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Computational aspects of alternative portfolio selection models in the presence of discrete asset choice constraints

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Abstract

We consider the mean-variance (M-V) model of Markowitz and the construction of the risk-return efficient frontier. We examine the effects of applying buy-in thresholds, cardinality constraints and transaction roundlot restrictions to the portfolio selection problem. Such discrete constraints are of practical importance but make the efficient frontier discontinuous. The resulting quadratic mixed-integer (QMIP) problems are NP-hard and therefore computing the entire efficient frontier is computationally challenging. We propose alternative approaches for computing this frontier and provide insight into its discontinuous structure. Computational results are reported for a set of benchmark test problems.

1. Introduction

We consider the portfolio selection model of Markowitz (1952, 1959, 1987) that laid the foundations of modern portfolio theory (MPT); see Constantinides and Malliaris (1995) for a survey. Markowitz shows how rational investors can construct optimal portfolios under conditions of uncertainty. The mean and variance of a portfolio's return represent the benefit and risk associated with the investment.

Markowitz shows that for a rational investor, maximizing expected utility, a chosen portfolio is optimal with respect to both expected return and variance of return. He defines such a non-dominated portfolio as *efficient*, that is, it offers the highest level of expected return for a given level of risk and the lowest level of risk for a given level of return. His normative mean-variance rule for investor behaviour both implies and justifies the observable phenomenon of diversification in investment. Determining the efficient set from the investment opportunity set, the set of all possible portfolios, requires the formulation and solution of a parametric quadratic program (QP). Plotted in risk-return space the *efficient set* traces out the *efficient frontier*.

Hanoch and Levy (1969) show that the M-V criterion is a valid efficiency criterion, for any individual's utility function, when the distributions considered are Gaussian normal. A study comparing alternative utility functions appears in Kallberg and Ziemba (1983). They show that portfolios with 'similar' absolute risk-aversion indices have 'similar' optimal compositions, regardless of the functional form and parameters of the utility function. Hence, M-V analysis is justified for any general concave utility function of the Von Neumann–Morgenstern type (Von Neumann and Morgenstern 1944).

The estimation of the underlying parameters (returns, variances and covariances) which are required as the input to M-V analysis is an important modelling step. Small changes in the inputs can have a large impact on the optimal asset weights. Chopra and Ziemba (1993) found that, for a

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typical investor's risk-tolerance level, errors in the forecast means are more than ten times as important as errors in the variances and about twenty times as important as errors in covariances. For practical aspects of portfolio analysis see Perold (1984), Hensel and Turner (1998) and Grinold and Kahn (1995). MPT has developed in tandem with simplifications to the QP required by M-V analysis. These simplifications centre around linearizing the quadratic objective function or reducing the number of parameters to be estimated. Both approaches involve either the approximation or decomposition of the covariance matrix.

Tobin (1958) developed the separation theorem which states that, in the presence of a risk-free asset, the optimal risky portfolio can be determined without any knowledge of investor preferences. Ziemba *et al* (1974) show that the solution to the portfolio problem involving a risk-free asset can be obtained by a two-stage process; first solving a deterministic linear complementarity problem and then a univariate stochastic program.

Sharpe (1963) proposed that the single-index, or 'market', model was a sufficient model of covariance. Subsequently, Sharpe (1964), Lintner (1965) and Mossin (1966) independently developed the capital asset pricing model. This linear model of equilibrium asset prices explains the covariance of asset returns solely through their covariance with the market. King (1966) presented evidence of the influence of industry factors that the market model did not take into account. Rosenberg (1974) presented a multifactor model that incorporated industry and other factors. Ross (1976) using factor analysis, developed the arbitrage pricing theory, which is a multi-index equilibrium model.

Index or factor models allow a simplification of the underlying QP. The covariance matrix can be expressed in a diagonal form and hence a linear approximation of the quadratic objective function can be obtained (see Sharpe 1971). In the case that an index or factor model is not employed, the nature of the covariance matrix also permits a natural decomposition of the quadratic objective term into a weighted sum of squares (see Vanderbei and Carpenter 1993). The technique of manipulating a quadratic objective function into a diagonal (variable separable) form and representing it by a piecewise linear polygonal approximation is also well known to practitioners.

A number of researchers have introduced alternative measures of risk for portfolio planning. In many cases these measurements are linear, leading to a corresponding simplification in the computational model. Konno and Yamazaki (1991) show that the mean absolute deviation (MAD) of returns is a risk measure equivalent to variance, under the assumption of multivariate normal returns. A generalization of the MAD model can be found in Worzel *et al* (1994). Speranza (1996) considers only the mean absolute value of negative deviations. Markowitz (1959) suggested that semivariance is the real cause for concern but variance is employed as the risk measure as it is more tractable computationally and reveals the same information. Downside risk measures are typically used in dynamic asset allocation problems (see Cariño and Ziemba (1998) and Zenios (1995) for a classification of asset management models). Multi-objective goal programming approaches have been proposed by Lee (1972) and Lee and Chesser (1980). Young (1998) employs a minimax investment rule measuring risk as the minimum return (maximum loss) that the portfolio would have achieved over all of the past observation periods. A survey of alternative portfolio selection models appears in Horniman *et al* (2000).

Simplifications to the basic quadratic programming problem have allowed the models to be extended to perform more realistic analysis, incorporating market imperfections. Rudd and Rosenberg (1979) consider linear transaction costs and Konno and Yamazaki (1991) claim that an advantage of the MAD model is that it limits the number of stocks held, allowing control of transaction costs. Adcock and Meade (1994) combine a modulus function for transaction costs with the usual quadratic objective. Young (1998) describes how the minimax model can be adapted to include linear transaction costs.

To capture the realism of portfolio planning a number of discrete restrictions have been considered by different researchers. These are summarized below.

(i) A *buy-in threshold* is defined as the minimum level below which an asset is not purchased. This requirement eliminates unrealistically small trades that can otherwise be included in an optimum portfolio.
(ii) *Cardinality constraints*: investors may wish to specify the number of assets in their portfolio for the purpose of monitoring and control.
(iii) *Roundlots* are defined as discrete numbers of assets which are taken as the basic unit of investment. Investors are restricted to making transactions only in multiples of these roundlots. This overcomes the assumption of the infinite divisibility of assets inherent in the M-V rule.

Cardinality constraints are intrinsically linked with buy-in thresholds; for example, a buy-in threshold of 5% of the value of a portfolio ensures that there can be no more than 20 stocks purchased. Representing the first two classes of problems requires the introduction of binary variables and the third class of problems requires the use of general integer variables. As a consequence the optimization model under consideration changes from a QP to a QMIP. The problem increases in size and becomes computationally complex. Mansini and Speranza (1999) have shown that finding a feasible solution to the portfolio selection problem with roundlots is NP-complete.

Bienstock (1996) and Lee and Mitchell (1997) use QMIP techniques to solve the portfolio selection problem with an upper limit on the size of the portfolios. Chang *et al* (1999) use heuristic algorithms (genetic algorithm, tabu search and simulated annealing) to solve cardinality constrained problems with specified portfolio sizes. Linear programming based heuristics are used by Speranza (1996), considering the negative semiMAD model with cardinality constraints. This model is extended in Mansini and Speranza (1999) to incorporate roundlots and then in Kellerer, Mansini and Speranza (1997) incorporating both roundlots and fixed costs.

The rest of this paper is organized as follows. In section 2 we introduce the underlying portfolio planning model and discuss the construction of the efficient frontier using two different QP models. The algebraic formulations of the M-V model including buy-in thresholds, cardinality constraints and roundlot restrictions are introduced in section 3. We also discuss some aspects of discontinuities and 'missing' sections of the efficient frontier in the presence of discrete constraints (DCEF). In section 4 we discuss our solution methods and present the computational results for data sets taken from five major stock markets. In addition, we also investigate the effects of the discrete constraints in the context of the portfolio rebalancing problem. In section 5 we discuss our research findings and present our conclusions. Appendix A contains the data set used to illustrate the shapes of some example DCEFs in section 4.2.

2. Mean-variance model

We introduce the basic notation, define the classical M-V model and describe an additional well known way to computing the 'Markowitz' efficient frontier (MEF).

Indices. $i, j = 1, \ldots, N$: denote the different risky assets.

Parameters. μ_i: the expected return of asset i. σ_{ij}: the covariance between asset i and asset j ($\sigma_{ii} = \sigma_i^2$ is the variance of asset i). ρ: the desired level of return for the portfolio.

Decision variables. x_i: the fraction of the portfolio value invested in asset i.

QP1

$$\text{Min } Z_{\text{QP1}} = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

subject to

$$\sum_{i=1}^{N} x_i \mu_i = \rho$$
$$\sum_{i=1}^{N} x_i = 1$$
$$x_i \geqslant 0 \qquad i = 1, \ldots, N$$

Varying the desired level of return, ρ, in QP1 and repeatedly solving the quadratic program identifies the minimum variance portfolio for each value of ρ. These are the efficient portfolios that compose the efficient set. Plotting the corresponding values of the objective function and ρ, variance and return respectively, traces the MEF in the mean-variance plane. Markowitz (1956) describes a 'critical line' solution algorithm tracing out the efficient frontier by identifying 'corner' portfolios—points at which a stock either enters or leaves the current portfolio. It is normal practice to use standard deviation rather than variance as the risk measure because the σ–ρ frontier is linear if a risk-free asset exists, see Tobin (1958) and Ziemba *et al* (1974).

An alternative formulation of QP1 explicitly trades risk against return in the objective function using the Arrow–Pratt absolute risk-aversion index R_A (see Kallberg and Ziemba 1983). R_A is defined as

$$R_A = -\frac{u''(w)}{u'(w)}$$

where w is portfolio wealth and u', u'' are the first and second derivatives of a Von Neumann–Morgenstern utility function u.

QP2

$$\text{Max } Z_{\text{QP2}} = \sum_{i=1}^{N} x_i \mu_i - \frac{R_A}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

subject to

$$\sum_{i=1}^{N} x_i = 1$$
$$x_i \geqslant 0 \qquad i = 1, \ldots, N$$

By increasing R_A from zero and solving the family of QPs we trace out the efficient frontier. Empirical results by Kallberg and Ziemba (1983) show that $R_A \geqslant 6$ leads to very risk-averse portfolios, $2 \leqslant R_A \leqslant 4$ represents moderate absolute risk aversion and $R_A \leqslant 2$ leads to risky portfolios. $R_A = 4$ corresponds approximately to pension fund management (typically, holdings of 60% stocks and 40% bonds). In practice it is common to model the risk-return trade-off using a parameter λ, $0 \leqslant \lambda \leqslant 1$, with the objective function

$$\text{Min } Z = \lambda \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij} - (1 - \lambda) \sum_{i=1}^{N} x_i \mu_i.$$

Setting

$$\frac{R_A}{2} = \frac{\lambda}{(1 - \lambda)}$$

shows equivalence with the objective function in QP2.

3. Efficient frontier with discrete constraints

3.1. Discontinuities in the DCEF

Discrete constraints, representing practical trading requirements, introduce discontinuities into the otherwise continuous efficient frontier. To illustrate the appearance of the discontinuities, we consider the small four-stock example from Chang *et al* (1999) with the following expected returns, standard deviations and correlations (see table 1).

The MEF for this data set is shown in figure 1. From the four stocks we are to choose a portfolio containing only two stocks. We can identify our opportunity set, figure 2, by considering the six pairwise combinations of stocks. Ordering by risk and return, we eliminate the inefficient portfolios to reveal the discontinuous DCEF shown in figure 3.

Table 1.

	Correlation matrix				Expected return	Standard deviation
Stock Number	1	2	3	4		
1	1				0.004 798	0.014 635
2	0.118 368	1			0.000 659	0.030 586
3	0.143 822	0.164 589	1		0.003 174	0.030 474
4	0.252 213	0.099 763	0.083 122	1	0.001 377	0.035 770

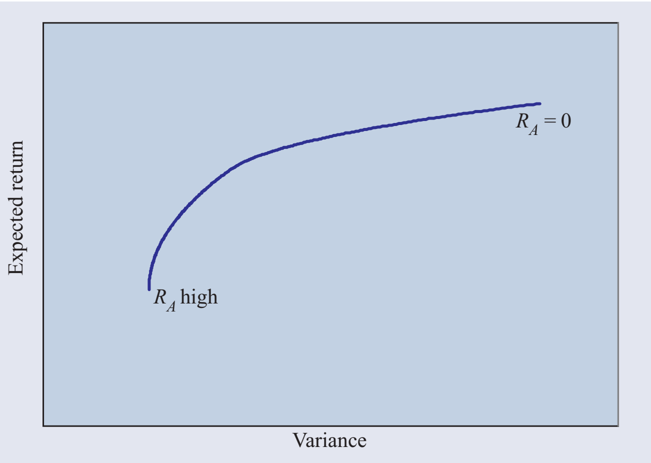

Figure 1. Four-stock example: MEF.

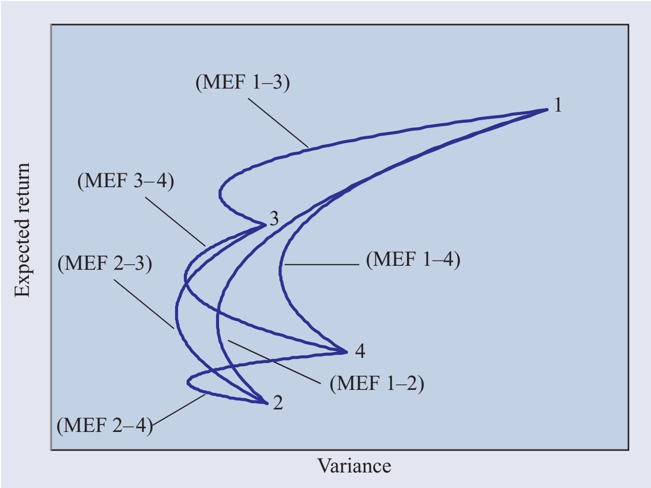

Figure 2. Four-stock example: investment opportunity set.

3.2. Representation of discrete constraints

In the presence of buy-in constraints the portfolio weights behave as semicontinuous variables (see Beale and Forrest 1976) and are modelled using variable upper and lower bounds in the following way. A binary variable, δ_i, and finite lower and upper bounds, l_i and u_i, respectively, are associated with each asset $i = 1, \ldots, N$. The buy-in thresholds are represented by the constraint pair

$$l_i\delta_i \leqslant x_i \leqslant u_i\delta_i \qquad \text{and} \qquad \delta_i = 0, 1 \qquad i = 1, \ldots, N.$$

We refer to model BUY-IN as QP1 with the above constraints added. Cardinality constraints require buy-in thresholds to be applied. They are simply modelled by constraining the sum of the binary variables to be equal to k,

$$\sum_{i=1}^{N} \delta_i = k,$$

where k represents the number of assets to be in the portfolio. Model CARD is model BUY-IN with this additional cardinality constraint.

CARD

$$\text{Min } Z_{\text{CARD}} = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

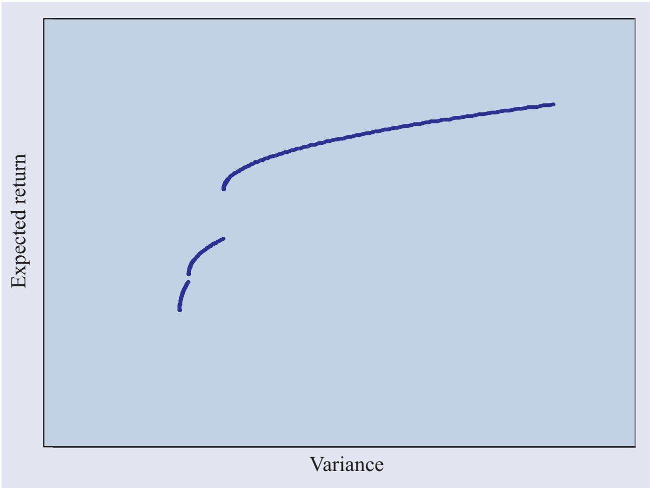

Figure 3. Four-stock example: DCEF.

subject to

$$\sum_{i=1}^{N} x_i \mu_i = \rho$$

$$\sum_{i=1}^{N} x_i = 1$$

$$l_i\delta_i \leqslant x_i \leqslant u_i\delta_i \qquad i = 1, \ldots, N$$

$$\sum_{i=1}^{N} \delta_i = k$$

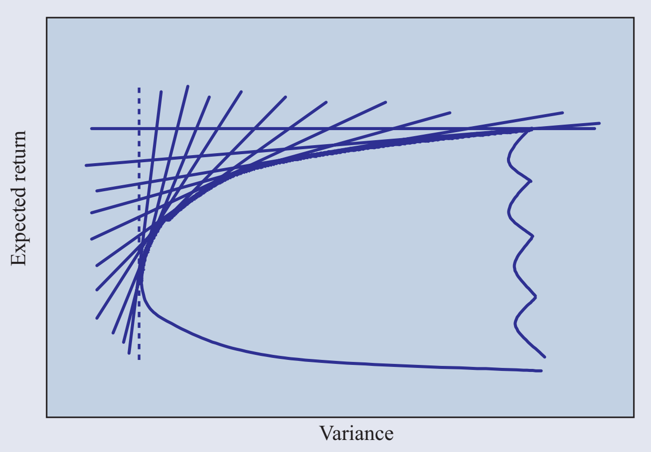

Figure 4. Tracing out the efficient frontier.

$$\delta_i = 0 \text{ or } 1 \qquad i = 1, \ldots, N.$$

Transaction roundlots, described as either numbers of stocks, or as a cash value, are expressed as a fraction, f_i, of the portfolio wealth. The portfolio weights are then defined in terms of f_i and an integer number of roundlots, y_i. Thus, portfolio weights x_i are re-expressed as $x_i = y_i f_i$, $i = 1, \ldots, N$. Applying roundlot constraints, it may not be possible to exactly satisfy the budget requirement $\sum_{i=1}^{N} x_i = 1$. Therefore, this restriction is made 'elastic' using undershoot and overshoot variables, ϵ^- and ϵ^+, respectively, which are penalized in the objective function with a high cost, γ. In an optimum solution ϵ^- and ϵ^+ are made as small as possible so that the fractional stock holdings x_i sum to a value 'as close as possible' to 1.

LOT

$$\text{Min } Z_{\text{LOT}} = \sum_{i=1}^{N} \sum_{j=1}^{N} y_i f_i y_j f_j \sigma_{ij} + \gamma\epsilon^- + \gamma\epsilon^+$$

subject to

$$\sum_{i=1}^{N} y_i f_i \mu_i = \rho$$

$$\sum_{i=1}^{N} y_i f_i + \epsilon^- - \epsilon^+ = 1$$

$$l_i \leqslant y_i f_i \leqslant u_i \qquad i = 1, \ldots, N$$

$$y_i \text{ integer} \qquad i = 1, \ldots, N$$

$$\epsilon^-, \epsilon^+ \geqslant 0$$

Buy-in thresholds and cardinality constraints can also be applied to model LOT.

3.3. Invisible sections of the DCEF

To generate the DCEF for the small example in section 3.1 we are able to use complete enumeration. The same frontier can be generated by repeatedly solving model CARD with $k = 2$. Model CARD is based on QP1, however, using QP2 as the underlying model prevents the full DCEF from being

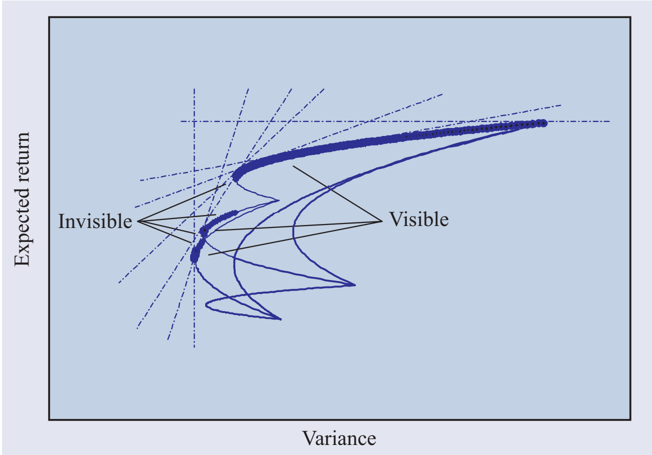

Figure 5. The gradient of the objective function of the 'lambda' formulation, model QP2.

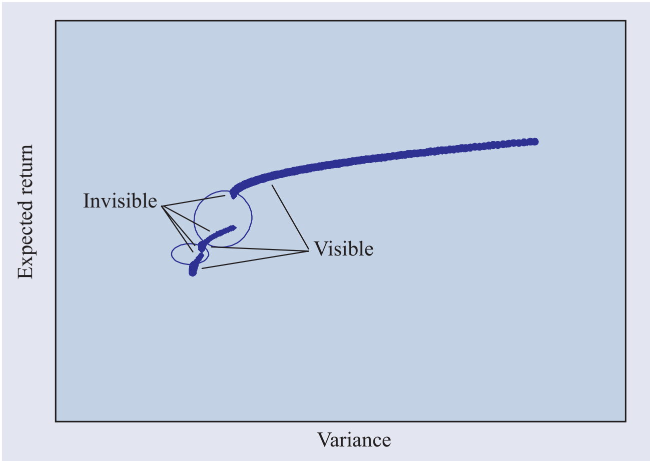

Figure 6. 'Invisible' sections of the DCEF.

generated. In order to explain the 'missing' sections, consider the objective function of QP2

$$\text{Max } Z_{\text{QP2}} = \sum_{i=1}^{N} x_i \mu_i - \frac{R_A}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}.$$

This can be rewritten as the equation of the straight line, $e = mv + c$, where e is the expected return, v is the variance, $m = \frac{R_A}{2}$ and Z_{QP2} is the e-intercept, c. Maximizing Z_{QP2}, for any given value of R_A, corresponds to maximizing the e-intercept, for a specified gradient. Drawing, over the feasible region, the family of lines described by any non-negative value of R_A, the e-intercept is maximized, uniquely, at the point of tangency with the upper-left border of the region. Systematically varying R_A, from zero upwards, changes the point of tangency and traces out the entire frontier; see figure 4.

Applying this method to the non-convex region in figure 2, we begin on the curve MEF 1–3 and slide continuously down until we find tangency with the curve MEF 2–4. The result is that our point of tangency 'jumps' to the lower curve maintaining smoothness in the increasing gradient but missing out some efficient points; see figure 5. The missing parts of the DCEF are also shown in figure 6.

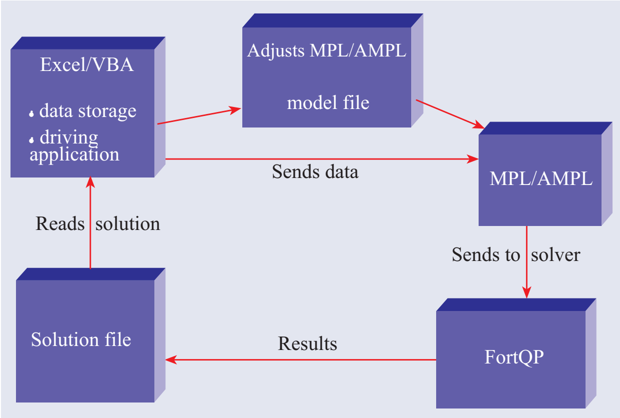

Figure 7. Data-modelling-solver architecture.

4. DCEF—a computational study

We investigate the shape of the DCEF for the BUY-IN, CARD and LOT models using 60 monthly returns (October 1994 to September 1999) for 30 stocks drawn randomly from the FTSE 100, see appendix A. We consider model CARD in detail for 5 data sets drawn from the Hang Seng, Dax, FTSE, S&P and Nikkei indices with 31, 85, 89, 98 and 225 stocks respectively (Beasley 1999). To compute the solutions to these models within an acceptable time frame we use two heuristic solution procedures; 'integer restart' and a two-stage 'reoptimization' heuristic. We compare our results to those of Chang *et al* (1999), obtained using modern heuristic methods (genetic algorithm, tabu search and simulated annealing). In our computational study we also examine the behaviour of the model and its solutions when the QP and QMIP are used to solve a portfolio rebalancing problem, see section 4.3.

4.1. Implementation

Software tools

The models are implemented using MPL, a mathematical programming language (Maximal 1999). We have also developed an alternative AMPL (Fourer *et al* 1993) environment. The solver system uses FortQP (Mitra *et al* 2000) and a VBA routine drives the system from EXCEL which is also used as the repository of the input data and output results, see figure 7. The system runs under Windows NT and in our computational work we have used a PC with a Pentium III, 500 MHz processor and 128 MB RAM.

Solution methods

Each point of the DCEF curve represents the global optimum solution of a 'discrete non-convex' optimization problem. Given that the quadratic form for the minimization problem is positive semidefinite, relaxing the discreteness restriction on the variables leads to a convex programming problem. This continuous variable QP relaxation of the problem provides a lower bound and is easily embedded (see Mitra (1976) and Lawler and Wood (1966)) in a branch-and-bound tree search paradigm.

The FortQP system implemented within the FortMP solver (Ellison *et al* 1999) has both interior point method (IPM) and sparse simplex (SSX) solution capabilities. The system is extensively tested using QLIB test data (Maros and Meszaros 1997) and models from the finance industry. For the given family of QMIP problems at hand the branch-and-bound algorithm has been specially constructed taking into consideration the following design issues:

SSX versus IPM. In medium-to-large test problems IPM performs better than SSX. Yet as an embedded solver of subproblems within branch-and-bound IPM is not well suited since the 'warm start' property is relatively poor. We have therefore chosen SSX as our embedded 'optimization engine' for solving subproblems. The dual algorithm is used to solve these subproblems efficiently.

Information sharing and algorithm choice. In solving the subproblems in the child node we share (reuse) the optimum basis information (basis list and the basis factors) of the parent node. We also apply the dual algorithm which reduces the total number of pivotal steps for reoptimization. These features also justify our choice of algorithm and vindicates the useful 'warm start' properties of the SSX.

Integer restart heuristic. In the construction of the DCEF involving, say, 500 points we are unlikely to solve all of these models to QMIP optimality. As a consequence, we are likely to lose the 'pareto efficient' property of the frontier and our experiments confirm this. We do, however, adopt a scheme of computing the DCEF from the highest return, and its corresponding risk, to lower return and reduced risk. We use the previous integer solution in this sequence as the 'first feasible and upper bounding QP value' for the next point (problem). Given the previous solution is feasible (or optimal), this solution is automatically a feasible solution for the current optimization problem, as we decrease the desired level of return from its highest value to the smallest and hence relax the constraint. This has the effect that we obtain an 'efficient' DCEF which is optimal (if all problems are solved to optimality) or suboptimal (if the algorithm is terminated at a feasible solution). However, the frontier we generate cannot contain inefficient points as we either stay at the previous solution or we improve on it. We believe, and our experimental results vindicate (see section 4.2), that this approach is preferable to applying modern heuristics to this discrete non-convex programming problem.

Reoptimization heuristic. To reflect common practice (in the absence of a QMIP solver) we employ a simple heuristic solving 2 continuous QP problems for each QMIP CARD problem. The portfolio size restriction and buy-in thresholds are initially ignored and the equivalent version of QP1 is solved. If there are at least k stocks in the optimal portfolio, QP1 is solved again using only the k stocks with the largest weights and at the same time we impose the buy-in thresholds as explicit lower bounds. The reoptimization of the auxiliary problem computes a portfolio with exactly k stocks above the appropriate buy-in thresholds.

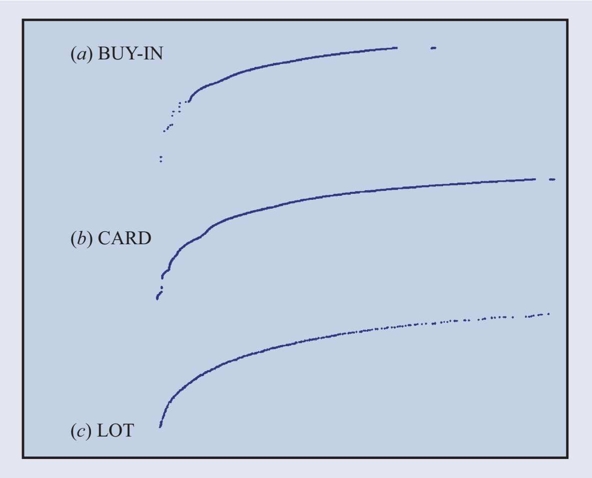

Figure 8. Shapes of DCEF frontiers.

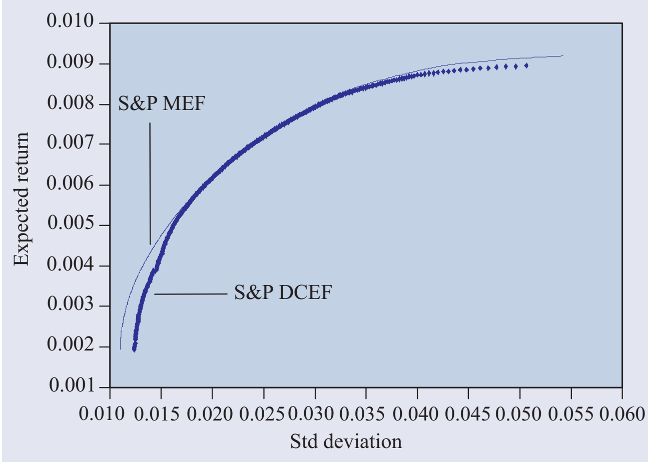

Figure 9. DCEF: S&P 1998 stocks.

4.2. Computational results

Shapes of DCEFs

The discrete efficient frontiers corresponding to the three models, BUY-IN, CARD and LOT, for the 30 stocks from the FTSE 100, are shown in figure 8. For each frontier, 2000 optimization problems were solved and in each instance the optimal solution was found. Figure 8(*a*) shows the DCEF for model BUY-IN with a 20% threshold. Figure 8(*b*) shows the DCEF for model CARD with $k = 4$ and a threshold of 10%. Figure 8(*c*) shows the DCEF for model LOT with a uniform lot size of 5% of the portfolio value. In each instance there are clear discontinuities in the frontiers.

'Integer restart' and 'reoptimization' heuristics

We investigate our heuristic approaches using model CARD for the 5 data sets drawn from the Hang Seng, DAX, FTSE, S&P and Nikkei indices. We set $l_i = 0.01, \quad i = 1, \ldots, N$ and use the cardinality restriction $k = 10$. To analyse the experimental results we follow the metric used in Chang *et al* (1999). The deviations of the points on the heuristically obtained DCEF are measured as the minimum absolute distance (vertical or horizontal) from the MEF. Since they do not calculate the *exact* DCEF but need to measure the usefulness of the heuristically computed frontier points, this deviation measure which they call 'error' provides a reasonable metric for comparison. These reported 'errors' mainly reflect the systematic deviations due to the discrete constraints. Using the same metric allows a comparison with the modern heuristic results of Chang *et al* (1999). For each data set and solution method we generate the frontiers by solving 500 optimization problems. This number is chosen arbitrarily and the points are equally spaced with respect to the decrease in the desired level of return, ρ.

Integer restart heuristic. The QMIP problems are solved to the second, improving, feasible integer solution subject to a limit of 500 nodes in the branch-and-bound algorithm. Figure 9 shows the DCEF for the S&P data set plotted against the MEF.

Table 2 presents the results for the integer restart method applied to the five data sets. The table includes the mean and median percentage errors, the total number of DCEF points computed, the number of integer optimal points and the total solution time in seconds. The number of optimal points obtained does not appear to influence the size of the errors observed, suggesting that when optimality is not reached, the second integer solution is a good approximation of the optimal solution.

For each data set the mean error is below 0.02% with the median error below 0.015%. In all instances, the mean is greater than the median indicating positively skewed error distributions. The size of the errors reported indicate that the DCEFs obtained are very close to the corresponding MEFs. This is borne out by a mean error of 0.008% (median error 0.006%) for the DCEF solved to optimality (3000 points) for the Hang Seng.

In order to establish the computational advantage of the integer restart heuristic we also calculate the DCEF without starting with the previous solution vector. The integer restart heuristic finds more non-dominated points and more optimal points with a smaller mean deviation in less time. To achieve similar error and optimality results the number of nodes to be searched in the B&B algorithm needs to be increased. For example, for the S&P data set the number of nodes has to be increased from 500 to 2500 but the solution time also increases fivefold.

Reoptimization heuristic. The results of the reoptimization heuristic are displayed in table 3 For each data set we consider again 500 points. The discrepancy between this number and the number of discrete points obtained corresponds to those portfolios with less than 10 stocks after the initial optimization and the infeasible solutions (not achieving the desired level of return) from the reoptimization. The number of DCEF points refers to the number of efficient discrete points.

When the reoptimization heuristic can be implemented it appears to offer a good approximation of the true solution. Again the mean errors are all below 0.02% with the median errors all below 0.015%. The reoptimization also reproduces the positive skewness in the error distributions.

Table 2. Results for the integer restart heuristic.

Index	Number of stocks	Total number of DCEF points	Number of integer optimal points	Solution time	Mean error	Median error
Hang Seng	31	500	492	57.55	0.014 15	0.009 97
DAX	85	500	228	8 405.33	0.013 99	0.011 59
FTSE	89	500	244	10 978.12	0.011 41	0.008 60
S&P	98	500	192	15 831.97	0.015 86	0.013 25
Nikkei	225	500	486	18 345.56	0.006 18	0.002 52
Hang Seng	31	3000	3000	382.21	0.008 26	0.006 28

Table 3. Results for the reoptimization heuristic.

Index	Number of stocks	Total number of MEF points	Number of discrete points	Number of DCEF points	Solution time	Mean error	Median error
Hang Seng	31	500	104	103	0.10	0.000 21	0.000 51
DAX	85	500	356	349	37.53	0.014 44	0.011 55
FTSE	89	500	375	355	36.18	0.010 14	0.007 15
S&P	98	500	356	278	44.93	0.016 52	0.013 56
Nikkei	225	500	376	374	280.92	0.003 16	0.001 51

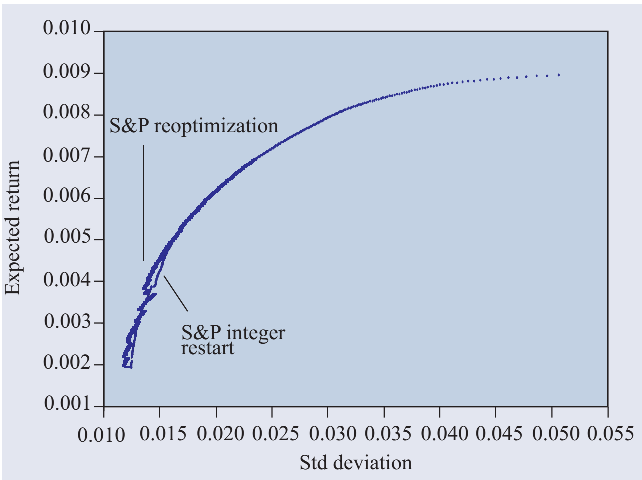

Figure 10. Restart and reoptimization DCEFs: S&P 1998 stocks.

Integer restart versus reoptimization heuristic. The reported errors are similar for both methods although the reoptimization heuristic is faster. However, the reoptimization method cannot generate the entire frontier if any of the portfolios on the MEF contain less than k stocks. Also, reoptimizing can generate inefficient and infeasible portfolios. Ignoring these inefficient portfolios leads to a coarser approximation of the true DCEF. Figure 10 shows all the discrete points (both efficient and inefficient) computed by the reoptimization heuristic plotted against the DCEF for the restart method.

Comparison with modern heuristic methods

Both the integer restart and reoptimization heuristics outperform the modern heuristic methods of Chang *et al* (1999) who report average mean and median deviations in excess of 1% (see table 4). Clearly this makes both of our heuristic schemes very attractive, from the point of view of the quality of the discrete solution. The computational times are difficult to compare. Unfortunately, it is not possible to further compare the results since their full DCEFs are not available (Beasley 2000).

4.3. Investigation of a portfolio rebalancing problem

We extend the CARD model to address the portfolio rebalancing problem. The aim is to identify the trades required to adjust the initial asset holdings such that the optimized portfolio tracks (in terms of variance) a target portfolio or index. The cardinality constraint restricts the number of trades that can be made.

The optimal portfolio weights, x_i, are now defined in terms of the initial holdings, n_i, and the amounts bought, b_i, and sold s_i. Thus, $x_i = n_i + b_i - s_i$. We introduce binary variables, δ_i^{b} and δ_i^{s}, to indicate if asset i is bought or sold. Constraining the sum of each pair to be at most 1 prevents an asset being both bought and sold:

$$\delta_i^{\mathrm{b}} + \delta_i^{\mathrm{s}} \leqslant 1 \qquad i = 1, \ldots, N.$$

Buy-in thresholds, LB_i^{b} and LB_i^{s}, and upper bounds, UB_i^{b} and UB_i^{s}, apply to the buying and selling variables respectively:

$$\delta_i^{\mathrm{b}} LB_i^{\mathrm{b}} \leqslant b_i \leqslant \delta_i^{\mathrm{b}} UB_i^{\mathrm{b}} \qquad i = 1, \ldots, N$$
$$\delta_i^{\mathrm{s}} LB_i^{\mathrm{s}} \leqslant s_i \leqslant \delta_i^{\mathrm{s}} UB_i^{\mathrm{s}} \qquad i = 1, \ldots, N.$$

The cardinality constraint restricts the sum of all the binary variables, the number of trades made, to be no greater than k:

$$\sum_{i=1}^{N} (\delta_i^{\mathrm{b}} + \delta_i^{\mathrm{s}}) \leqslant k.$$

Table 4. Comparison with modern heuristic approaches.

Index	Number of stocks	Solution method	Number of efficient points	Mean error	Median error
Hang Seng	31	Integer restart heuristic	500	0.014 15	0.009 97
			3000	0.008 26	0.006 28
		Rounding heuristic	103	0.000 21	0.000 51
		GA heuristic	1317	0.945 70	1.181 90
		TS heuristic	1268	0.990 80	1.199 20
		SA heuristic	1003	0.989 20	1.208 20
		Pooled (GA, TS, SA)	2491	0.933 20	1.189 90
DAX	85	Integer restart heuristic	500	0.013 99	0.011 59
		Rounding heuristic	349	0.014 44	0.011 55
		GA heuristic	1270	1.951 50	2.126 20
		TS heuristic	1467	3.063 50	2.538 30
		SA heuristic	1135	2.429 90	2.467 50
		Pooled (GA, TS, SA)	2703	2.192 70	2.462 60
FTSE	89	Integer restart heuristic	500	0.011 41	0.008 60
		Rounding heuristic	355	0.010 14	0.007 15
		GA heuristic	1482	0.878 40	0.596 00
		TS heuristic	1301	1.390 80	0.713 70
		SA heuristic	1183	1.134 10	0.636 10
		Pooled (GA, TS, SA)	2538	0.779 00	0.593 80
S&P	98	Integer restart heuristic	500	0.015 86	0.013 25
		Rounding heuristic	278	0.016 52	0.013 56
		GA heuristic	1560	1.715 70	1.144 70
		TS heuristic	1587	3.167 80	1.148 70
		SA heuristic	1284	2.697 00	1.128 80
		Pooled (GA, TS, SA)	2759	1.310 60	1.068 60
Nikkei	225	Integer restart heuristic	500	0.006 18	0.002 52
		Rounding heuristic	374	0.003 16	0.001 51
		GA heuristic	1823	0.643 1	0.606 2
		TS heuristic	1701	0.989 1	0.591 4
		SA heuristic	1655	0.637	0.629 2
		Pooled (GA, TS, SA)	3648	0.569	0.584 4

In practice it is common to employ a factor model to describe asset returns. Tracking a target portfolio then involves replicating the risk profile (the vector of factor sensitivities) of the target portfolio. Let $c = 1, \ldots, C$ denote the factors and f_c the level of the cth factor, β_{ic} the sensitivity of asset i to factor c, α_i the mean return of asset i and ϵ_i the specific return of asset i. The asset returns r_i, are given by the linear form

$$r_i = \alpha_i + \sum_{c=1}^{C} \beta_{ic} f_c + \epsilon_i.$$

The factors are constructed such that there is no correlation between the factors, no correlation between the factors and specific returns and it is assumed that the specific returns are uncorrelated. The variance of returns is given by

$$\mathrm{Var}(r_i) = \sigma_i^2 = \sum_{c=1}^{C} \beta_{ic}^2 \sigma_{f_c}^2 + \sigma_{\epsilon_i}^2.$$

Denoting the sensitivity of the index to factor c by I_c, the portfolio rebalancing model can be stated as follows.

REBALANCE

$$\text{Min } Z_{\text{REB}} = \sum_{c=1}^{C} y_{P,c}^2 \sigma_{f_c}^2 + \sum_{i=1}^{N} x_i^2 \sigma_{\epsilon_i}^2$$

subject to

$$y_{P,c} = \Big(\sum_{i=1}^{N} x_i \beta_{ic}\Big) - I_c \qquad c = 1, \ldots, C$$

$$\sum_{i=1}^{N} x_i \mu_i \geqslant \rho$$

$$\sum_{i=1}^{N} x_i = 1$$

$$x_i = n_i + b_i - s_i \qquad i = 1, \ldots, N$$

$$\delta_i^{\mathrm{b}} LB_i^{\mathrm{b}} \leqslant b_i \leqslant \delta_i^{\mathrm{b}} UB_i^{\mathrm{b}} \qquad i = 1, \ldots, N$$

$$\delta_i^{\mathrm{s}} LB_i^{\mathrm{s}} \leqslant s_i \leqslant \delta_i^{\mathrm{s}} UB_i^{\mathrm{s}} \qquad i = 1, \ldots, N$$

$$\delta_i^{\mathrm{b}} + \delta_i^{\mathrm{s}} \leqslant 1$$

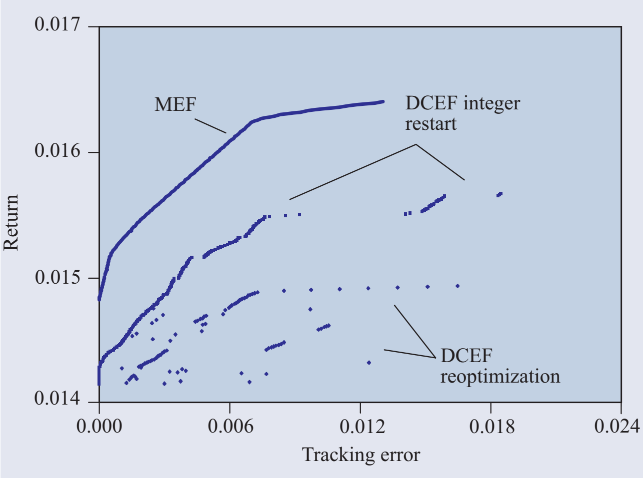

Figure 11. Rebalancing: 20 bonds, 6 bonds traded.

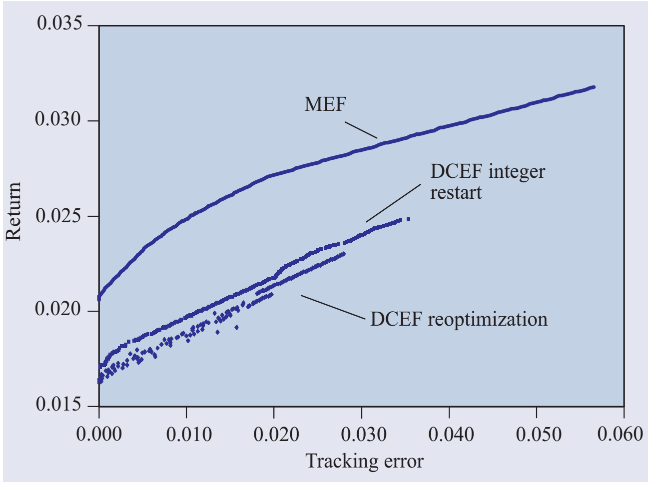

Figure 12. Rebalancing: 49 bonds, 10 bonds traded.

$$\sum_{i=1}^{N}(\delta_i^{\mathrm{b}} + \delta_i^{\mathrm{s}}) \leqslant k$$

$$x_i, b_i, s_i \geqslant 0 \qquad i = 1, \ldots, N$$

$$\delta_i^{\mathrm{b}}, \delta_i^{\mathrm{s}} = 0 \text{ or } 1 \qquad i = 1, \ldots, N.$$

We implement this model using three factors (explaining approximately 98% of the total variance) for two fixed income data sets. The first problem is to rebalance a given 20 bond portfolio, using at most six trades, to track an index of 330 bonds. The second problem involves a portfolio of 49 bonds tracking an index of 391 bonds, with a limit of 10 trades. For each problem we plot 200 points on the DCEF using the same integer restart and reoptimization heuristics as before. In the optimizations, residual risk is ignored, making the factor risk of the target indices the focus of the model. Figures 11 and 12 show the DCEFs for these problems. For the smaller of the two data sets all of the restart solutions are integer optimal. For the larger data set most solutions are integer feasible.

When constructing a new portfolio, the relative positions of the MEF and DCEF, are determined by the number of stocks in a portfolio on the MEF and the size of the cardinality constraint. For the rebalancing problem it is the number of trades identified by a portfolio on the MEF, the cardinality constraint and also the initial holding that affect the relative positions of the frontiers.

5. Discussion and conclusions

5.1. Comparative results

The use of QMIP enables us to use discrete constraints and capture important features of real-world problems. For the DCEF we highlight the discontinuities which follow as a consequence of imposing discrete constraints. We also explain why the risk-return trade-off cannot be used to construct the entire DCEF. Computing the entire 'DCEF to optimality' for even a reasonable size model remains a computationally intractable task. We show that by 'integer restarting' the QMIP with the previous solution we are able to generate a reasonable number of optimal and near optimal points within a restricted branch-and-bound search. We introduce a simple 'reoptimization' heuristic which proves to be computationally very efficient in constructing parts of the DCEF. We have identified two papers (Chang *et al* 1999 and Bienstock 1996) which have addressed the computational aspects of QMIP. The computational results show that our method outperforms the modern heuristic approach of the former. The paper by Bienstock looks at solving QMIP problems with heuristics for managing the cardinality constraint. This work is mainly concerned with the optimization algorithm for a single instance of a QMIP problem. We are more interested in exploiting a parametric approach to solving the discrete efficient frontier by warm starting the next QMIP problem. The warm start feature of Bienstock's paper applies to an individual problem after the disjunctive cut has been added.

The integer restart heuristic appears to be a valid method for investigating the problem of portfolio rebalancing. The reoptimization heuristic performs relatively poorly as the restart DCEF dominates the reoptimization DCEF in both cases studied. We also observe that in real applications the interest is not so much to construct the entire frontier accurately but to identify (and 'zoom in' to) an appropriate 'risk-return' segment where a number of alternative exact portfolios can be constructed.

5.2. Solution systems: current state-of-the-art

The solution of convex quadratic programs by sparse simplex (Mitra 1976) or by the interior point method (see Vanderbei (1994), Jones and Mitra (1997)) are now well established. Although there exist large sets of test data for QP problems they are not immediately relevant in the context of portfolio planning. It is also possible to apply any nonlinear solver such as the ones provided by NAG (1999) or that found within EXCEL. There are many commercially available solvers, such as those provided by MathSoft (NUOPT 1998), Operations Research Systems (GIANO 1999), Advanced Portfolio Technology (APT 2000) amongst others. Commercial LP systems such as CPLEX (1997) and OSL (IBM 1995) also

Table A1. Average monthly returns and covariances from five years' data for 30 FTSE stocks.

Stock number	1	2	3	4	5	6	7	8	9	10
Expected return	1.9231	0.3950	0.9231	2.1234	0.5822	2.4004	2.3603	0.7932	1.1049	0.7388
Stock 1	44.3591									
Stock 2	12.5813	55.4118								
Stock 3	5.9673	0.4586	41. 0694							
Stock 4	13.1064	5.3978	−1.1127	38.8121						
Stock 5	4.8330	11.8629	3.8800	9.3633	40.1504					
Stock 6	31.8484	15.7188	2.3163	7.4252	10.3679	61.8480				
Stock 7	36.4109	23.3858	−9.0213	21.4845	11.4553	45.0002	89.0782			
Stock 8	15.4496	10.2141	8.5785	5.3165	7.5599	26.1761	28.0091	44.2418		
Stock 9	3.3790	−3.9218	11.2122	−0.8193	−4.1408	3.8793	2.3141	−2.3792	49.2997	
Stock 10	14.4884	14.0008	0.3506	1.7166	25.1077	16.9762	27.0829	19.7745	−4.6766	60.4822
Stock 11	4.8061	15.6431	−8.1843	1.6400	8.6543	14.2046	20.1257	10.4054	−4.0006	23.8506
Stock 12	6.3673	−1.7339	9.3644	−0.0434	7.7353	13.6164	8.3009	13.2075	8.7405	1.2429
Stock 13	17.5273	19.4895	10.0226	13.3435	21.5076	19.8401	26.1493	21.5748	−13.9164	31.2236
Stock 14	20.3714	21.4225	−0.9602	−0.9443	17.0215	35.5623	51.5795	31.6692	−13.5125	35.2425
Stock 15	4.6626	7.7249	13.9976	−3.2001	7.0223	11.9670	3.3668	−1.0326	11.2959	9.3192
Stock 16	6.9818	−4.8496	9.8336	−0.1453	7.4180	4.3640	5.3660	12.1749	9.4513	19.1732
Stock 17	9.2554	−0.9581	4.8525	−1.8828	−0.6854	11.8507	12.8461	9.9904	9.7687	−3.3282
Stock 18	9.9860	9.5638	15.6003	11.0239	20.3285	20.9279	22.5954	25.0349	11.5507	21.2028
Stock 19	10.8073	21.7486	0.5542	11.4746	9.7190	14.1490	14.9537	8.9985	0.0905	7.1867
Stock 20	6.3428	−1.3307	−7.7636	11.6846	8.2683	11.3679	22.5338	20.3737	−8.8334	12.1690
Stock 21	17.4619	10.1502	10.6081	3.9556	9.6999	19.9152	23.9791	19.4550	5.5415	14.0175
Stock 22	0.9183	2.6363	3.9260	12.6739	17.1122	12.3284	9.5956	11.0185	−0.2397	7.5793
Stock 23	6.2159	7.2150	5.8835	4.5602	15.4986	23.7688	27.1578	25.5365	3.2994	14.9532
Stock 24	8.0790	15.9699	5.8752	3.5692	14.7254	14.1447	19.9735	16.9202	1.1718	19.2443
Stock 25	−2.7174	−9.7647	−11.1966	−8.4836	10.5996	11.2000	−4.4236	12.7476	−9.8964	13.8600
Stock 26	17.9332	1.1008	−5.1822	1.5520	4.8314	30.4307	31.6643	29.9974	−2.5327	21.1830
Stock 27	−0.5185	2.0338	0.3383	−4.4157	3.6868	4.7662	6.2970	15.4424	−0.2073	19.4279
Stock 28	12.0854	28.2566	−8.7473	1.9600	23.9601	19.8818	29.4014	15.2403	−16.0066	34.3006
Stock 29	13.8039	3.9858	5.1420	16.6629	2.9495	17.1986	19.2041	3.3403	4.7296	2.7432
Stock 30	10.3900	16.6060	3.5069	6.4344	18.8809	25.5144	34.9891	32.0006	−12.3490	26.4902

Table A1. Continued.

Stock number	11	12	13	14	15	16	17	18	19	20
Expected return	1.1954	0.5775	2.5019	0.3152	1.5001	1.8989	1.7460	1.2366	1.2509	0.7493
Stock 11	37.8261									
Stock 12	−6.3669	31.9311								
Stock 13	16.4499	−1.6659	77.1391							
Stock 14	19.1531	11.9854	33.9238	82.2687						
Stock 15	6.4534	−4.7699	11.4253	3.9857	64.0410					
Stock 16	3.8130	8.1015	8.9752	4.5364	−4.8198	38.4227				
Stock 17	−13.3500	9.8185	−10.6832	6.7398	5.3612	−0.7394	40.6584			
Stock 18	4.1351	0.5254	28.4416	24.1109	20.8041	7.0093	14.5095	79.4799		
Stock 19	8.3512	3.0824	7.2041	9.2856	1.0705	−7.2643	0.9579	6.7248	37.7253	
Stock 20	5.5548	3.6094	17.8511	22.7147	0.1103	6.4249	5.3646	18.4547	3.0081	44.6980
Stock 21	−1.3558	8.5733	15.4246	19.6629	14.3351	11.1846	12.1726	16.6755	7.1091	3.6814
Stock 22	−5.9742	3.5035	7.5169	8.8428	13.3204	−7.5054	4.7289	23.5466	9.2018	15.2742
Stock 23	7.0658	13.5514	18.1989	22.4374	4.8242	2.7644	14.6693	27.5381	6.5745	14.7446
Stock 24	7.8282	4.9043	19.0235	28.8589	8.9305	5.2109	6.3461	28.5846	10.7705	18.1423
Stock 25	7.3210	3.6247	13.0249	−0.2227	−4.4157	14.0825	−4.7384	10.5420	−1.0552	13.0105
Stock 26	20.5619	11.7268	22.7388	48.3054	−7.4816	17.5917	3.7492	12.1592	−0.0140	30.0701
Stock 27	5.7230	−4.7336	20.7437	11.8125	10.9219	12.5256	−4.0824	18.8931	−3.6541	14.9177
Stock 28	17.7291	3.7221	26.8055	36.4593	−8.8789	13.6664	−3.0415	12.5306	17.3668	19.5915
Stock 29	0.3909	8.1764	4.9850	8.1445	6.0819	0.0887	4.8677	2.7280	11.7927	5.9487
Stock 30	9.0255	9.7111	19.1408	39.2370	1.7386	6.9190	5.4667	22.9398	17.0197	21.6440

provide QP solver capability. Chang *et al* (1999) provide heuristics for obtaining good suboptimal solutions but they are not able to solve a discrete portfolio problem to optimality for a given data set. The authors are aware of only one other QMIP system (GIANO) but no scientific description or performance figures are available in the publish literature.

Our experience vindicates that a branch-and-bound solver which uses SSX for solving QPs is the most attractive avenue. In this approach the 'warm start' feature of SSX is exploited which, unfortunately, rules out using IPM since its restart properties are relatively poor. Finally, we would like to highlight that our relative success in computing the 'near optimal' DCEF is due to the use of 'warm start' and dual SSX and 'integer restart' using the previous solution which speeds up the computation of the next efficient point.

5.3. Future directions

Although the integer restart method produces good suboptimal results there is some scope for improving solver performance using preprocessing techniques. The success of the very simple reoptimizing heuristic suggests that there may be some benefit

Table A1. Continued.

Stock number	21	22	23	24	25	26	27	28	29	30
Expected return	1.2468	2.4568	2.3528	0.6651	3.4346	0.6886	1.3782	2.2393	1.8350	1.4522
Stock 21	39.3485									
Stock 22	6.4136	48.9834								
Stock 23	12.5524	16.4556	60.0240							
Stock 24	12.5216	9.8386	9.2266	34.4239						
Stock 25	0.4526	−7.3964	5.3785	1.1923	84.3889					
Stock 26	7.5221	−3.0085	17.4780	18.4134	21.2663	88.3961				
Stock 27	11.7879	10.2016	6.4771	12.9987	2.3873	14.2432	42.6852			
Stock 28	11.2792	7.0790	8.6710	22.0286	15.9701	26.1601	15.5715	65.1371		
Stock 29	9.7819	6.5569	3.8675	9.8203	−7.4017	9.4172	1.2209	2.0285	33.3388	
Stock 30	14.5880	23.7346	27.0986	20.3379	10.7548	19.9260	12.9230	30.2109	6.6499	51.9320

in investigating a refined approach that can overcome some of the drawbacks.

Our study shows that the portfolio rebalancing problem with discrete constraints should be investigated more thoroughly. Particularly, the relationship between the cardinality constraint and the initial holding requires further exploration.

The QMIP capability will also allow the discrete models to be extended to incorporate short sales and transaction costs. It would also be interesting to observe the effect these extensions have on the shape of the efficient frontier.

A natural extension is to implement the practical, discrete, constraints in dynamic multiperiod asset management models (Ziemba and Vickson 1975, Ziemba and Mulvey 1998).

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Appendix A. Illustrative data set

Average monthly returns and covariances from 5 years' data for 30 FTSE stocks (see table A1).

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AU: Please supply caption to table 1.

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