Infiltration in Porous Media with Dynamic Capillary Pressure: Travelling Waves

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Abstract

We consider a model for non-static groundwater flow where the saturation-pressure relation is extended by a dynamic term. This approach together with a convective term due to gravity, results in a pseudo-parabolic Burgers type equation. We give a rigorous study of global travelling wave solutions, with emphasis on the role played by the dynamic term and the appearance of fronts.

1 Introduction

Capillary pressure is an essential characteristic of two-phase flow in porous media. It is the empirical macroscopic description of the pressure differences, due to surface tension, between the phases in the pores of an elementary representation volume.
In the standard approach, capillary pressure is expressed as a monotone function of one of the phase saturations: decreasing in terms of the wetting phase saturation and increasing in terms of the non-wetting phase saturation. Such expressions are based on static conditions within an elementary representative volume, see for instance Bear [3, 4] or Bedrikovetsky [5] .
Recently, see Gray and Hassanisadeh [11], new and more realistic models have been proposed to include non-static conditions as well. In its simplest form this leads to a capillary pressure, which is now a function of the wetting phase saturation and its time derivative. The formulation, with this improved capillary pressure, results in a transport equation containing higher order mixed derivatives. This equation is subject of study in this paper.
We confine ourselves to the particular case of unsaturated groundwater flow, where imbibition takes places under influence of gravity. The two phases are water (wetting phase) and air (non-wetting phase). Throughout we assume relative small values of the water saturation, so that regions where the porous medium is fully saturated, as described by Hulshof [12] and van Duijn and Peletier [8] for example, do not occur.
According to Bear [3, 4] we have for water in a homogeneous and isotropic porous medium the momentum balance equation

$$q = -K(S)(\nabla p_w + \rho g e_z) \quad \text{(Darcy's law)}, \tag{1.1}$$

and the mass balance

$$\phi\partial_t(\rho S) + \mathrm{div}(\rho q) = 0. \tag{1.2}$$

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Here q denotes volumetric water flux, S water saturation, $K(S)$ hydraulic conductivity, p_w water pressure, ρ water density, ϕ porosity and g gravity constant. Further e_z is the unit vector in positive z-direction, i.e. against the direction of gravity. To solve equation (1.1) and (1.2) an additional relation between p_w and S is needed. This relation is based on the assumption that the air pressure, p_a, is constant in space and time. This assumption is common practice in unsaturated groundwater flow and valid for small air viscosity.

Under static conditions, see Bear [3, 4] and Bedrikovetsky [5], one uses

$$p_a - p_w = p_c(S), \tag{1.3}$$

where p_c denotes the monotone capillary pressure. Under non-static conditions, Gray and Hassanisadeh [11] proposed to use

$$p_a - p_w = p_c(S) - \phi L(S)\partial_t S, \tag{1.4}$$

to capture the difference between drainage and imbibition. Here $L(S)$ is a nonlinear damping term.

Equations (1.1), (1.2) and (1.4) can be combined into a single equation for S, which reads

$$\phi\partial_t(\rho S) = \operatorname{div}\{\rho K(S)\rho g e_z + \rho K(S)\nabla(-p_c(S) + \phi L(S)\partial_t S)\}. \tag{1.5}$$

Assuming now a one-dimensional flow in vertical z-direction, with ρ and ϕ constant, and applying a straightforward scaling, equation (1.5) reduces to

$$\partial_t S = \partial_z\{K(S) + K(S)\partial_z(-p_c(S) + L(S)\partial_t S)\}. \tag{1.6}$$

In a previous work, see [2] and [1], G.I. Barenblatt proposed a different model to describe the non-static situation. He modified (1.1) and (1.3) by replacing S by $S + \tau\partial_t S$ ($\tau > 0$), into $p_c(S)$ and $K(S)$. The resulting equation then is of the form

$$\partial_t S = \partial_{zz}\{\Phi_1(S + \tau\partial_t S)\} + \partial_z\{\Phi_2(S + \tau\partial_t S)\},$$

where Φ_1 and Φ_2 are nonlinear positive functions and degenerate at $S + \tau\partial_t S = 0$. This equation admits a splitting into two equations which decouple the space and time derivatives. Namely,

$$\begin{cases} w = S + \tau S_t \\ -\tau(\Phi_1(w)_{zz} + \Phi_2(w)_z) + w = S. \end{cases}$$

Thus it can be studied as an ODE problem in Banach spaces. Moreover maximum principles are applicable, see [7]. There is a similar splitting for (1.6), which reads

$$\begin{cases} w = S_t - p_c(S) \\ -(K(S)w_z)_z + w = K(S)_z - p_c(S) \end{cases}$$

The appearance of $K(S)$ as a coefficient in the w-equation, however does not allow the maximum principle to be applied as in [7].

We note here that if the Barenblatt ansatz ($S \leftrightarrow S + \tau\partial S$) is applied only in (1.3), then (1.6) would result with

$$L(S) = -\frac{\tau}{\phi}\frac{dp_c}{dS},$$

implying that $L(S)$ becomes unbounded as $S \downarrow 0$. However, experiments carried out by Smiles et al [14], see also Hassanizadeh [10] for an overview, show that $L(S)$ vanishes as $S \downarrow 0$.

To investigate the role of the nonlinear terms (i.e. $K(S)$, $p_c(S)$ and $L(S)$), we replace them by power-law relations. Note that this is consistent with the assumption of small water saturation. We approximate

$$\begin{array}{rcll} K(S) & = & S^\alpha & (\alpha > 1), \\ p_c(S) & = & -1 + S^{-\beta} & (\beta > 0), \\ L(S) & = & \varepsilon S^\gamma & (\gamma > 0), \end{array}$$

where $\varepsilon > 0$ is introduced as a parameter to investigate the consequence of the third order mixed term. The parameter ranges are chosen to capture the relevant physical properties of unsaturated flow. In particular, we want K and L to be non-negative, with $L(0) = K(0) = K'(0) = 0$, and $p_c(0^+) = \infty$. Using these power law relations, equation (1.6) reduces to

$$\partial_t S = \partial_z\{S^\alpha + \beta S^{\alpha-\beta-1}\partial_z S + \varepsilon S^\alpha \partial_z(S^\gamma \partial_t S)\}. \tag{1.7}$$

The static capillary pressure relation (1.3), would have resulted in the convection-diffusion equation

$$\partial_t S = \partial_z\{S^\alpha + \beta S^{\alpha-\beta-1}\partial_z S\}. \tag{1.8}$$

It is well-known, see e.g. Gilding [9], that this equation has finite speed of propagation if and only if

$$\int_0^\delta \frac{D(S)}{S} dS < \infty \text{ for some } \delta > 0, \text{ with } D(S) = \beta S^{\alpha-\beta-1}. \tag{1.9}$$

This requires $\alpha - \beta > 1$. Because occurrence of fronts has our special interest we analyse equation (1.7) in the parameter range

$$\beta,\ \gamma > 0 \quad \text{and} \quad \beta < \alpha - 1. \tag{1.10}$$

In this paper we analyse travelling waves solutions of (1.7). They are conjectured to describe the large time behaviour of solutions resulting from a certain class of initial conditions. The stability properties of travelling waves are subject of a separated future study.

Thus we consider

$$S(z,t) = f(\eta) \quad \text{with} \quad \eta = z + ct, \tag{1.11}$$

subject to the boundary conditions

$$f(\infty) = A > 0, \quad f(-\infty) = \delta \geq 0 \quad (\delta < A). \tag{1.12}$$

Hence the fluid moves downwards whenever the wave speed c is positive. For f we obtain the equation

$$cf' = \{f^\alpha + \beta f^{\alpha-\beta-1} f' + c\varepsilon f^\alpha (f^\gamma f')'\}' \quad \text{on} \quad \mathbb{R}, \tag{1.13}$$

Integration and application of boundary conditions give the equation

$$c(f-\delta) = f^\alpha - \delta^\alpha + \beta f^{\alpha-\beta-1} f' + c\varepsilon f^\alpha (f^\gamma f')', \tag{1.14}$$

and the second order boundary value problem

$$(TW) \begin{cases} c(f-\delta) = f^\alpha - \delta^\alpha + \beta f^{\alpha-\beta-1} f' + c\varepsilon f^\alpha (f^\gamma f')' \quad \text{on} \quad \mathbb{R} \\ \\ f(-\infty) = \delta, \quad f(+\infty) = A, \end{cases}$$

where c is given by

$$c = \frac{A^\alpha - \delta^\alpha}{A - \delta}, \tag{1.15}$$

which is the Rankine-Hugoniot wave speed if we interpret (1.7) as a regularization of the hyperbolic equation $\partial_t S = \partial_x S^\alpha$. Note that $c \downarrow A^{\alpha-1}$ as $\delta \downarrow 0$, $c \uparrow \alpha A^{\alpha-1}$ as $\delta \uparrow A$, with $\frac{dc}{d\delta} > 0$ for $0 < \delta < A$.

In Section 2 we show existence of travelling waves for fixed positive values of ε and δ. They are unique up to translations in η. This analysis also shows an oscillatory, but non-periodic, behaviour of the profiles. Here the value of ε is crucial; for ε sufficiently small (depending on $\alpha, \beta, \gamma, \delta$ and A) we obtain monotone profiles, this is shown in Section 3.

In Section 4 we study the limit case $\varepsilon \to 0$, while $\delta > 0$ is kept fixed. Using essentially monotonicity for small ε, we obtain convergence to travelling waves of equation (1.8), i.e. the standard model based on the static pressure saturation relation (1.3).

In Section 5 we analyse existence of front solutions to Problem TW with $\delta = 0$. It turns out that there are two relevant ranges of powers α, β, γ for which fronts appear. In the range $2\beta > \alpha - \gamma - 2$ there exists a family of solutions which degenerate at a finite value of η. When $2\beta = \alpha - \gamma - 2$ existence of fronts is shown provided $\varepsilon \leq \frac{\beta}{4A^{2(\alpha-1)}}$. Uniqueness does not hold. Nevertheless we have discerned in each of the previous cases a unique (up to translations in η) exceptional profile, which is the limit profile to (TW) when letting $\delta \to 0$. This is shown in Section 6. In the other cases, $2\beta = \alpha - \gamma - 2$, $\varepsilon \geq \frac{\beta}{4A^{2(\alpha-1)}}$, and $2\beta < \alpha - \gamma - 2$, bounded travelling wave solutions do not exist. These results correspond to the formal asymptotic analysis made in [13].

It is worthy to observe that the limits $\varepsilon \to 0$ and $\delta \to 0$ do not commute. We can always take the limit $\varepsilon \to 0$ followed by $\delta \to 0$. However, the converse order is only possible if $2\beta \geq \alpha - \gamma - 2$.

Remark 1.1 *Fixing $\delta \in (0, A)$, the existence is demonstrated for $\alpha > 1$ and $\beta, \gamma > 0$. All other results require in addition $\beta < \alpha - 1$.*

2 Existence and Uniqueness

The main result of this section is:

Theorem 2.1 *Let $\alpha > 1$, $\beta, \gamma, \varepsilon > 0$ and $0 < \delta < A$. Further, let c be given by (1.15). Then there exists a C^∞ solution of Problem TW, unique up to translations in η.*

Proof. We transform equation (1.14) into a planar system and apply a phase plane analysis. First we set $u = f^{1+\gamma}$, which gives

$$cu^{\frac{1-\alpha}{1+\gamma}} - 1 = (c\delta - \delta^\alpha)u^{-\frac{\alpha}{1+\gamma}} + \frac{\beta}{1+\gamma}u^{(-\frac{\beta}{1+\gamma}-1)}u' + \frac{\varepsilon c}{1+\gamma}u'' \quad \text{on} \quad \mathbb{R} \tag{2.1}$$

with boundary conditions

$$u(-\infty) = \delta^{1+\gamma}, \quad u(\infty) = A^{1+\gamma}.$$

Next we put this equation in the Liénard phase-plane, by considering u and $v := \frac{\varepsilon c}{1+\gamma}u' - u^{-\frac{\beta}{1+\gamma}}$, as independent variables. This results in the system

$$(P)\begin{cases} \varepsilon u' = F(u,v) = \frac{1+\gamma}{c}\left(v + u^{-\frac{\beta}{1+\gamma}}\right) \\ v' = G(u) = -1 + cu^{\frac{1-\alpha}{1+\gamma}} - (c\delta - \delta^\alpha)u^{-\frac{\alpha}{1+\gamma}}. \end{cases}$$

A solution f of (TW) is an integral curve of (P) connecting the equilibria $p_- = (\delta^{1+\gamma}, -\delta^{-\beta})$ and $p_+ = (A^{1+\gamma}, -A^{-\beta})$. The phase plane, with the isoclines $\Gamma_u = \{(u,v) : F(u,v) = 0\}$ and $\Gamma_v = \{(u,v) : G(u) = 0\}$, is drawn in Figure 1.

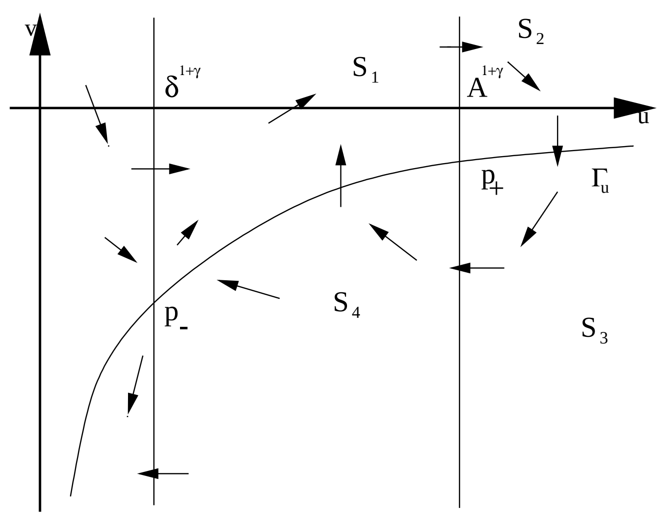


Figure 1: Phase plane for (P)

The matrix of the linearised system is

$$\begin{pmatrix} -\frac{\beta}{\varepsilon c}X^{-\beta-1-\gamma} & \frac{1+\gamma}{\varepsilon c} \\ \frac{X^{-\alpha-\gamma-1}}{1+\gamma}\left(c(1-\alpha)X+(cX-X^{\alpha})\alpha\right) & 0 \end{pmatrix},$$

where $X=\delta$ corresponds to the equilibrium p_-, and $X=A$ to p_+. The eigenvalues in p_- and p_+ are, expressed in X,

$$\lambda=-a\pm\sqrt{a^2+b} \text{ with } a=\frac{\beta}{2\varepsilon c}\frac{1}{X^{\beta+\gamma+1}} \text{ and } b=\frac{c-\alpha X^{\alpha-1}}{X^{\alpha+\gamma}\varepsilon c}. \tag{2.2}$$

It is clear that $a>0$ for positive X, and b has the sign of $c-X^{\alpha-1}\alpha$. Introducing its primitive

$$i(X)=cX-X^{\alpha}, \tag{2.3}$$

we note that A and δ are related by $i(A)=i(\delta)>0$, so that $b>0$ in $X=\delta$ and $b<0$ in $X=A$. Therefore at p_- the two eigenvalues are real and of opposite sign, whence p_- is a saddle point. The point p_+ is a sink and

$$\begin{aligned} &0\le a^2+b<a^2 \text{ implies two real eigenvalues, } \lambda_1\le\lambda_2<0, \text{ and} \\ &a^2+b<0 \text{ implies two complex eigenvalues, with negative real part.} \end{aligned} \tag{2.4}$$

To prove Theorem 2.1 we have to show that an orbit leaving p_- connects to p_+. Because p_- is a saddle point there exist locally two orbits containing solutions $(u(\eta),v(\eta))$ of (P), satisfying

$$\lim_{\eta\to-\infty}(u(\eta),v(\eta))=p_-.$$

Let C be the orbit for which $u',v'>0$. Inspection of Figure 1 shows that the other orbit will never reach p_+. The only possibility giving existence of a travelling wave is for C to end up in p_+. The corresponding solutions will satisfy (TW), and uniqueness up to translations in η will hold.

To show that C connects to p_+ we consider the sets, see also Figure 1,

$$\begin{aligned} S_1&=\{(u,v)\in\mathbb{R}^2:\delta^{\gamma+1}<u<A^{\gamma+1}, v>-u^{-\frac{\beta}{1+\gamma}}\}, \\ S_2&=\{(u,v)\in\mathbb{R}^2:u>A^{\gamma+1}, v>-u^{-\frac{\beta}{1+\gamma}}\}, \\ S_3&=\{(u,v)\in\mathbb{R}^2:u>A^{\gamma+1}, v<u^{-\frac{\beta}{1+\gamma}}\}, \\ S_4&=\{(u,v)\in\mathbb{R}^2:\delta^{\gamma+1}<u<A^{\gamma+1}, v<-u^{-\frac{\beta}{1+\gamma}}\}. \end{aligned}$$

Note that the boundaries of S_1,S_2,S_3,S_4 are the isoclines of system (P). Furthermore C enters S_1 from p_-.

We have the following two possibilities.

Lemma 2.1 *The orbit C rotates around p_+, going from S_i to $S_{i+1 \bmod 4}$, or it enters p_+ from S_1 or S_3.*

Proof. Let $(u(\eta), v(\eta))$ be a solution of (P) parametrising C. Further, let $\eta_{max} \leq \infty$ be the maximum η-value for which the solution can be extended to $(-\infty, \eta_{max})$. Near points where $u'(\eta) \neq 0$, we can express any solution of (P) locally as $v = v(u)$, satisfying

$$\frac{dv}{du} = \frac{v'}{u'} = \varepsilon \frac{G(u)}{F(u,v)}. \tag{2.5}$$

Below we exhaust all possibilities.

1. Suppose $(u(\eta), v(\eta)) \in S_1$ for all $0 < \eta < \eta_{max}$. Then two cases need to be considered:

> **(i)**: $u \uparrow \bar{u}$ and $v \uparrow \bar{v}$, with $\bar{u}, \bar{v} < \infty$. This implies $\eta_{max} = +\infty$ and $(\bar{u}, \bar{v}) = p_+$.
> **(ii)**: $u \uparrow \bar{u}$ and $v \uparrow \infty$. Since $F(u,v)$ becomes unbounded as $v \to \infty$, while $G(u)$ remains bounded, (2.5) directly gives a contradiction.

Thus either $(u(\eta), v(\eta)) \to p_+$ as $\eta \to \infty$ or C crosses $u = A^{1+\gamma}$ at $\eta = \eta_1 \in (0, \eta_{max})$ and enters S_2.

2. Suppose $(u(\eta), v(\eta))$ remains in S_2 for all $\eta \in (\eta_1, \eta_{max})$. Again two cases are possible.

> **(i)**: $u \uparrow \bar{u}$ and $v \downarrow \bar{v}$ with $\bar{u} < \infty$ and, of course, $\bar{v} > -\infty$. Then $\eta_{max} = +\infty$ and $(\bar{u}, \bar{v})$ is an equilibrium point, which is impossible because there are no equilibria to the right of p_+.
> **(ii)**: $u \uparrow +\infty$ and $v \downarrow \bar{v}$, with $\bar{v} \geq 0$. Then the v-equation implies $v' \to -1$ contradicting $\bar{v} \geq 0$.

Thus, there exists $\eta_2 \in (\eta_1, \eta_{max})$ such that $(u(\eta), v(\eta))$ crosses Γ_u and enters S_3.

3. Suppose $(u(\eta), v(\eta)) \in S_3$ for all $\eta \in (\eta_2, \eta_{max})$. Completely analogous to **1.**, we find that either $(u(\eta), v(\eta)) \to p_+$ or C crosses $A^{1+\gamma}$ at some $\eta_3 \in (\eta_0, \eta_{max})$ and enters S_4.

4. Suppose $(u(\eta), v(\eta)) \in S_4$ for all $\eta \in (\eta_3, \eta_{max})$. Again we distinguish

> **(i)**: $u \downarrow \bar{u}$ and $v \uparrow \bar{v}$. As before $\eta_{max} = +\infty$ and consequently $(\bar{u}, \bar{v}) = p_-$. Thus C is a homoclinic orbit. Next we consider the domain D enclosed by C. Since its boundary is smooth, except at p_- where it is Lipschitz, we may apply Gauss' theorem. Using the fact that C is an orbit of (P) we get
>
> $$0 = \oint_C \left(\frac{1}{\varepsilon} F(u,v) dv - G(u) du \right) = \iint_D \operatorname{div} \left(\frac{F(u,v)}{\varepsilon}, G(u) \right),$$
>
> contradicting
>
> $$\operatorname{div} \left(\frac{1}{\varepsilon} F(u,v), G(u) \right) = -\frac{\beta}{c\varepsilon} u^{-\frac{\beta+\gamma+1}{\gamma+1}} < 0 \quad \text{for all } u > \delta^{\gamma+1}.$$
>
> **(ii)**: C crosses $u = \delta^{\gamma+1}$ below p_- at $(\delta^{1+\gamma}, \tilde{v})$. Consider the closed curve $C \cup C_1$, where C_1 is the straight line segment parametrised by
>
> $$\begin{cases} u(s) = \delta^{\gamma+1}, \\ v(s) = s \quad s \in [\tilde{v}, -\delta^{-\beta}]. \end{cases}$$

As before we call D the interior of $C \cup C_1$, and apply Gauss' theorem. This gives

$$0 > \iint_D \operatorname{div}\left(\frac{F(u,v)}{\varepsilon}, G(u)\right) = \oint_{C\cup C_1}\left(\frac{1}{\varepsilon}F(u,v)dv - G(u)du\right) =$$
$$\oint_{C_1}\left(\frac{1}{\varepsilon}F(u,v)dv - G(u)du\right) = \int_{\tilde{v}}^{-\delta^{-\beta}} \frac{1}{\varepsilon}F(\delta^{\gamma+1}, s)ds > 0,$$

a contradiction.

Hence there exists $\eta_4 \in (\eta_3, \eta_{max})$ at which $(u(\eta), v(\eta))$ crosses Γ_u and enters S_1. □
To complete the proof of the theorem we use Lemma 2.1 and the Poincaré-Bendixon theorem. This leave us with two possibilities. Either $(u(\eta), v(\eta)) \to p_+$ as $\eta \to \infty$ or C approaches a periodic orbit. The latter is impossible by the same argument as in **4(i)**, now applied to the periodic orbit instead of C. The C^∞ regularity for f follows from a bootstrap argument. □

3 Monotonicity

In this section we derive a sufficient condition for the travelling wave solution to have a monotone profile. This condition is related to the value of ε. Therefore we write $C = C_\varepsilon$ and $P = P_\varepsilon$ whenever appropriate.
The eigenvalues at p_+ are (see expressions (2.2))

$$\lambda_{1,2} = \frac{-\beta}{2\varepsilon c}\frac{1}{A^{\beta+\gamma+1}} \pm \frac{1}{2}\sqrt{\frac{\beta^2}{(\varepsilon c)^2}\frac{1}{A^{2(\beta+\gamma+1)}} + 4\frac{(c - A^{\alpha-1}\alpha)}{A^{\alpha+\gamma}\varepsilon c}},$$

with corresponding eigenvectors

$$(u_{1,2}, v_{1,2}) = \left(1, \frac{1}{2(1+\gamma)}\frac{\beta \pm \sqrt{\beta^2 + 4(cA - A^\alpha\alpha)\varepsilon c A^{2\beta-\alpha+\gamma+1}}}{A^{1+\gamma+\beta}}\right).$$

Thus for all ε such that

$$\varepsilon < \varepsilon^* := \frac{\beta^2}{4c(A^{\alpha-1}\alpha - c)}A^{\alpha-2\beta-\gamma-2} \quad (\varepsilon^* > 0), \tag{3.1}$$

λ_1, λ_2 are real and strictly negative, which is a necessary condition for the travelling wave profile to be monotone. Henceforth we suppose being in this situation. With $\lambda_2 < \lambda_1 < 0$ we call (u_1, v_1) the slow eigenvector and (u_2, v_2) the fast eigenvector at p_+. By standard local analysis, see e.g. [6], there exist exactly two orbits entering p_+ tangent to the (u_2, v_2)-direction: one along (u_2, v_2), the other along $(-u_2, -v_2)$. The connecting orbit goes around p_+ at most a finite number of times.

Lemma 3.1 *Let (1.10) be satisfied. For $\delta > 0$ fixed, there exists $0 < \bar{\varepsilon} < \varepsilon^*$, such that for every $0 < \varepsilon < \bar{\varepsilon}$ the travelling wave obtained in Theorem 2.1 is strictly increasing on $\mathbb{R}$.*

Lemma 3.1 is a direct consequence of Proposition 3.1 below, in which we construct an invariant region which contains C and which itself is contained in S_1.
More specifically, for fixed $\delta \in (0, A)$ and $\mu \in (0, 1)$, let S^δ_μ denote the set enclosed by the curves

$$u = \delta^{1+\gamma}, v = -u^{-\frac{\beta}{1+\gamma}} \quad \text{and} \quad v = g_\mu(u) := (-\mu u^{-\frac{\beta}{1+\gamma}} - (1-\mu)A^{-\beta}).$$

We will show in Proposition 3.1 that for ε sufficiently small (i.e. $\varepsilon < \varepsilon^\delta_\mu$), S^δ_μ is invariant for Problem P_ε or, equivalently, $C_\varepsilon \in S^\delta_\mu$. Proposition 3.1 will also be helpful in the study of the limits $\varepsilon \to 0$ and $\delta \to 0$. In particular we want to bound ε^δ_μ away from zero as $\delta \to 0$. It will appear that this is only possible if $2\beta + \gamma + 2 - \alpha \geq 0$.

Proposition 3.1 *For any fixed δ in $(0, A)$ and $\mu \in (0, 1)$ there exists $\varepsilon^\delta_\mu \in (0, \varepsilon^*)$, such that for every $\varepsilon \in (0, \varepsilon^\delta_\mu)$, $C_\varepsilon \in S^\delta_\mu$. Further, if $2\beta + \gamma + 2 - \alpha \geq 0$,*

$$\lim_{\delta\to 0} \varepsilon^\delta_\mu = 4\mu(1-\mu) \lim_{\delta \to 0} \varepsilon^* = \mu(1-\mu)\frac{\beta^2}{4(\alpha-1)}A^{-2\beta-\gamma-1},$$

and

$$\lim_{\delta\to 0} S^\delta_\mu \quad \textit{is invariant for Problem TW with} \quad \delta = 0.$$

Proof. Observe that the eigenvectors at p_+ satisfy

$$(u_1, v_1) \to (1, \frac{\beta}{(1+\gamma)A^{1+\gamma+\beta}}), \text{ and } (u_2, v_2) \to (1, 0) \text{ as } \varepsilon \to 0,$$

where $(1, \frac{\beta}{(1+\gamma)A^{1+\gamma+\beta}})$ is the tangent vector at p_+ to Γ_u. The invariant region S^δ_μ is below the horizontal line $v = -A^{-\beta}$, and only contains orbits entering p_+ along the slow eigenvector (u_1, v_1). Observe that $g_\mu(u) > -u^{-\frac{\beta}{1+\gamma}}$ for all $0 < u < A^{1+\gamma}$, and $g_\mu(A^{1+\gamma}) = -A^{-\beta}$. Obviously the vector field is pointing inwards at boundary points of S^δ_μ on $u = \delta^{1+\gamma}$ and Γ_u. It remains to examine the vector field on $v = g_\mu(u)$. We clearly must have

$$\frac{dv}{du} = \frac{\varepsilon c}{1+\gamma}\frac{G(u)}{g_\mu(u) + u^{-\frac{\beta}{1+\gamma}}} \leq g'_\mu(u) \quad \text{for all } u \in (\delta^{1+\gamma}, A^{1+\gamma}), \tag{3.2}$$

for S^δ_μ to be invariant. This is equivalent to

$$\varepsilon H_\delta(u) \leq \mu(1-\mu) \tag{3.3}$$

where

$$H_\delta(u) = \frac{c}{\beta}\,\frac{-1 + cu^{\frac{1-\alpha}{1+\gamma}} - (c\delta - \delta^\alpha)u^{-\frac{\alpha}{1+\gamma}}}{(u^{-\frac{\beta}{1+\gamma}} - A^{-\beta})}u^{1+\frac{\beta}{1+\gamma}}. \tag{3.4}$$

Note that $H_\delta(\delta^{1+\gamma}) = 0$ and, by L'Hôspital's rule,

$$H_\delta(A^{1+\gamma}) = \lim_{u \to A^{1+\gamma}} H_\delta(u) = c\,\frac{\alpha A^{\alpha-1} - c}{\beta^2}A^{2\beta+\gamma+2-\alpha}$$
$$\to \frac{(\alpha-1)}{\beta^2}A^{2\beta+\gamma+\alpha} \text{ as } \delta \to 0.$$

For $u > 0$ fixed, $\lim_{\delta\to 0} H_\delta(u)$ behaves as $u^{\frac{2\beta+\gamma+2-\alpha}{1+\gamma}}$ near 0, which suggests to write $H_\delta(u)$ as

$$H_\delta(u) = \frac{c}{\beta} u^{\frac{2\beta+\gamma+2-\alpha}{1+\gamma}} h_\delta(u) \tag{3.5}$$

$$\text{where} \quad h_\delta(u) = \frac{-u^{\frac{\alpha}{1+\gamma}} + cu^{\frac{1}{1+\gamma}} - (c\delta - \delta^\alpha)}{u^{\frac{1}{1+\gamma}}(1 - A^{-\beta}u^{\frac{\beta}{1+\gamma}})}.$$

Observe that for every $u \in (0, A^{1+\gamma})$ and every $\delta > 0$, recalling (1.15),

$$h_0(u) - h_\delta(u) = \frac{(c\delta - \delta^\alpha) - u^{\frac{1}{1+\gamma}}(c - A^{\alpha-1})}{u^{\frac{1}{1+\gamma}}(1 - A^{-\beta}u^{\frac{\beta}{1+\gamma}})} > 0. \tag{3.6}$$

So that

$$h_\delta(u) < h_0(u) = \frac{-u^{\frac{\alpha}{1+\gamma}} + A^{\alpha-1}u^{\frac{1}{1+\gamma}}}{u^{\frac{1}{1+\gamma}}(1 - A^{-\beta}u^{\frac{\beta}{1+\gamma}})} = A^{\alpha-1}\frac{1 - A^{1-\alpha}u^{\frac{\alpha-1}{1+\gamma}}}{1 - A^{-\beta}u^{\frac{\beta}{1+\gamma}}}, \tag{3.7}$$

which is increasing for $u \in (0, A^{1+\gamma})$. Here we used $\beta < \alpha - 1$ from (1.10).
Setting

$$M := \sup_{u\in(0,A^{1+\gamma})} h_0(u) = \frac{(\alpha-1)A^{\alpha-1}}{\beta},$$

an upper bound for $H_\delta(u)$ is given by

$$H_\delta(u) < M\frac{c}{\beta}A^{2\beta+\gamma+2-\alpha} \quad \text{if} \quad 2\beta+\gamma+2-\alpha \geq 0,$$

and

$$H_\delta(u) < M\frac{c}{\beta}\delta^{2\beta+\gamma+2-\alpha} \quad \text{if} \quad 2\beta+\gamma+2-\alpha \leq 0.$$

Thus a sufficient condition for (3.3) to hold is

$$\varepsilon < \varepsilon_\mu^\delta := \begin{cases} \mu(1-\mu)\frac{\beta^2}{c(\alpha-1)}A^{-2\beta-\gamma-1} & \text{if} \quad 2\beta+\gamma+2-\alpha \geq 0 \\ \\ \mu(1-\mu)\frac{\beta^2}{c(\alpha-1)A^{\alpha-1}}\delta^{\alpha-2\beta-\gamma-2} & \text{if} \quad 2\beta+\gamma+2-\alpha \leq 0. \end{cases} \tag{3.8}$$

This completes the proof of the first statement. The statements about the $\delta \to 0$ limit follow immediately from (3.8) with $2\beta \geq \alpha - \gamma - 2$, (3.1) and (1.15). □

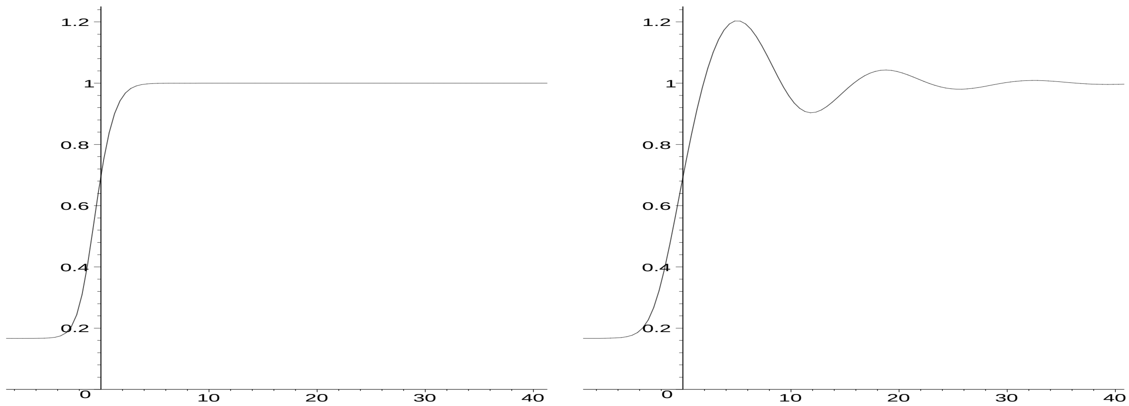
4 The $\varepsilon \to 0$ limit case

Let $\delta \in (0, A)$ be fixed and (1.10) be satisfied. In this section we examine the behaviour of the connecting orbit C_ε and that of the corresponding travelling wave $f = f_\varepsilon$ as $\varepsilon \to 0$. For $\varepsilon < \bar{\varepsilon}$ we denote C_ε by

$$v = \varphi_\varepsilon(u), \quad \delta^{\gamma+1} \leq u \leq A^{\gamma+1}. \tag{4.1}$$

As a first convergence result we have

Proposition 4.1 $\varphi_\varepsilon(u) \to -u^{-\frac{\beta}{1+\gamma}}$ *uniformly on* $[\delta^{1+\gamma}, A^{1+\gamma}]$ *as* $\varepsilon \to 0$.



(a) monotone profile: $\varepsilon = \varepsilon^* - 0.01 \approx 0.048$

(b) oscillatory profile: $\varepsilon = 1$

Figure 2: Travelling wave solutions for different values of ε, where $f(0) = \frac{A+\delta}{2}$, $\alpha = \frac{4}{3}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$, $A = 1$, $\delta = \frac{1}{6}$

Proof. Proposition 3.1 implies

$$-u^{\frac{-\beta}{1+\gamma}} < \varphi_\varepsilon(u) < g_\mu(u) \tag{4.2}$$

for all $u \in (\delta^{1+\gamma}, A^{1+\gamma})$ and for all $\varepsilon \in (0, \varepsilon_\mu^\delta)$. Since $g_\mu(u) \to -u^{-\frac{\beta}{1+\gamma}}$ as $\mu \uparrow 1$, the result is immediate. □

For the travelling waves f_ε we have

Theorem 4.1 *Translate f_ε so that $f_\varepsilon(0) = \frac{\delta+A}{2}$ for all $\varepsilon > 0$. Then $f_\varepsilon \to f \in C^\infty(\mathbb{R})$ as $\varepsilon \to 0$, uniformly on $\mathbb{R}$, where f satisfies Problem TW with $\varepsilon = 0$.*

Proof. First we employ the scaling $\eta = \varepsilon\tau$, so that in the τ-variable (P_ε) reads

$$(\tilde{P}_\varepsilon) \begin{cases} \dot{u} = F(u, v), \\ \dot{v} = \varepsilon G(u). \end{cases} \tag{4.3}$$

Unlike (P_0) the limit system $(\tilde{P}_0)$, is well defined. The one-dimensional manifold of critical points

$$M_0 = \{F(v, u) = 0\} = \{v = -u^{\frac{-\beta}{1+\gamma}}\}$$

is invariant and normally hyperbolic in the sense of geometric singular perturbation theory, see [15], because for $(\tilde{P}_0)$ the only pure imaginary eigenvalue is zero, and has a one-dimensional eigenspace tangential to M_0. Let K be a neighbourhood of $\{(u, v) \in M_0 : \ \delta^{1+\gamma} \le u \le A^{1+\gamma}\}$, and choose $0 < \delta_1 < \delta^{1+\gamma} < A^{1+\gamma} < A_1$ such that $\{(u, v) \in M_0 : \ \delta_1 \le u \le A_1\} \subset\subset K$. By Fenichel's invariant manifold theorem [15], there exists, for given $k \in \mathbb{N}$, a number $\varepsilon_0 > 0$ and a function $h \in C^k([\delta_1, A_1] \times [0, \varepsilon_0])$ with $h(u, 0) = -u^{-\frac{\beta}{1+\gamma}}$, such that for every $0 < \varepsilon < \varepsilon_0$

$$M_\varepsilon = \{(u, v) \in K : \ v = h(u, \varepsilon), \ \delta_1 \le u \le A_1\}$$

is locally invariant. The manifold M_ε is not uniquely determined. However between $u = \delta^{1+\gamma}$ and $u = A^{1+\gamma}$ it must coincide with the connecting orbit $v = \varphi_\varepsilon(u)$, because this is the only orbit which remains close to $\{v = -u^{-\frac{\beta}{1+\gamma}} : \ \delta_1 < u < A_1\}$. Using (4.1) and the v-equation in (P_ε), we note that $u_\varepsilon = f_\varepsilon^{1+\gamma}$ satisfies,

$$u' = \frac{G(u)}{\varphi_\varepsilon'(u)}, \tag{4.4}$$

and connects the two zeros of G.
Since $h \in C^k$ and $h(u,\varepsilon) = \varphi_\varepsilon(u)$ for $\delta^{1+\gamma} \leq u \leq A^{1+\gamma}$, we have as a result of Proposition 4.1, that $\varphi_\varepsilon'(u) \to \frac{\beta}{1+\gamma} u^{-\frac{\beta}{1+\gamma}-1}$ uniformly on $[\delta^{1+\gamma}, A^{1+\gamma}]$ and thus

$$\frac{G(u)}{\varphi_\varepsilon'(u)} \to \frac{G(u)}{\frac{\beta}{1+\gamma} u^{-\frac{\beta}{1+\gamma}-1}} \quad \text{uniformly}$$

on $[\delta^{1+\gamma}, A^{1+\gamma}]$ as $\varepsilon \to 0$. In this limit the differential equation (4.4) is identical to equation (2.1) with $\varepsilon = 0$. Using the fact that $u_\varepsilon(0)$ is fixed for all $\varepsilon > 0$, standard arguments imply that u_ε converges uniformly on $\mathbb{R}$ to the corresponding solution of the limit equation. □

5 $\delta = 0$ system

In this section we consider the limit case $\delta = 0$ directly. Thus we study the system

$$(P_\varepsilon^0) \left\{ \begin{array}{l} \varepsilon u' = F_0(u,v) = \frac{1+\gamma}{c} \left(v + u^{-\frac{\beta}{1+\gamma}} \right), \\ v' = G_0(u) = -1 + c u^{\frac{1-\alpha}{1+\gamma}}, \end{array} \right.$$

where $c = A^{\alpha-1}$, and we look for orbits connecting $u = 0$ to $u = A^{1+\gamma}$. The critical point corresponding to the latter now has real eigenvalues, see also (3.1), for

$$0 < \varepsilon \leq \varepsilon^* = \frac{\beta^2}{4(\alpha-1)} A^{-\alpha-2\beta-\gamma}.$$

The phase plane, see Figure 1, clearly implies, that the desired orbit has to be originated from the segment $\{(u,v) : \ u = 0, v \leq 0\}$ where the equations are singular. Since we are interested in (P_ε^0) as limit of (P_ε^δ), and in particular of a possible limit orbit of the connecting orbit C, we expect such a limit orbit, if it exists, to behave as $v \sim -d\, u^{-\frac{\beta}{1+\gamma}}$, $0 < d \leq 1$, as $u \to 0$. Thus a convenient new dependent variable is $Z = u^q v$, where q, for later purposes, is not fixed yet. Whenever $u' \neq 0$, Z satisfies the equation

$$u \frac{dZ}{du} = qZ + \frac{\varepsilon c}{1+\gamma} u^{1+2q+\frac{1-\alpha}{1+\gamma}} \frac{c - u^{\frac{\alpha-1}{1+\gamma}}}{Z + u^{q-\frac{\beta}{1+\gamma}}}. \tag{5.1}$$

Below we investigate the solvability of (5.1) for $0 < u < A^{1+\gamma}$. The analysis and results critically depend on the value of the parameters α, β and γ. In particular the value of $2\beta - \alpha + \gamma + 2$ plays a crucial role, which is to be expected considering the results of the formal analysis by Hulshof and King [13]. With q appropriately chosen, we consider the cases:

5.1 $2\beta > \alpha - \gamma - 2$.

Here we take $q = \frac{\beta}{1+\gamma}$, and set $W = u^{\frac{2\beta+\gamma+2-\alpha}{1+\gamma}}$. Then (5.1) becomes

$$(2\beta + \gamma + 2 - \alpha)\frac{dZ}{dW} = \frac{\beta Z}{W} + \frac{\varepsilon c(c - W^{\frac{\alpha-1}{2\beta+\gamma+2-\alpha}})}{Z+1}. \tag{5.2}$$

We look for solutions of (5.2) with $Z > -1$ as $W \to 0$ (i.e. $u' > 0$ as $u \to 0$). In Figure 3 we sketch the (W, Z)-phase plane. Equation (5.2) and the phase plane imply that $Z \to Z_0 \in \{0, -1\}$ as $W \to 0$, where orbits with $Z_0 = 0$ have $v = o(-u^{-\frac{\beta}{1+\gamma}})$ while orbits with $Z_0 = -1$ have $v \sim -u^{-\frac{\beta}{1+\gamma}}$.

Proposition 5.1 *For $2\beta > \alpha - \gamma - 2$ there is a unique orbit C^0 with $u \to 0$ and $v \sim -u^{-\frac{\beta}{1+\gamma}}$ as η decreases. This orbit reaches $(u, v) = (0, -\infty)$ at some finite η-value, implying the existence of a travelling wave with a front. The local behavior of the front is determined by the relation*

$$f' \sim \frac{c}{\beta} f^{\beta+2-\alpha} \quad \text{as} \quad f \to 0.$$

Proof. We first prove existence. Choose W_0 small and denote the solution of (5.2) with $Z = \xi$ in $W = W_0$ by $Z = Z(W, \xi)$. Let

$$\begin{aligned}
S_+ &= \{\xi \in (-1, 0) : \ Z(W, \xi) \to 0 \text{ as } W \downarrow 0\} \\
S_- &= \{\xi \in (-1, 0) : \exists W_* \in (0, W_0) \ Z(W, \xi) \to -1 \text{ as } W \downarrow W_* > 0\} \\
S_0 &= \{\xi \in (-1, 0) : \ Z(W, \xi) \to -1 \text{ as } W \downarrow 0\}.
\end{aligned}$$

By standard arguments we have for W_0 sufficiently small that

$$(0, -1) = S_- \cup S_0 \cup S_+$$

and S_- and S_+ are nonempty and open. Hence S_0 is nonempty, which gives existence. We observe that for such an orbit, see (5.2),

$$Z + 1 \sim aW \text{ as } W \downarrow 0 \text{ where } a = \frac{\varepsilon c^2}{\beta}. \tag{5.3}$$

Next we prove uniqueness. Suppose there are two solutions $Z = Z_1(W)$ and $Z = Z_2(W)$ with $Z \to -1$ as $W \downarrow 0$. Since

$$\frac{1}{Z_1 + 1} - \frac{1}{Z_2 + 1} = -\frac{1}{(\tilde{Z} + 1)^2}(Z_1 - Z_2),$$

where $\tilde{Z}$ lies between Z_1 and Z_2, we have for $Y = Z_1 - Z_2 > 0$, say,

$$\frac{dY}{dW} - b\frac{Y}{W} \sim -b'\frac{Y}{W^2} \quad \text{as } W \to 0,$$

where $b = \frac{\beta}{2\beta+\gamma+2-\alpha}$, and $b' = \frac{\beta}{\varepsilon c^2(2\beta+\gamma+2-\alpha)}$. Here we used (5.3). Hence $Y \to \infty$ as $W \downarrow 0$, contradicting (5.3).

Expressing W and Z in terms of f we observe that (5.3) implies the behaviour

$$f' \sim \frac{c}{\beta} f^{\beta+2-\alpha} \quad \text{as } f \to 0.$$

Since $\beta + 2 - \alpha < 1$ we find that f reaches zero at some finite η-value; i.e. the travelling wave has a finite front. □

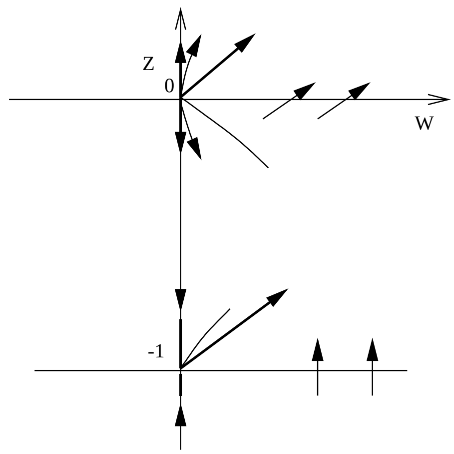


Figure 3: Phase plane realted to (5.2), with $q = \frac{\beta}{1+\gamma}$.

Note that we did not put any restriction on ε. Thus the conclusion about the behaviour as $f \to 0$, provided $2\beta > \alpha - \gamma - 2$, is valid for any value of $\varepsilon \geq 0$. This ε-uniformity is lost in the next case.
Indeed, the behaviour of the travelling wave near the front when it vanishes corresponds to criterion (1.9) in the case where the capillary damping is absent.

5.2 $2\beta = \alpha - \gamma - 2$.

Now we take $q = \frac{\beta}{1+\gamma}$, and set $W = u^{\frac{\alpha-1}{1+\gamma}}$. Then (5.1) becomes

$$(\alpha - 1)\frac{dZ}{dW} = \left(\beta Z + \frac{\varepsilon c(c - W)}{(Z + 1)}\right)\frac{1}{W}. \tag{5.4}$$

Again we look for solutions satisfying $Z > -1$ and $Z \to Z_0$ as $W \to 0$. It follows that

$$Z_0^{\pm} = -\frac{1}{2} \pm \frac{1}{2}\left(1 - \frac{4\varepsilon c^2}{\beta}\right)^{\frac{1}{2}},$$

implying $\varepsilon \leq \frac{\beta}{4c^2}$. With reference to Figure 4 we have

Proposition 5.2 *(i) If $\varepsilon < \frac{\beta}{4c^2}$, there exists a family of orbits satisfying $v \sim Z_0^+ u^{-\frac{\beta}{1+\gamma}}$ as $u \to 0$, and a unique orbit, denoted by C^0, which satisfies $v \sim Z_0^- u^{-\frac{\beta}{1+\gamma}}$ as $u \to 0$. All cases give travelling waves with finite fronts. In particular the orbit C^0 implies*

$$f' \sim \frac{1}{\varepsilon c}\left(\frac{1}{2} - \left(\frac{1}{2} - \frac{4\varepsilon c^2}{\beta}\right)^{\frac{1}{2}}\right) f^{\beta+2-\alpha}, \quad \textit{as } f \to 0.$$

(ii) If $\varepsilon = \frac{\beta}{4c^2}$ *there exists a family of orbits having* $v \sim -\frac{1}{2}u^{-\frac{\beta}{1+\gamma}}$ *as* $u \to 0$. *In particular there is a unique orbit, again denoted by* C^0, *satisfying* $u^{\frac{\beta}{1+\gamma}}v \uparrow -\frac{1}{2}$ *as* $u \to 0$. *The orbit* C^0 *implies again a traveling wave with a finite front, such that*

$$f' \sim \frac{2c}{\beta} f^{\beta+2-\alpha}, \quad \text{as } f \to 0.$$

(iii) If $\varepsilon > \frac{\beta}{4c^2}$, *there is no orbit with* $v > -u^{\frac{-\beta}{1+\gamma}}$ *and* $u \to 0$.

Remark 5.1 *(i) Comparing (3.1) and (3.8) we observe that*

$$\varepsilon^0_{\frac{1}{2}} = \varepsilon^* = \frac{\beta^2}{4(\alpha-1)c^2} < \frac{\beta}{4c^2},$$

implying that, depending on the ε*-value, monotone and oscillatory waves with finite fronts occur.*
(ii) Since now $\beta + 2 - \alpha = -\beta - \gamma < 0$, *finite front waves have* $f' \to \infty$ *as* $f \to 0$.

Proof. *(i)* It is immediate from the phase-plane in Figure 4 that $(0, Z_0^+)$ is a source, and $(0, Z_0^-)$ is a saddle, with one unique orbit $Z = Z(W)$ leaving in the direction $W > 0$. It behaves as $Z - Z_0^- \sim \left(\frac{1}{2} - \frac{1}{2}(1 - \frac{4\varepsilon c^2}{\beta})^{\frac{1}{2}}\right)$ as $W \to 0$.
(ii) Now $Z_0 = -\frac{1}{2}$ at local analysis shows that $(0, Z_0)$ is a saddle-node, with a unique orbit in the direction of $W > 0$. This orbit does not cross the isocline, and behaves as $Z - Z_0 \sim \frac{1}{2}W$.
(iii) Now the segment $\{(W, Z) : \ W = 0, Z \in (-1, 0)\}$ is disconnected from the isocline and hence no connecting orbit exists. □

5.3 $2\beta < \alpha - \gamma - 2$.

Here we choose $q = \frac{\alpha-2-\gamma}{2(1+\gamma)} > \frac{\beta}{1+\gamma}$ and $W = u^{q-\frac{\beta}{1+\gamma}}$, yielding the equation

$$(\alpha - \gamma - 2 - 2\beta)\frac{dZ}{dW} = (\alpha - \gamma - 2)\frac{Z}{W} + \frac{2\varepsilon c(c - W^{\frac{2(\alpha-1)}{\alpha-\gamma-2-2\beta}})}{(W+Z)W} \tag{5.5}$$

Proposition 5.3 *There exists no orbits with* $v > -u^{-\frac{\beta}{1+\gamma}}$ *and* $u \to 0$.

Proof. Suppose such an orbit exists. Then we would have

$$(\alpha - \gamma - 2 - 2\beta)\frac{dZ}{dW} \sim \frac{1}{W}\left((\alpha - \gamma - 2)Z + \frac{2\varepsilon c^2}{Z}\right), \ Z > 0,$$

as $W \downarrow 0$. Since $(\alpha - \gamma - 2)Z + \frac{2\varepsilon c^2}{Z}$ is negative and bounded away from zero, this gives a contradiction. □

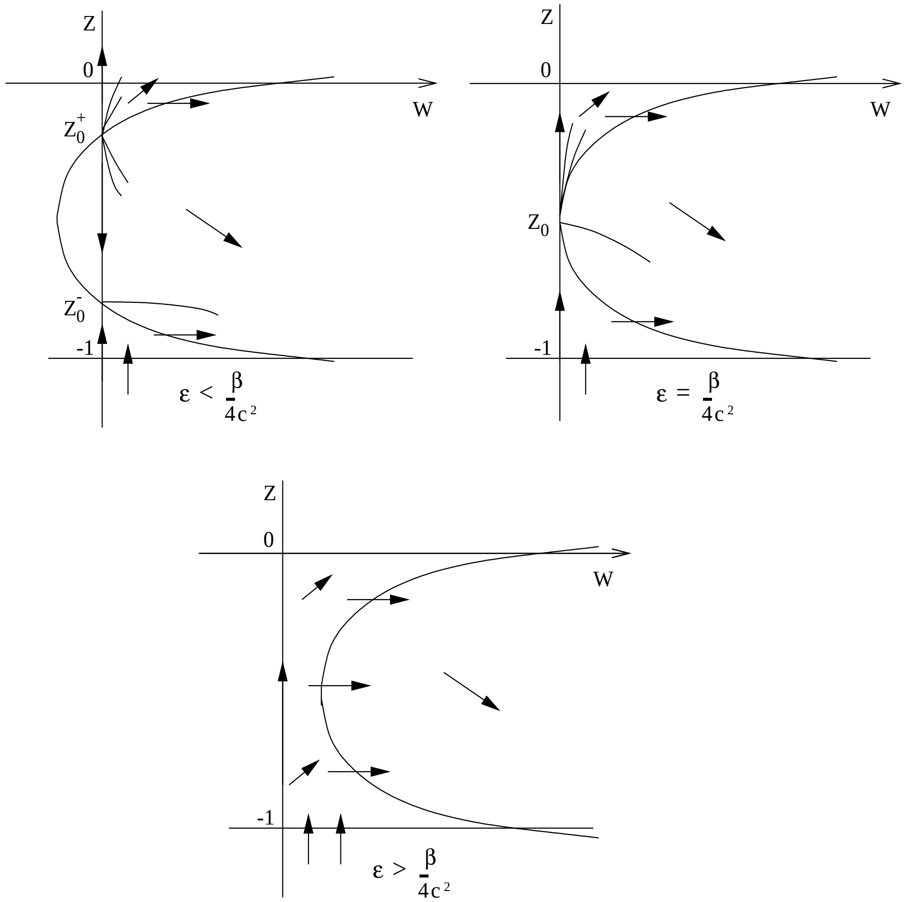


Figure 4: $q = \frac{\beta}{1+\gamma}$, $2\beta = \alpha - \gamma - 2$ phase-plane for different ε-values.

6 $\delta \to 0$ limit

For the purpose of this section we denote the connecting orbit of (P_ε^δ) by $v = \varphi_\delta(u)$ and, in the cases for which C^0 from Proposition 5.1 and 5.2 exists, we call its graph $v = \varphi_0(u)$.

Lemma 6.1 *There exists $\delta^* > 0$ such that, with $c = c(\delta) = \frac{A^\alpha - \delta^\alpha}{A - \delta}$,*

$$\mathcal{F}_u(\delta) = c(\delta) G_\delta(u) = c(\delta)\left(-1 + c(\delta) u^{\frac{1-\alpha}{1+\gamma}} - c(\delta) A - A^\alpha u^{-\frac{\alpha}{1+\gamma}}\right)$$

is decreasing in $0 \le \delta < \delta^*$ for any fixed $0 < u \le \left(\frac{A}{2}\right)^{1+\gamma}$.

Proof. Since

$$\frac{d\mathcal{F}_u}{d\delta} = \frac{dc}{d\delta}\left(-1 + 2cu^{\frac{1-\alpha}{1+\gamma}} + (A^\alpha - 2cA)u^{-\frac{\alpha}{1+\gamma}}\right),$$

and $\frac{dc}{d\delta} > 0$, we need to show that the term between brackets is negative for small δ. At $\delta = 0$ it becomes

$$-1 + A^{\alpha-1} u^{-\frac{\alpha}{1+\gamma}}(2u^{\frac{1}{1+\gamma}} - A) < 0,$$

for all $0 < u < \left(\frac{A}{2}\right)^{1+\gamma}$. □

Proposition 6.1 *For α, β, γ such that $2\beta > \alpha - \gamma - 2$, or $2\beta = \alpha - \gamma - 2$ and $\varepsilon \in (0, \frac{\beta}{4c^2})$ fixed, translate f_δ such that $f_\delta(0) = \frac{A}{2}$ for all $\delta \in (0, \delta^*)$. Then $f_\delta \to f \in C^\infty(\mathbb{R})$ uniformly on $\mathbb{R}$. Hence f satisfies Problem TW with $\delta = 0$.*

Proof. It will be sufficient to show that $\varphi_\delta(u) \to \varphi_0(u)$ locally uniformly. By Lemma 6.1 we have for any $0 < \bar{\delta} \leq \delta^*$

$$-u^{-\frac{\beta}{1+\gamma}} < \varphi_{\delta_1}(u) < \varphi_{\delta_2}(u) < \varphi_0(u)$$

for $0 < \delta_1 < \delta_2 < \delta^*$ and $u \in (\bar{\delta}, \left(\frac{A}{2}\right)^{1+\gamma})$. Also

$$\frac{\varepsilon c(\delta)}{1+\gamma}\frac{G_\delta(u)}{F_\delta(u,v)} \to \frac{\varepsilon A^{\alpha-1}}{1+\gamma}\frac{G_0(u)}{F_0(u,v)} \text{ as } \delta \to 0$$

uniformly on $[\bar{\delta}^{1+\gamma}, \left(\frac{A}{2}\right)^{1+\gamma}]$. Therefore

$$\varphi_\delta(u) \uparrow \bar{\varphi}(u) \leq \varphi_0(u)$$

where $v = \bar{\varphi}_0(u)$ is a solution of (P_ε^0). The reasoning above holds for every $0 < \bar{\delta} \leq \delta^*$, which implies that $\bar{\varphi}_0(u)$ exists for all $u \in (0, \left(\frac{A}{2}\right)^{1+\gamma})$. In view of Section 5 and $\bar{\varphi} \leq \varphi_0$ this implies that $\bar{\varphi}_0(u) = \varphi_0(u)$.
Using that $f_\delta(0) = \frac{A}{2}$ is fixed for all $0 < \delta < \delta^*$, standard arguments imply that f_δ converges uniformly on $(-\infty, \frac{A}{2})$ to the corresponding solution of the limit equation. Existence of global travelling waves and uniqueness of the initial value problem for all $0 \leq \delta < \delta^*$, implies the uniform convergence on $\mathbb{R}$.

□

7 Concluding Remarks

In this paper we present a model for unsaturated groundwater flow which includes an expression for the non-static phase pressure difference, see (1.4). Replacing the nonlinearities in the transport equation by power-law expressions we arrive at (1.7). We study travelling wave solutions representing moisture profiles moving downwards due to gravity.

For positive initial saturation ($\delta > 0$) we demonstrate existence and uniqueness (up to translations). Small values of the damping coefficient ε result in monotone saturation profiles. Large values of ε result in profiles which exhibit oscillatory behaviour near the injection saturation A.

When initially no moisture is present ($\delta = 0$) existence of bounded travelling waves depends critically on the exponents of the power-law expressions. This is related to the occurrence of finite fronts in the moisture profiles: i.e. descending planes (in the direction of gravity) below which the water saturation remains zero. Related to equation (1.7) we have shown the following.

If $2\beta > \alpha - \gamma - 2$, then travelling wave solutions with fronts exist for all $\varepsilon > 0$. In other words, for $S(z,t) = f(\eta)$, with $\eta = z + ct$, there exists $\eta_0 \in \mathbb{R}$ such that $f(\eta) = 0$ for all $\eta \leq \eta_0$. Moreover near $\eta = \eta_0$ the profile satisfies

$$f' \sim \frac{A^{\alpha-1}}{\beta} f^{\beta+2-\alpha}.$$

This corresponds to the front behaviour of solutions of the convection diffusion equation under static conditions (equation (1.8)); i.e. ε and γ are absent in this asymptotic expression.

If $2\beta = \alpha - \gamma - 2$ we obtain a similar result provided the damping coefficient ε is sufficiently small: i.e. $\varepsilon \leq \frac{\beta}{4A^2(\alpha-1)}$. For larger values of ε, no waves exist satisfying $f(-\infty) = 0$. Finally, if $2\beta < \alpha - \gamma - 2$, again no such waves exist, regardless the value of $\varepsilon > 0$.

We also investigate the limit $\varepsilon \to 0$ (for $\delta > 0$, fixed) and $\delta \to 0$ (for $\varepsilon > 0$ fixed). In particular the latter provides a uniqueness criterion for the degenerate case when $\delta = 0$. We also note that the limits $\varepsilon \to 0$ and $\delta \to 0$ do not commute: $\varepsilon \to 0$ followed by $\delta \to 0$ is always possible, while $\delta \to 0$ followed by $\varepsilon \to 0$ is only possible when $2\beta \geq \alpha - \gamma - 2$.

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