(\sqrt{q},\sqrt{q})$ in $D^a/\mathrm{Z}_{W^{\mathfrak{s}}}(a)$, the affine line attached to the cell $\mathbf{d}_1$, to the one point $y_a \in D/W^{\mathfrak{s}}$ since $(1,q^{-1})$, $(q,1)$ are in the same $W^{\mathfrak{s}}$-orbit: $(1,q^{-1}) \cong_{baba} (q,1)$.

We have

$$\pi^{-1}_{\sqrt{q}}(y_a) = \left\{\rho^{\mathfrak{s}}(e,(1,q^{-1})), \rho^{\mathfrak{s}}((ab)^2,(1,1)), \rho^{\mathfrak{s}}(a,(1,1)), \rho^{\mathfrak{s}}(a,(q,q))\right\}.$$

It follows that

$$|\pi^{-1}_{\sqrt{q}}(y_a)| = 4 = |i^G_{T\subset B}(y_a)|.$$

Similarly we obtain that $|\pi^{-1}_{\sqrt{q}}(y'_a)| = |\pi^{-1}_{\sqrt{q}}(y''_a| = 4 = |i^G_{T\subset B}(y''_a)| = |i^G_{T\subset B}(y'_a)|$. $\square$

Lemma 7.4. *Part (4) of Theorem* 1.5 *is true for the point*

$$\mathfrak{s} = [T,\chi\otimes\chi]_G \in \mathfrak{B}(G_2)$$

when χ is a cubic ramified character of $F^\times$.

Proof. The semisimple elements y, σ below are always related as follows:

$$\sigma = \pi_{\sqrt{q}}(y).$$

Let $\eta^{\mathfrak{s}} \colon (T^\vee//W^{\mathfrak{s}}) \to \mathrm{Irr}(J^{\mathfrak{s}})$ be the bijection which is induced by the Morita equivalences in (51). Then the definition (13) of $\mu^{\mathfrak{s}} \colon (T^\vee//W^{\mathfrak{s}}) \to \mathrm{Irr}(G)^{\mathfrak{s}}$ gives:

$$\mu^{\mathfrak{s}}(y) = \begin{cases} \mathcal{V}^{\mathfrak{s}}_{\sigma,1,1}\,, & \text{if } y \in T^\vee/W^{\mathfrak{s}}; \\ \mathcal{V}^{\mathfrak{s}}_{\sigma,u_1,1}\,, & \text{if } y \in (T^\vee/W^{\mathfrak{s}})_{\mathbf{d}_1}; \end{cases}$$

the three isolated points are sent to the elements in the discrete series which have inertial support $\mathfrak{s}$:

$$\mu^{\mathfrak{s}}(pt_1) = \mathcal{V}^{\mathfrak{s}}_{y_a,u_e,1}, \quad \mu^{\mathfrak{s}}(pt_2) = \mathcal{V}^{\mathfrak{s}}_{y'_a,u_e,1}, \quad \mu^{\mathfrak{s}}(pt_3) = \mathcal{V}^{\mathfrak{s}}_{y''_a,u_e,1}.$$

Now the infinitesimal character of $\mathcal{V}^{\mathfrak{s}}_{\sigma,u,\rho}$ is σ, therefore the map $\mu^{\mathfrak{s}}$ satisfies

$$inf.ch. \circ \mu^{\mathfrak{s}} = \pi_{\sqrt{q}}. \qquad \square$$

Lemma 7.5. *Part (5) of Theorem* 1.5 *is true in this case.*

Proof. It follows from [21, Prop. 1.1, p. 469] that $\delta(\chi)$ (viewed as a representation of M_α) is the unique subrepresentation of $I^\alpha(\nu^{1/2}\chi \otimes \nu^{-1/2}\chi)$. So it has inertial support $[T, \nu^{1/2}\chi \otimes \nu^{-1/2}\chi]_{M_\alpha}$. It implies that $I_\alpha(0, \delta(\chi))$ has inertial support $[T, \nu^{1/2}\chi \otimes \nu^{-1/2}\chi]_G = [T, \chi \otimes \chi]_G = \mathfrak{s}$.

If we look at M_β, still from [21, Prop. 1.1, p. 469], we see that $\delta(\chi)$ (here viewed as a representation of M_β) is the unique subrepresentation of $I^\beta(\nu^{-1/2}\chi \otimes \nu)$. Hence it has inertial support $[T, \nu^{-1/2}\chi \otimes \nu]_{M_\beta}$. It follows that $I_\beta(0, \delta(\chi))$ has inertial support $[T, \nu^{-1/2}\chi \otimes \nu]_G = [T, \chi \otimes 1]_G$, which is not equal to $\mathfrak{s}$, because $\chi \otimes 1$ does not belong to the W-orbit of $\chi \otimes \chi$ (see also Proposition 5.4).

Hence the compact extended quotient is accounted for as follows: The compact quotient E/W is sent to the unitary principal series

$$\{I(\psi_1\chi \otimes \psi_2\chi : \psi_1, \psi_2 \in \Psi(F^\times)\}/W^{\mathfrak{s}},$$

the component $U(1)$ to the intermediate unitary series

$$\{I_\alpha(0, \delta(\psi\chi)) : \psi \in \Psi^{\mathrm{t}}(F^\times)\},$$

and the 3 isolated points pt_1, pt_2, pt_3 are sent to the 3 elements in the discrete series

$$\pi(\chi) \subset I(\nu\chi \otimes \chi), \quad \pi(\omega\chi) \subset I(\nu\omega\chi \otimes \omega\chi), \quad \pi(\omega^2\chi) \subset I(\nu\omega^2\chi \otimes \omega^2\chi). \qquad \square$$

8. The case $H^{\mathfrak{s}} = \mathrm{SO}(4, \mathbb{C})$

We assume in this section that $\chi_1 = \chi_2 = \chi$ with χ a ramified quadratic character. It follows from (31) that

$$\mathfrak{s} = [T, \chi \otimes \chi]_G = [T, \chi \otimes 1]_G = [T, 1 \otimes \chi]_G.$$

From (28), we get

$$\begin{aligned} W^{\mathfrak{s}} &= \{e, a, babab, bababa\} = \left\{e, a, r^3, ar^3\right\} \\ &= \langle s_\alpha, s_{3\alpha+2\beta}\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \end{aligned} \tag{52}$$

We recall that $Q^{\mathfrak{s}} \subset X(T)$ denotes the root lattice of $\Phi^{\mathfrak{s}}$. We have

$$Q^{\mathfrak{s}} = \mathbb{Z}\alpha \oplus \mathbb{Z}(3\alpha + 2\beta). \tag{53}$$

Hence, $Q^{\mathfrak{s}}$ is strictly contained in $X(T)$, see (26). This shows that the group $H^{\mathfrak{s}}$ is not simply connected. Setting $V := Q^{\mathfrak{s}} \otimes_{\mathbb{Z}} \mathbb{Q}$, we define the weight lattice $P^{\mathfrak{s}}$, as in [8, Chap. VI, 1.9], by

$$P^{\mathfrak{s}} := \{x \in V \ :< x, \gamma > \in \mathbb{Z}, \, \forall \gamma \in \Phi^{\mathfrak{s}\vee}\}\,.$$

We have

$$P^{\mathfrak{s}} = \frac{1}{2}\mathbb{Z}\alpha \oplus \frac{1}{2}\mathbb{Z}(3\alpha + 2\beta).$$

Hence, $X(T)$ is strictly contained in $P^{\mathfrak{s}}$. This shows that the $H^{\mathfrak{s}}$ is not of adjoint type. Now, let X_0 denote the subgroup of $X(T)$ orthogonal to $\Phi^{\mathfrak{s}\vee}$. We see that $X_0 = \{0\}$. This means that the group $H^{\mathfrak{s}}$ is semisimple. Hence, $H^{\mathfrak{s}}$ is isomorphic to $\mathrm{SO}(4, \mathbb{C})$. The group $\mathrm{SO}(4, \mathbb{C})$ is isomorphic to the quotient group $(\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}))/\langle -\mathrm{I}, -\mathrm{I}\rangle$, where I is the identity in $\mathrm{SL}(2, \mathbb{C})$.

The group $H^{\mathfrak{s}}$ admits 4 unipotent classes $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_1'$, $\mathbf{e}_e$. The closure order on unipotent classes is the following:

$$\begin{array}{ccc} & \mathbf{e}_e & \\ \diagup & & \diagdown \\ \mathbf{e}_1 & & \mathbf{e}_1' \\ \diagdown & & \diagup \\ & \mathbf{e}_0 & \end{array}\,.$$

We have

$$J^{\mathfrak{s}} = J^{\mathfrak{s}}_{\mathbf{e}_e} \oplus J^{\mathfrak{s}}_{\mathbf{e}_1} \oplus J^{\mathfrak{s}}_{\mathbf{e}_1'} \oplus J^{\mathfrak{s}}_{\mathbf{e}_0}.$$

Definition. We define the following partition of $T^\vee/\!/W^{\mathfrak{s}}$:

$$T^\vee/\!/W^{\mathfrak{s}} = (T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_e} \sqcup (T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_1} \sqcup (T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_1'} \sqcup (T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_0}, \tag{54}$$

where

$$\begin{array}{rcl} (T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_e} & := & pt_1 \sqcup pt_2, \\ (T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_1} & := & (T^\vee)^a/\mathrm{Z}(a) \cong \mathbb{A}^1, \\ (T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_1'} & := & (T^\vee)^{ar^3}/\mathrm{Z}(ar^3) \cong \mathbb{A}^1, \\ (T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_0} & := & T^\vee/W \sqcup pt_*, \end{array}$$

with $pt_1 := (1,1)$, $pt_2 := (-1,-1)$ and $pt_* := (1,-1) \sim_{W^{\mathfrak{s}}} (-1,1)$.

Note. By contrast to the definitions in the previous cases, the above definition does not correspond to a partition of the conjugacy classes in W indexed by unipotent classes in $G^\vee$. Indeed, we get the following correspondence between conjugacy classes in W and unipotent classes in $G^\vee$:

- (a) corresponds to the unipotent class in $G^\vee$ corresponding to $\mathbf{e}_1$,
- (ar^3) corresponds to the unipotent class in $G^\vee$ corresponding to $\mathbf{e}_1'$,
- (e) corresponds to the class in $G^\vee$ corresponding to $\mathbf{e}_0$,
- (r^3) corresponds to **both** the unipotent class in $G^\vee$ corresponding to $\mathbf{e}_e$ and the unipotent class in $G^\vee$ corresponding to $\mathbf{e}_0$.

Lemma 8.1. *We have*

$$\begin{array}{rcl} T^\vee/\!/W^{\mathfrak{s}} & = & (pt_1 \sqcup pt_2) \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^1 \sqcup (T^\vee/W^{\mathfrak{s}} \sqcup pt_*), \\ E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} & = & (pt_1 \sqcup pt_2) \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup (E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup pt_*). \end{array}$$

Moreover, we have a ring isomorphism

$$\mathbb{C}[T^\vee/W^{\mathfrak{s}}] \sim \mathbb{C}[X, Y]_0,$$

where $\mathbb{C}[X,Y]_0$ denotes the coordinate ring of the quotient of $\mathbb{A}^2$ by the action of $\mathbb{Z}/2\mathbb{Z}$ which reverses each vector.

We have $J^{\mathfrak{s}} \asymp \mathcal{O}(T^\vee/\!/W^{\mathfrak{s}})$, where

$$J^{\mathfrak{s}}_{\mathbf{e}_e} \sim_{\text{morita}} \mathcal{O}((T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_e}), \quad J^{\mathfrak{s}}_{\mathbf{e}_1} \sim_{\text{morita}} \mathcal{O}((T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_1}),$$
$$J^{\mathfrak{s}}_{\mathbf{e}'_1} \sim_{\text{morita}} \mathcal{O}((T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}'_1}), \qquad J^{\mathfrak{s}}_{\mathbf{e}_0} \asymp \mathcal{O}((T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_0}).$$

Proof. 1. Extended quotient: Let $D = D^{\mathfrak{s}} \cong T^\vee$. We give the case-by-case analysis.

- $c = e$. $D^c/\mathrm{Z}(c) = D/W^{\mathfrak{s}}$.
- $c = a$. $D^c = \{(t,t) : t \in \mathbb{C}^\times\}$.
$$D^c/\mathrm{Z}(c) = \{\{(t,t),(t^{-1},t^{-1})\} : t \in \mathbb{C}^\times\} \cong \mathbb{A}^1.$$
- $c = r^3$. $D^c = \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. Therefore, $D^c/\mathrm{Z}(c) = pt_1 \sqcup pt_2 \sqcup pt_*$.
- $c = ar^3$. $D^c = \{(t,t^{-1}) : t \in \mathbb{C}^\times\}$.
$$D^c/\mathrm{Z}(c) = \{\{(t,t^{-1}),(t^{-1},t)\} : t \in \mathbb{C}^\times\} \cong \mathbb{A}^1.$$

Let $\mathbb{M}[u] := \mathbb{C}[u,u^{-1}]$ denote the $\mathbb{Z}/2\mathbb{Z}$-graded algebra of Laurent polynomials in one indeterminate u. Let a denote the generator of $\mathbb{Z}/2\mathbb{Z}$. The group $\mathbb{Z}/2\mathbb{Z}$ acts as automorphism of $\mathbb{M}[u]$, with $a(u) = u^{-1}$. We define

$$\mathbb{L}[u] := \{P \in \mathbb{M}[u] \, : \, a(P) = P\}$$

as the algebra of balanced Laurent polynomials in u.

Let $\bar{T}^\vee$ be the maximal torus of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. Then the coordinate ring $\mathbb{C}[\bar{T}^\vee/W^{\mathfrak{s}}]$ is $\mathbb{L}[u] \otimes \mathbb{L}[v]$. The map

$$(u + 1/u, v + 1/v) \mapsto (X,Y)$$

sends $\mathbb{C}[\bar{T}^\vee/W^{\mathfrak{s}}]$ to $\mathbb{C}[X,Y]$ (the polynomial algebra in two indeterminates X, Y), a ring isomorphism.

Recall that $T^\vee$ is the standard maximal torus in $H^{\mathfrak{s}} \cong (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))/\langle -\mathrm{I}, -\mathrm{I}\rangle$. Hence it follows that $\mathbb{C}[T^\vee/W^{\mathfrak{s}}]$ is the ring of balanced polynomials in u, v which are fixed under $(u,v) \mapsto (-u,-v)$. These polynomials correspond to those polynomials in X, Y which are fixed under $(X,Y) \mapsto (-X,-Y)$. Therefore, we have a ring isomorphism $\mathbb{C}[T^\vee/W^{\mathfrak{s}}] \sim \mathbb{C}[X,Y]_0$.

2. Compact extended quotient:

$$E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup pt \sqcup pt \sqcup pt = E^{\mathfrak{s}}/\!/W^{\mathfrak{s}}.$$

The group $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ (see (3)) is the extended affine Weyl group of the p-adic group $(\mathrm{SL}(2,F) \times \mathrm{SL}(2,F))/\langle -\mathrm{I}, -\mathrm{I}\rangle$, which admits $H^{\mathfrak{s}}$ as Langlands dual.

3. Extended affine Weyl group:

We will describe the group $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$. Let $\bar{X}$ be the cocharacter group of a maximal torus of $\mathrm{PGL}(2,F) \times \mathrm{PGL}(2,F)$, and let $\bar{W}^{\mathfrak{s}}_{\mathrm{a}} := W^{\mathfrak{s}} \ltimes \bar{X}$. Then $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ is a subgroup of $\bar{W}^{\mathfrak{s}}_{\mathrm{a}}$.

Let W_2 be the extended affine Weyl group corresponding to $\mathrm{PGL}(2,F)$, that is, $W_2 = \mathbb{Z}/2\mathbb{Z} \ltimes X_2$, where X_2 is the cocharacter group of a maximal torus of $\mathrm{PGL}(2,F)$ and $\mathbb{Z}/2\mathbb{Z} = \{e,a\}$. Let $t \neq a$ be the other simple reflection in W_2. In W_2 there exists a unique element g of order 2 such that $gag = t$. It is known that the length of g is 0. Let W'_2 be a copy of W_2. The simple reflections in W'_2 will be denoted by $a' = ar^3$, t' correspondingly. Denote by g' the element corresponding to g, then $(g')^2 = e$ and $g'a'g' = t'$. Then $\bar{W}^{\mathfrak{s}}_{\mathrm{a}} = W_2 \times W'_2$ and $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ is the subgroup of $\bar{W}^{\mathfrak{s}}_{\mathrm{a}}$ generated by gg', a, t, a', t'.

4. Asymptotic Hecke algebra: Since $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ is a subgroup of $\bar{W}^{\mathfrak{s}}_{\mathrm{a}}$, its based ring of $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ can be described as a subring of the based ring of the latter.

• From the above description, we see that the two-sided cell $\mathbf{e}_e$ of $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ consists of e and gg' and that its based ring is isomorphic to the group algebra of $\mathbb{Z}/2\mathbb{Z}$. It follows that $J_{\mathbf{e}_e}$ is Morita equivalent to $\mathbb{C} \oplus \mathbb{C}$. Hence we have

$$J^{\mathfrak{s}}_{\mathbf{e}_e} \sim_{\mathrm{morita}} \mathcal{O}((T^\vee//W^{\mathfrak{s}})_{\mathbf{e}_e}). \tag{55}$$

• Let U_2 be the subgroup of $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ generated by gg', a, t, then the map which sends gg' to g, a to a, t to t defines an isomorphism from U_2 to W_2. Thus $\mathbf{e}_1$ is the lowest two-sided cell of U_2, which equals $U_2 - \{e, gg'\}$. Therefore, the based ring of $\mathbf{e}_1$ is isomorphic to $\mathrm{M}_2(R)$, where R denotes the representation ring of $\mathrm{SL}(2,\mathbb{C})$. Hence the based ring of $\mathbf{e}_1$ is clearly Morita equivalent to R, but R is isomorphic to the polynomial ring $\mathbb{Z}[u]$ in one variable. It follows that $J_{\mathbf{e}_1}$ is Morita equivalent to $\mathbb{C}[u]$, where u is an indeterminate. From the first part of the lemma, we get that

$$J^{\mathfrak{s}}_{\mathbf{e}_1} \sim_{\mathrm{morita}} \mathcal{O}((T^\vee//W^{\mathfrak{s}})_{\mathbf{e}_1}). \tag{56}$$

• Similarly, let U'_2 be the subgroup of $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ generated by gg', a', t', then the map which sends gg' to g, a' to a, t' to t defines an isomorphism from U'_2 to W_2. Thus $\mathbf{e}'_1$ is the lowest two-sided cell of U'_2, which equals $U'_2 - \{e, gg'\}$. So the based ring of $\mathbf{e}'_1$ is isomorphic to $\mathrm{M}_2(R)$, which is also Morita equivalent to the polynomial ring $\mathbb{Z}[v]$ in one variable. From the first part of the lemma, we obtain that

$$J^{\mathfrak{s}}_{\mathbf{e}'_1} \sim_{\mathrm{morita}} \mathcal{O}((T^\vee//W^{\mathfrak{s}})_{\mathbf{e}'_1}). \tag{57}$$

• Let x denote the fundamental cocharacter of X_2 and write the operation in X_2 by multiplication. For the fundamental cocharacter x, we still use x if it is regarded as an element in W_2 and denote it by x' if it is regarded as an element in W'_2. Let a be the nonunit element of $\mathbb{Z}/2\mathbb{Z}$. For the element a, we still use a if it is regarded as an element in W_2 and denote it by a' if it is regarded as an element in W'_2. In this way $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ is the subgroup of $\bar{W}^{\mathfrak{s}}_{\mathrm{a}}$ which consists of the elements $(a^m x^i)(a'^n x'^j)$ with $m+n$ even and $i, j \in \{0, 1\}$.

The lowest two-sided cell $\mathbf{e}_0$ of $\widetilde{W}^{\mathfrak{s}}_{\mathrm{a}}$ consists of the 16 elements which are listed in the first column of Table 2. (Removing the restriction on $m+n$, these elements form the lowest two-sided cell $\bar{\mathbf{e}}_0$ of $\bar{W}^{\mathfrak{s}}_{\mathrm{a}}$).

Let $V(\ell)$ be the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of highest weight ℓ and let $V_{ij}(\ell)$ be the element in $\mathrm{M}_2(R)$ whose (i,j) entry is $V(\ell)$ and other entries are 0. Then it follows from [28, Theorem 1.10 and its proof in §4.4] that the based ring of $\bar{\mathbf{e}}_0$ is isomorphic to $\mathrm{M}_2(R) \otimes \mathrm{M}_2(R)$, and that the element in $\mathrm{M}_2(R) \otimes \mathrm{M}_2(R)$, denoted by V_w, corresponding to t_w, for $w \in \mathbf{e}_0$, is given by Table 2.

Then the based ring of $\mathbf{e}_0$ is isomorphic to the subring of $\mathrm{M}_2(R) \otimes_{\mathbb{Z}} \mathrm{M}_2(R)$ spanned by the elements $(\tau_{ij} V(m_{ij})) \otimes (\tau'_{kl} V(n_{kl}))$, with condition $m_{ij} + n_{kl} + i + j + k + l$ is even, where all τ_{ij} and τ'_{kl} are integers. Hence, $J_{\mathbf{e}_0}$ is isomorphic to the subring of $\mathrm{M}_2(\mathbb{C}[X]) \otimes_{\mathbb{C}} \mathrm{M}_2(\mathbb{C}[Y])$ spanned by the elements $(\tau_{ij} z^{m_{ij}}) \otimes (\tau'_{kl} z^{n_{kl}})$, with condition $m_{ij} + n_{kl} + i + j + k + l$ is even, where all τ_{ij} and τ'_{kl} are integers.

The above parity condition is: $m_{ij} + b_{kl} + i + j + k + l$ even. For example, $m_{11} + n_{11}$ is even, $m_{22} + n_{11}$ is even, $m_{12} + n_{11}$ is odd, $m_{21} + n_{11}$ is odd, $m_{12} + n_{12}$ is even, $m_{21} + n_{21}$ is even, $m_{22} + n_{22}$ is even; so we are allowed monomials of even degree

TABLE 2. The map $w \mapsto V_w$.

$w \in \mathbf{e}_0$	V_w
$(ax^m)(a'x'^n)$, $m, n \geq 0$, $m+n$ even	$V_{11}(m) \otimes V_{11}(n)$
$(ax^m)(x'^n)$, $m \geq 0$, $n \geq 1$, $m+n$ even	$V_{11}(m) \otimes V_{21}(n-1)$
$(ax^m)(a'x'^n)$, $m \geq 0$, $n \leq -2$, $m+n$ even	$V_{11}(m) \otimes V_{22}(-n-2)$
$(ax^m)(x'^n)$, $m \geq 0$, $n \leq -1$, $m+n$ even	$V_{11}(m) \otimes V_{12}(-n-1)$
$(x^m)(a'x'^n)$, $m \geq 1$, $n \geq 0$, $m+n$ even	$V_{21}(m-1) \otimes V_{11}(n)$
$(x^m)(x'^n)$, $m \geq 1$, $n \geq 1$, $m+n$ even	$V_{21}(m-1) \otimes V_{21}(n-1)$
$(x^m)(a'x'^n)$, $m \geq 1$, $n \leq -2$, $m+n$ even	$V_{21}(m-1) \otimes V_{22}(-n-2)$
$(x^m)(x'^n)$, $m \geq 1$, $n \leq -1$, $m+n$ even	$V_{21}(m-1) \otimes V_{12}(-n-1)$
$(ax^m)(a'x'^n)$, $m \leq -2$, $n \geq 0$, $m+n$ even	$V_{22}(-m-2) \otimes V_{11}(n)$
$(ax^m)(x'^n)$, $m \leq -2$, $n \geq 1$, $m+n$ even	$V_{22}(-m-2) \otimes V_{21}(n-1)$
$(ax^m)(a'x'^n)$, $m \leq -2$, $n \leq -2$, $m+n$ even	$V_{22}(-m-2) \otimes V_{22}(-n-2)$
$(ax^m)(x'^n)$, $m \leq -2$, $n \leq -1$, $m+n$ even	$V_{22}(-m-2) \otimes V_{12}(-n-1)$
$(x^m)(a'x'^n)$, $m \leq -1$, $n \geq 0$, $m+n$ even	$V_{12}(-m-1) \otimes V_{11}(n)$
$(x^m)(x^n)$, $m \leq -1$, $n \geq 1$, $m+n$ even	$V_{12}(-m-1) \otimes V_{21}(n-1)$
$(x^m)(a'x'^n)$, $m \leq -1$, $n \leq -2$, $m+n$ even	$V_{12}(-m-1) \otimes V_{22}(-n-2)$
$(x^m)(x'^n)$, $m \leq -1$, $n \leq -1$, $m+n$ even	$V_{12}(-m-1) \otimes V_{12}(-n-1)$

on the diagonals of both matrices, monomials of odd degree on the off-diagonals of both matrices OR the same thing with reversed parity. Taking the span, we realize all even polynomials on the diagonal, all odd polynomials on the off-diagonal in both matrices OR the same thing with reversed parity.

In other words, the ring $\mathrm{M}_2(\mathbb{C}[X])$ is $\mathbb{Z}_2$-graded:

- $(\mathrm{M}_2(\mathbb{C}[X]))_0$: $=$ even polynomials on the diagonal, odd polynomials on the off-diagonal;
- $(\mathrm{M}_2(\mathbb{C}[X]))_1$: $=$ odd polynomials on the diagonal, even polynomials on the off-diagonal.

Consider the $\mathbb{Z}_2$-graded tensor product $\mathbb{B}[X, Y] := \mathrm{M}_2(\mathbb{C}[X]) \otimes_{\mathbb{C}} \mathrm{M}_2(\mathbb{C}[Y])$. Then $J_{\mathbf{e}_0}$ is isomorphic to the even part $\mathbb{B}[X, Y]_0$ of $\mathbb{B}[X, Y]$.

Give $\mathbb{C}[X, Y]$ a $\mathbb{Z}_2$-grading by the convention that a monomial $X^m Y^n$ is even (odd) according to the parity of $m + n$. Form the algebra $\mathrm{M}_4(\mathbb{C}[X, Y])$. Give this a $\mathbb{Z}_2$-grading by saying that the even (resp. odd) elements are those which have a 2×2-block in the upper left corner consisting of even (resp. odd) polynomials, a 2×2-block in the lower right corner consisting of even (resp. odd) polynomials, a 2×2-block in the lower left corner consisting of odd (resp. even) polynomials, a 2×2-block in the upper right corner consisting of odd (resp. even) polynomials.

Then the even part of $\mathrm{M}_2(\mathbb{C}[X]) \otimes \mathrm{M}_2(\mathbb{C}[Y])$ is isomorphic to the even part of $\mathrm{M}_4(\mathbb{C}[X, Y])$; i.e., as $\mathbb{Z}_2$-graded algebras $\mathrm{M}_2(\mathbb{C}[X]) \otimes \mathrm{M}_2(\mathbb{C}[Y])$ and $\mathrm{M}_4(\mathbb{C}[X, Y])$ are isomorphic.

Let $\mathrm{M}_4(\mathbb{C}[X, Y])_0$ consist of all 4×4 matrices with entries in $\mathbb{C}[X, Y]$ such that: the upper left 2×2 block and the lower right 2×2 block are 2×2 matrices with entries in $\mathbb{C}[X, Y]_0$ and the lower left 2×2 block and the upper right 2×2 block are 2×2 matrices with entries in $\mathbb{C}[X, Y]_1$. Let $\bar{P}(X, Y) = (P_{i,j}(X, Y))_{1 \leq i,j \leq 4}$ be an element of $\mathrm{M}_4(\mathbb{C}[X,Y])$. We write $\bar{P}(X,Y)$ as a 2×2 block matrix as

$$\bar{P}(X,Y) = \begin{pmatrix} \bar{P}(X,Y)_{1,1} & \bar{P}(X,Y)_{1,2} \\ \bar{P}(X,Y)_{2,1} & \bar{P}(X,Y)_{2,2} \end{pmatrix},$$

where $\bar{P}(X,Y)_{i,j} \in \mathrm{M}_2(\mathbb{C}[X,Y])$, for $i,j \in \{1,2\}$. Hence the matrix $\bar{P}(X,Y)$ is in $\mathrm{M}_4(\mathbb{C}[X,Y])_0$ if and only if we have, for $i,j \in \{1,2\}$,

$$\bar{P}(X,Y)_{i,i} \in \mathrm{M}_2(\mathbb{C}[X,Y]_0) \text{ and } \bar{P}(X,Y)_{i,j} \in \mathrm{M}_2(\mathbb{C}[X,Y]_1) \text{ if } i \neq j.$$

If (z,z') is a pair of complex numbers, then evaluation at (z,z') gives an algebra homomorphism

$$\begin{aligned} \mathrm{ev}_{(z,z')}\colon \quad \mathrm{M}_4(\mathbb{C}[X,Y])_0 &\longrightarrow \mathrm{M}_4(\mathbb{C}) \\ \bar{P}(X,Y) &\mapsto \bar{P}(z,z'). \end{aligned}$$

The algebra homomorphism $\mathrm{ev}_{(z,z')}$ is surjective except when $(z,b)=(0,0)$. In the case $(z,z')=(0,0)$ the image of $\mathrm{ev}_{(z,z')}$ is the subalgebra of $\mathrm{M}_4(\mathbb{C})$ consisting of all 4×4 matrices of complex numbers such that the lower left 2×2 block and the upper right 2×2 block are zero; i.e., the image is $\mathrm{M}_2(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C})$ embedded in the usual way in $\mathrm{M}_4(\mathbb{C})$. So, except for $(z,z')=(0,0)$ we have a simple module, say $M_{(z,z')}$. When $(z,z')=(0,0)$ we have a module which is the direct sum of two simple modules, that we denote by $M'_{(0,0)}$ and $M''_{(0,0)}$.

Since we have

$$\begin{aligned} \begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix} \bar{P}(z,z') \begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix}^{-1} &= \begin{pmatrix} \bar{P}(z,z')_{1,1} & -\bar{P}(z,z')_{1,2} \\ -\bar{P}(z,z'b)_{2,1} & \bar{P}(z,z')_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} \bar{P}(-z,-z')_{1,1} & \bar{P}(-z,-z')_{1,2} \\ \bar{P}(-z,-z')_{2,1} & \bar{P}(-z,-z')_{2,2} \end{pmatrix}, \end{aligned}$$

the matrix $\begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix}$ conjugates the simple modules $M_{(z,z')}$ and $M_{(-z,-z')}$.

On the other hand, let (z_1,z_1') be a pair of complex numbers such that $(z_1,z_1') \notin \{(z,z'),(-z,-z')\}$. Then there exists an even polynomial $Q(X,Y)$ such that $Q(z_1,z_1') \neq Q(z,z')$. Indeed,

- if $z_1 \notin \{z,-z\}$, we can take $Q(X,Y) = X^2$,
- if $z_1 = z$, we can take $Q(X,Y) = XY$ (since then $z_1' \neq z'$),
- if $z_1 = -z$, we can take $Q(X,Y) = XY$ (since then $z_1' \neq -z'$).

Consider the matrix

$$\bar{Q}(X,Y) := \begin{pmatrix} Q(X,Y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathrm{ev}_{(z,z')}(\bar{Q}(X,Y)) \neq \mathrm{ev}_{(z_1,z_1')}(\bar{Q}(X,Y))$. It follows that the simple modules $M_{(z,z')}$ and $M_{(z_1,z_1')}$ are not isomorphic.

Let $A = \mathrm{M}_4(\mathbb{C}[X,Y])$ and let $\Gamma := \{1,\varepsilon\}$, where $\varepsilon\colon A \to A$ is defined by

$$\varepsilon(\bar{P}(X,Y)) = \begin{pmatrix} \bar{P}(-X,-Y)_{1,1} & -\bar{P}(-X,-Y)_{1,2} \\ -\bar{P}(-X,-Y)_{2,1} & \bar{P}(-X,-Y)_{2,2} \end{pmatrix}.$$

From Lemma 5.6, we know that the unital $\mathbb{C}$-algebras A^Γ and $(A \rtimes \Gamma)e_\Gamma(A \rtimes \Gamma)$ are Morita equivalent. Here $e_\Gamma = \frac{1}{2}(1+\varepsilon)$.

We embed A into the crossed product algebra $A \rtimes \Gamma$ by sending $\bar{P}(X,Y)$ to $\bar{P}(X,Y)[1]$. For $1 \le i,j \le 4$, let $E_{i,j} \in A$ be the matrix with entry (i,j) equal to 1 and all the other entries equal to 0. We have

$$E_{3,1}([1]+[\varepsilon]) = E_{3,1}[1] + E_{3,1}[\varepsilon] \text{ and } ([1]+[\varepsilon])E_{3,1} = E_{3,1}[1] - E_{3,1}[\varepsilon].$$

We get

$$E_{3,1}[1] = (\frac{1}{2}E_{3,1})([1]+[\varepsilon]) + ([1]+[\varepsilon])(\frac{1}{2}E_{3,1}),$$

it follows that $E_{3,1}[1]$ belongs to the two-sided ideal $([1]+[\varepsilon])$.

Since $E_{i,1}[1] = E_{i,3}(E_{3,1}[1])$, we get that $E_{i,1}[1] \in ([1]+[\varepsilon])$ for each i. Since $E_{i,j}[1] = (E_{i,1}[1])E_{1,j}$, then we get that $E_{i,j}[1] \in ([1]+[\varepsilon])$ for any i,j. Hence, we have proved that

$$(A \rtimes \Gamma)e_\Gamma(A \rtimes \Gamma) = A \rtimes \Gamma.$$

It follows that the unital $\mathbb{C}$-algebras A^Γ and $A \rtimes \Gamma$ are Morita equivalent.

Thus we have proved that for $\mathrm{M}_4(\mathbb{C}[X,Y])_0 = A^\Gamma$ the $M_{z,z'}$ with $(z,z') \neq (0,0)$, and $M'_{(0,0)}$, $M''_{(0,0)}$ are (up to isomorphism) all the simple modules and that they are distinct except that $M_{(z,z')}$ and $M_{(-z,-z')}$ are isomorphic.

Let $\mathcal{J}$ be the ideal in $\mathrm{M}_4(\mathbb{C}[X,Y])_0$ which (by definition) is the pre-image with respect to $\mathrm{ev}_{(0,0)}$ of $\mathrm{M}_2(\mathbb{C}) \oplus \{0\}$. Then $\mathrm{ev}_{(z,z')}$ surjects $\mathcal{J}$ onto $\mathrm{M}_4(\mathbb{C})$ except at $(z,z') = (0,0)$, and $\mathrm{ev}_{(0,0)}$ surjects $\mathcal{J}$ onto $\mathrm{M}_2(\mathbb{C}) \oplus \{0\}$. In $\mathbb{C}[X,Y]_0 \oplus \mathbb{C}$ let $\mathcal{I}$ be the ideal $\mathbb{C}[X,Y]_0 \oplus \{0\}$. Consider the two filtrations

$$\{0\} \subset \mathcal{J} \subset \mathrm{M}_4(\mathbb{C}[X,Y])_0,$$
$$\{0\} \subset \mathcal{I} \subset \mathbb{C}[X,Y]_0 \oplus \mathbb{C}.$$

Let δ_0 be the algebra homomorphism

$$\begin{aligned} \delta_0\colon \qquad \mathbb{C}[X,Y]_0 \oplus \mathbb{C} &\longrightarrow \mathrm{M}_4(\mathbb{C}[X,Y])_0, \\ (P(X,Y),z) &\mapsto \begin{pmatrix} P(X,Y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where the complex number z is viewed as an even polynomial of degree zero.

We view the ideal $\mathcal{J}$ as an algebra. We recall that $k = \mathbb{C}[X,Y]_0$. Then k is a unital finitely generated free commutative algebra. Hence, k is the coordinate algebra of an affine variety, the variety $\mathbb{C}^2/(z,z') \sim (-z,-z')$, thus k is noetherian. It follows that $B := \mathrm{M}_4(\mathbb{C}[X,Y]_0)$ is a k-algebra of finite type. This implies that $\mathcal{J}$ is a k-algebra of finite type. Therefore, any simple $\mathcal{J}$-module, as a vector space over $\mathbb{C}$, is finitely dimensional.

Hence any simple $\mathcal{J}$-module gives a surjection

$$\mathcal{J} \twoheadrightarrow \mathrm{M}_n(\mathbb{C}),$$

and, by using Lemma 5.7, extends uniquely to a simple B-module.

It follows that δ_0 is spectrum preserving with respect to these filtrations.

This proves that $J^{\mathfrak{s}}_{\mathbf{e}_0}$ is geometrically equivalent to $\mathbb{C}[X,Y]_0 \oplus \mathbb{C}$. This is the coordinate ring of the extended quotient $(\mathbb{A}^2)/\!/(\mathbb{Z}/2\mathbb{Z})$, see the first part of the lemma. Hence we have

$$J^{\mathfrak{s}}_{\mathbf{e}_0} \asymp \mathcal{O}((T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_0}). \tag{58}$$

□

Note. When the complex reductive group is simply-connected, we show in [2, Theorem 4] that we have

$$J_{\mathbf{c}_0} \asymp \mathcal{O}(T^\vee/W)$$

where $\mathbf{c}_0$ is the lowest two-sided cell. This geometrical equivalence is a Morita equivalence. This result depends on results of Lusztig and Xi. The above phenomenon, namely

$$J^{\mathfrak{s}}_{\mathbf{e}_o} \asymp \mathcal{O}(T^\vee/W^{\mathfrak{s}}) \oplus \mathbb{C}$$

where $\mathbf{e}_0$ is the lowest two-sided cell, is a consequence of the fact that $H^{\mathfrak{s}}$ is not simply-connected. This geometrical equivalence is not a Morita equivalence, nor is it spectrum-preserving. It is spectrum-preserving with respect to a filtration of length 2.

The algebraic variety $(T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_0}$ has two irreducible components, the primitive ideal space of $J^{\mathfrak{s}}_{\mathbf{e}_0}$ does not. Hence the bijection $\mu^{\mathfrak{s}}$ is not a homeomorphism. This implies, in particular, by using [4, Theorem 2], that there cannot exist a spectrum preserving morphism from $(T^\vee/\!/W^{\mathfrak{s}})_{\mathbf{e}_0}$ to $J^{\mathfrak{s}}_{\mathbf{e}_0}$.

Lemma 8.2. *The flat family is given by*

$$\mathfrak{X}_\tau : (x - \tau^2 y)(1 - \tau^2 xy) = 0 \quad \cup \quad \{(1, -1)\}.$$

Proof. We now write down all the **nonunitary** quasicharacters of T which obey the reducibility conditions (29):

$$\psi^{-1}\chi \otimes \psi\nu\chi, \;\; \psi\chi \otimes \psi^{-1}\nu^{-1}\chi, \;\; \psi\chi \otimes \psi\nu^{-1}\chi, \;\; \psi\chi \otimes \psi\nu\chi, \quad \text{with } \psi \in \Psi(F^\times).$$

Note that

- the last two characters are in one W-orbit, namely

  $$\{\psi\chi \otimes \psi\nu\chi : \psi \in \Psi(F^\times)\}$$

  which, with the same change of variable is

  $$\{\phi\nu^{-1/2}\chi \otimes \phi\nu^{1/2}\chi : \psi \in \Psi(F^\times)\}.$$

  Since

  $$babab(\phi\nu^{-1/2}\chi \otimes \phi\nu^{1/2}\chi) = \phi^{-1}\nu^{-1/2}\chi \otimes \phi^{-1}\nu^{1/2}\chi,$$

  the induced representations

  $$\{I(\phi\nu^{-1/2}\chi \otimes \phi\nu^{1/2}\chi) : \phi \in \Psi(F^\times)\}$$

  are parametrized by an algebraic curve $\mathfrak{C}_1$. A point on $\mathfrak{C}_1$ has coordinates the unordered pair $\{z\sqrt{q}, z/\sqrt{q}\}$.
- the first two characters are in another W-orbit, namely

  $$\{\psi^{-1}\chi \otimes \psi\nu\chi : \psi \in \Psi(F^\times)\}$$

  which with the change of variable $\phi := \psi\nu^{1/2}$ is

  $$\{\phi^{-1}\nu^{1/2}\chi \otimes \phi\nu^{1/2}\chi : \phi \in \Psi(F^\times)\}.$$

  Since

  $$a(\phi^{-1}\nu^{1/2}\chi \otimes \phi\nu^{1/2}\chi) = \phi\nu^{1/2}\chi \otimes \phi^{-1}\nu^{1/2}\chi$$

  the induced representations

  $$\{I(\phi^{-1}\nu^{1/2}\chi \otimes \phi\nu^{1/2}\chi) : \phi \in \Psi(F^\times)\}$$

are parametrized by the algebraic curve $(\mathbb{C}^\times)/\mathbb{Z}/2\mathbb{Z}$. We shall refer to this curve as $\mathfrak{C}_1'$. A point on $\mathfrak{C}_1'$ has coordinates the unordered pair $\{z^{-1}/\sqrt{q}, z/\sqrt{q}\}$ with $z \in \mathbb{C}^\times$.

The coordinate ring of $\mathfrak{C}_1$ or $\mathfrak{C}_1'$ is the ring of balanced Laurent polynomials in one indeterminate t. The map $t + t^{-1} \mapsto x$ then secures an isomorphism

$$\mathbb{C}[t, t^{-1}]^{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{C}[x]$$

and so

$$\mathfrak{C}_1 \cong \mathfrak{C}_1' \cong \mathbb{A}^1,$$

the affine line.

The algebraic curves $\mathfrak{C}_1$ and $\mathfrak{C}_1'$ intersect in two points, namely

$$z_a := \chi \otimes \nu\chi = (1, q^{-1}), \qquad z_c := \epsilon\chi \otimes \nu\epsilon\chi = (-1, -q^{-1}).$$

According to the next paragraph, these are the points of length 4 and multiplicity 1.

On the other hand, we note that

$$I(\epsilon\chi \otimes \chi) := \mathrm{Ind}_{TU}^G(\epsilon\chi \otimes \chi) = \pi^+ \oplus \pi^-$$

by [14, G_2 Theorem], since $\chi, \epsilon\chi$ are distinct characters of order 2. $\square$

We define z_b and z_d as

$$z_b := \nu^{1/2}\psi^{-1}\chi \otimes \nu^{1/2}\psi\chi = (z^{-1}/\sqrt{q}, z/\sqrt{q}),$$
$$z_d := \nu^{-1/2}\psi \otimes \nu^{1/2}\psi = (z\sqrt{q}, z/\sqrt{q}).$$

Lemma 8.3. *Define, for each two-sided cell* $\mathbf{e}$ *of* $\widetilde{W}_a^{\mathfrak{s}}$*, cocharacters* $h_{\mathbf{e}}$ *as follows:*

$$h_{\mathbf{e}}(\tau) = (1, \tau^{-2}) \quad \text{if } \mathbf{e} = \mathbf{e}_e, \mathbf{e}_1, \mathbf{e}_1', \quad \text{and } h_{\mathbf{e}_0} = 1,$$

and define $\pi_\tau(x) = \pi(h_{\mathbf{e}}(\tau) \cdot x)$ *for all* x *in the* $\mathbf{e}$*-component. Then, for all* $t \in T^\vee/W^{\mathfrak{s}}$ *we have*

$$|\pi_{\sqrt{q}}^{-1}(t)| = |i_{T\subset B}^G(t)|.$$

Proof. We observe that

$$I(\nu^{1/2}\psi^{-1}\chi \otimes \nu^{1/2}\psi\chi) = I(aba(\nu^{1/2}\psi^{-1}\chi \otimes \nu^{1/2}\psi\chi)) = I(\nu^{-1/2}\psi\chi \otimes \nu).$$

Then Lemma 5.5 gives:

$$|i_{T\subset B}^G(t)| = 2 \quad \text{if } t = z_b,\ z_d,$$
$$|i_{T\subset B}^G(t)| = 4 \quad \text{if } t = z_a,\ z_c.$$

This leads to the following points of length 4:

$$\chi \otimes \nu\chi, \quad \epsilon\chi \otimes \nu\epsilon\chi, \quad \nu^{-1}\chi \otimes \chi, \quad \nu^{-1}\epsilon\chi \otimes \epsilon\chi.$$

Since

$$babab(\nu^{-1}\chi \otimes \chi) = \chi \otimes \nu\chi,$$
$$babab(\nu^{-1}\epsilon\chi \otimes \epsilon\chi) = \epsilon\chi \otimes \nu\epsilon\chi,$$

this leads to exactly 2 points in the Bernstein variety $\Omega^{\mathfrak{s}}(G)$ which parametrize representations of length 4, namely $[T, \chi\otimes\nu\chi]_G$ and $[T, \epsilon\chi\otimes\nu\epsilon\chi]_G$. The coordinates of these points in the algebraic surface $\Omega^{\mathfrak{s}}(G)$ are $(1, q^{-1})$ and $(-1, -q^{-1})$.

We will now compute a correcting system of cocharacters (see the paragraph after Theorem 1.5). The extended quotient $T^\vee/\!/W^{\mathfrak{s}}$ is a disjoint union of 6 irreducible components $Z_1, Z_2, \ldots, Z_6$. Our notation is such that $Z_1 = pt_1$, $Z_2 = pt_2$, $Z_3 \simeq \mathbb{A}^1$, $Z_4 \simeq \mathbb{A}^1$, $Z_5 = pt_*$, $Z_6 = T^\vee/W^{\mathfrak{s}}$.

In the following table, the first column comprises 6 irreducible components $X_1, \ldots, X_6$ of $\widetilde{T^\vee}$ for which $\rho^{\mathrm{i}}(X_j) = Z_j,\ j = 1, \ldots, 6$. Let $[z_1, z_2]$ denote the image of (z_1, z_2) via the standard quotient map $p^{\mathrm{i}}\colon T^\vee \to T^\vee/W^{\mathfrak{s}}$, so that $[z_1, z_2] = W^{\mathfrak{s}} \cdot (z_1, z_2)$. When the first column itemizes the pairs $(w, t) \in \widetilde{T^\vee}$, the second column itemizes $p^{\mathfrak{s}}(h_j(\tau)t)$. The third column itemizes the corresponding correcting cocharacters.

$X_1 = (r^3, (1, 1))$	$[1, \tau^{-2}]$	$h_1(\tau) = (1, \tau^{-2})$
$X_2 = (r^3, (-1, -1))$	$[-1, -\tau^{-2}]$	$h_2(\tau) = (1, \tau^{-2})$
$X_3 = \{(a, (z, z)) : z \in \mathbb{C}^\times\}$	$[z, \tau^{-2}z]$	$h_3(\tau) = (1, \tau^{-2})$
$X_4 = \{(ar^3, (z, z^{-1})) : z \in \mathbb{C}^\times\}$	$[z, \tau^{-2}z]$	$h_4(\tau) = (1, \tau^{-2})$
$X_5 = (r^3, (1, -1))$	$[1, -1]$	$h_5(\tau) = 1$
$X_6 = \{(e, (z_1, z_2)) : z_1, z_2 \in \mathbb{C}^\times\}$	$\{[z_1, z_2] : z_1, z_2 \in \mathbb{C}^\times\}$	$h_6(\tau) = 1$

It is now clear that cocharacters can be assigned to two-sided cells as follows: $h_{\mathbf{e}}(\tau) = (1, \tau^{-2})$ if $\mathbf{e} = \mathbf{e}_e, \mathbf{e}_1, \mathbf{e}_1'$, and $h_{\mathbf{e}_0} = 1$.

We have, for instance,

$$\pi_{\sqrt{q}}^{-1}(z_a) = \left\{\rho^{\mathfrak{s}}(e, (1, q^{-1})), \rho^{\mathfrak{s}}(r^3, (1, 1)), \rho^{\mathfrak{s}}(a, (1, 1)), \rho^{\mathfrak{s}}(ar^3, (1, 1))\right\},$$

$$\pi_{\sqrt{q}}^{-1}(z_c) = \left\{\rho^{\mathfrak{s}}(e, (-1, -q^{-1})), \rho^{\mathfrak{s}}(r^3, (-1, -1)), \rho^{\mathfrak{s}}(a, (-1, -1)), \rho^{\mathfrak{s}}(ar^3, (-1, -1))\right\}.$$

We have

$$|\pi_{\sqrt{q}}^{-1}(t)| = 2 \quad \text{if } t = z_b, z_d,$$
$$|\pi_{\sqrt{q}}^{-1}(t)| = 4 \quad \text{if } t = z_a, z_c.$$

Finally, we define $z_* := \epsilon\chi \otimes \chi = (-1, 1)$. Then we have

$$\pi_{\sqrt{q}}^{-1}(z_*) = \left\{\rho^{\mathfrak{s}}(e, (-1, 1)), \rho^{\mathfrak{s}}(r^3, (-1, 1))\right\}.$$

Hence

$$|\pi_{\sqrt{q}}^{-1}(z_*)| = |i_{T\subset B}^G(z_*)|. \qquad \square$$

Lemma 8.4. *Part (4) of Theorem* 1.5 *is true for the point*

$$\mathfrak{s} = [T, \chi \otimes \chi]_G \in \mathfrak{B}(G_2)$$

where χ is a ramified quadratic character of $F^\times$.

Proof. This proof requires a detailed analysis of the associated KL-parameters. We recall (see the beginning of Section 8) that

$$H^{\mathfrak{s}} = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})/\langle -\mathrm{I}, -\mathrm{I}\rangle.$$

We recall also the beginning of Section 8 that the group $H^{\mathfrak{s}}$ admits 4 unipotent classes $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_1'$, $\mathbf{e}_0$. We have the corresponding decomposition of the asymptotic algebra into ideals:

$$J^{\mathfrak{s}} = J^{\mathfrak{s}}_{\mathbf{e}_e} \oplus J^{\mathfrak{s}}_{\mathbf{e}_1} \oplus J^{\mathfrak{s}}_{\mathbf{e}_1'} \oplus J^{\mathfrak{s}}_{\mathbf{e}_0}.$$

We will write

$$\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow H^{\mathfrak{s}}, \quad (x,y) \mapsto [x,y],$$

$$s_\tau := \begin{bmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{bmatrix}, \quad \text{for } \tau \in \mathbb{C}^\times,$$

$$u := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Recall that $T^\vee$ is the standard maximal torus in $H^{\mathfrak{s}}$. We will write

$$[s_\tau, s_{\tau'}] \in T^\vee.$$

The group $W^{\mathfrak{s}}$ is generated by the element which exchanges τ and τ^{-1}, and the element which exchanges τ' and ${\tau'}^{-1}$.

We will now consider separately the 4 unipotent classes in $H^{\mathfrak{s}}$.

8.0.3. *Case 1.* We consider

$$S := [s_{\sqrt{q}}, s_{\sqrt{q}}] \in T^\vee, \quad U := [u,u] \in H^{\mathfrak{s}}.$$

These form a semisimple-unipotent pair, i.e.,

$$SUS^{-1} = U^q.$$

We note that the component group of the simultaneous centralizer of S and U is given by

$$\mathrm{Z}(S,U) = \mathrm{Z}([s_{\sqrt{q}}, s_{\sqrt{q}}],[u,u]) = \{[\mathrm{I},\mathrm{I}],[\mathrm{I},-\mathrm{I}]\} = \mathbb{Z}/2\mathbb{Z}.$$

We also have

$$\mathrm{Z}([s_{\sqrt{q}}, s_{-\sqrt{q}}],[u,u]) = \{[\mathrm{I},\mathrm{I}],[\mathrm{I},-\mathrm{I}]\} = \mathbb{Z}/2\mathbb{Z}.$$

In each case, the associated variety of Borel subgroups is a point, namely $[b,b]$ where b is the standard Borel subgroup of $\mathrm{SL}(2,\mathbb{C})$. The KL-parameters are given by

$$([s_{\sqrt{q}}, s_{\sqrt{q}}],[u,u],1), \qquad ([s_{\sqrt{q}}, s_{-\sqrt{q}}],[u,u],1).$$

These two KL-parameters correspond to the ideal $J^{\mathfrak{s}}_{\mathbf{e}_e}$ in $J^{\mathfrak{s}}$, for which we have

$$J^{\mathfrak{s}}_{\mathbf{e}_e} \asymp \mathbb{C} \oplus \mathbb{C}.$$

This is *not* an L-packet in the principal series of $H^{\mathfrak{s}}$.

8.0.4. *Case 2.* For each $\tau \in \mathbb{C}^\times$, we have

$$\mathrm{Z}([s_{\sqrt{q}}, s_\tau],[u,\mathrm{I}]) = \mathbb{Z}/2\mathbb{Z}.$$

The associated variety of Borel subgroups comprises two points, namely $[b,b]$ and $[b,b^o]$ where b^o is the opposite Borel subgroup, i.e., the lower-triangular matrices in $\mathrm{SL}(2,\mathbb{C})$. The component group $\mathbb{Z}/2\mathbb{Z}$ acts on the homology of the two points as the trivial 2-dimensional representation. The KL-parameters in this case are given by

$$([s_{\sqrt{q}}, s_\tau],[u,\mathrm{I}],1).$$

These parameters correspond to the ideal $J^{\mathfrak{s}}_{\mathbf{e}_1}$ of $J^{\mathfrak{s}}$:

$$J^{\mathfrak{s}}_{\mathbf{e}_1} \asymp \mathcal{O}(\mathbb{A}^1).$$

8.0.5. *Case 3.* We also have the KL-parameters

$$([s_\tau, s_{\sqrt{q}}], [\mathrm{I}, u], 1)$$

with $\tau \in \mathbb{C}^\times$. These parameters correspond to the ideal $J^{\mathfrak{s}}_{\mathbf{e}'_1}$ of $J^{\mathfrak{s}}$:

$$J^{\mathfrak{s}}_{\mathbf{e}'_1} \asymp \mathcal{O}(\mathbb{A}^1).$$

8.0.6. *Case 4.* We need to consider the component group of the semisimple-unipotent pair

$$([s_\tau, s_{\tau'}], [\mathrm{I}, \mathrm{I}]).$$

The component group of this semisimple-unipotent pair is trivial unless $\tau = \tau' = \mathbf{i}$, where $\mathbf{i} = \sqrt{-1}$ denotes a square root of 1. In that case we have

$$\mathrm{Z}([s_{\mathbf{i}}, s_{\mathbf{i}}], [\mathrm{I}, \mathrm{I}]) = \mathbb{Z}/2\mathbb{Z}.$$

The associated variety of Borel subgroups of $H^{\mathfrak{s}}$ comprises 4 points:

$$[b, b], \quad [b^o, b^o], \quad [b, b^o], \quad [b^o, b].$$

The generator of the component group $\mathbb{Z}/2\mathbb{Z}$ switches b and b^o. The 4 points span a vector space of dimension 4 on which $\mathbb{Z}/2\mathbb{Z}$ acts by switching basis elements as follows:

$$[b, b] \to [b^o, b^o], \qquad [b, b^o] \to [b^o, b].$$

Therefore, $\mathbb{Z}/2\mathbb{Z}$ acts as the direct sum of two copies of the regular representation $1 \oplus \mathrm{sgn}$ of $\mathbb{Z}/2\mathbb{Z}$.

We recall that the equivalence relation among the KL-parameters for $H^{\mathfrak{s}}$ is conjugacy in $H^{\mathfrak{s}}$. The KL-parameters in this case are

$$([s_\tau, s_{\tau'}], [\mathrm{I}, \mathrm{I}], 1)$$

with $\tau, \tau' \in \mathbb{C}^\times$, and

$$([s_{\mathbf{i}}, s_{\mathbf{i}}], [\mathrm{I}, \mathrm{I}], \mathrm{sgn}).$$

This corresponds to the ideal $J^{\mathfrak{s}}_{\mathbf{e}_0} \subset J^{\mathfrak{s}}$ for which we have shown in equation (58) that

$$J^{\mathfrak{s}}_{\mathbf{e}_0} \asymp \mathcal{O}(T^\vee / W^{\mathfrak{s}}) \oplus \mathbb{C}.$$

There is an L-packet in the principal series of $H^{\mathfrak{s}}$ with the following KL-parameters:

$$([s_{\mathbf{i}}, s_{\mathbf{i}}], [\mathrm{I}, \mathrm{I}], 1), \qquad ([s_{\mathbf{i}}, s_{\mathbf{i}}], [\mathrm{I}, \mathrm{I}], \mathrm{sgn}).$$

The representations indexed by these KL-parameters are tempered. The corresponding representations of G itself still belong to an L-packet (see the end of Subsection 2.3). These are the representations denoted π^+ and π^- in the proof of Lemma 8.5.

Throughout §8 we have been using the ring isomorphism

$$\mathbb{C}[X, Y]_0 \cong \mathbb{C}[T^\vee / W^{\mathfrak{s}}]$$

induced by the map $\zeta\colon T^\vee \to \mathbb{A}^2$ defined by

$$\zeta((z_1, z_2)) := (z_1 + z_1^{-1}, z_2 + z_2^{-1}).$$

Note that this isomorphism sends $(\mathbf{i}, \mathbf{i})$ to $(0, 0) \in \mathbb{A}^2$. This is the unique point in the affine space $\mathbb{A}^2$ which is fixed under the map $(x, y) \mapsto (-x, -y)$.

Consider the map

$$\begin{aligned} M_{z,z'} &\mapsto \mathcal{V}^{\mathfrak{s}}_{[s_{\zeta(z)},s_{\zeta(z')}],[U,U],1}, \quad \text{if } (z,z') \neq (0,0), \\ \{M'_{0,0}, M''_{0,0}\} &\mapsto \{\mathcal{V}^{\mathfrak{s}}_{[s_{\mathbf{i}},s_{\mathbf{i}}],[\mathrm{I},\mathrm{I}],1}, \mathcal{V}^{\mathfrak{s}}_{[s_{\mathbf{i}},s_{\mathbf{i}}],[\mathrm{I},\mathrm{I}],\mathrm{sgn}}\}, \end{aligned}$$

from the set of simple $J^{\mathfrak{s}}_{\mathbf{e}}$-modules to the subset of $\mathrm{Irr}(G)^{\mathfrak{s}}$ such that $[U,U]$ corresponds to the two-sided cell $\mathbf{e}$. This map induces a bijection which corresponds, at the level of modules of the Hecke algebra, to the bijection induced by the Lusztig map ϕ_q, by the uniqueness property of ϕ_q. $\square$

Lemma 8.5. *Part (5) of Theorem* 1.5 *is true in this case.*

Proof. We start with the list of all those tempered representations of G_2 which admit inertial support $\mathfrak{s}$ (see Proposition 5.4):

$$I(\psi_1\chi \otimes \psi_2\chi) \cup I_\alpha(0, \delta(\psi\chi)) \cup \pi(\chi) \cup \pi(\epsilon\chi) \cup I_\beta(0, \delta(\phi\chi))$$

where

$$\psi_1 := z_1^{\mathrm{val}_F}, \;\; \psi_2 := z_2^{\mathrm{val}_F}, \;\; \psi := z^{\mathrm{val}_F}, \;\; \phi := w^{\mathrm{val}_F}$$

are unramified characters of $F^\times$, and

$$\pi(\chi) \subset I(\nu\chi \otimes \chi), \quad \pi(\epsilon\chi) \subset I(\nu\epsilon\chi \otimes \epsilon\chi)$$

are the elements in the discrete series described in [21, Prop. 4.1].

We recall from Lemma 8.1 that

$$E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} = (E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup pt_*) \sqcup \mathbb{I} \sqcup (pt_1 \sqcup pt_2) \sqcup \mathbb{I}.$$

Then the restriction of $\mu^{\mathfrak{s}}$ to $\mathrm{Irr}^{\mathfrak{s}}(G)^{\mathrm{t}}$ is as follows:

$$W^{\mathfrak{s}} \cdot (z_1, z_2) \mapsto I(\psi_1\chi_1 \otimes \psi_2\chi_2),$$

unless $z_1 = -1, z_2 = 1$, in which case

$$\begin{aligned} W^{\mathfrak{s}} \cdot (-1,1) \cup pt_* &\mapsto \pi^+ \oplus \pi^-, \\ pt_1 \sqcup pt_2 &\mapsto \pi(\chi) \sqcup \pi(\epsilon\chi). \end{aligned}$$

We note that $(\psi\chi)^\vee = \psi^{-1}\chi$, so that $I_\gamma(0, \delta(\psi\chi)) \cong I_\gamma(0, \delta(\psi^{-1}\chi))$ by [21], where $\gamma = \alpha, \beta$. Finally,

$$z \mapsto I_\alpha(0, \delta(\psi\chi)) \quad \text{and} \quad w \mapsto I_\beta(0, \delta(\phi\chi)). \qquad \square$$

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