A Life Cycle Theory on Firm Finance and Investment

Yizhou Xiao

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Abstract

This paper studies the evolution of the external financing environment for SMEs and how it affects firms’ growth decisions. Asymmetric information on qualities of projects makes external financing costly. Collateral mitigates this problem, but its availability is limited by the size of the firm. As a firm grows, more collateral becomes available, broadening the firms access to external financing channels and lowering its cost of capital. The firms growth decision is affected by how effective additional collateral can lower its future cost of capital, and the distance to the next financing stage. Small firm may optimally choose to stay small when it is financially constrained and far from access to formal lending stage. When small firm approaches the formal lending stage, the strong incentive to expand makes it locally risk loving. This framework also enables one to evaluate firm growth rate as a popular empirical proxy for firm value. I show that high growth rate is not necessarily associated with high firm value.

Keywords: Firm Financial Growth Cycle; Collateral; Growth Decision; Risk Taking

JEL Classification: G21, G32.
1 Introduction

Small and medium enterprises (SMEs) have played an important role in economic growth and employment (Neumark et al., 2011; Fort et al., 2013). However, while some small businesses grow substantially as they age, most small firms remain to be small for decades (Cabrai and Mata, 2003; Angelini and Andrea, 2008). A potential explanation is that small businesses suffers from financial frictions, and existing literature analyzes the financial friction by simply assume some exogenous financing cost or constraint (Cabrai and Mata, 2003). However, those exogenous assumption miss probably the most striking feature for entrepreneurship finance that SMEs at various growth stages may face different external financing environments (Berger and Udell, 1998; Robb and Robinson, 2012). In particular, nascent firms mainly rely on informal lending channels such as funding from family members, friends and trade partners. As firms grow, they gain access to financial intermediaries. When they grow further and become established corporations, firms are able to borrow from corporate bond market. Different from a constant fixed cost or constraint assumptions, firm growth extends its access to external financing channels and lowers its future cost of capital. The expectation of the evolution of its external financing opportunities thus may be crucial in firm growth decision.

In this paper I study the evolution of the external financing environment for SMEs and how it affects firm’s growth decision. Even though various empirical studies show that the financing environment may play an important role in a firm’s performance and investment decisions (King and Levine, 1993; Rajan and Zingales, 1998; Beck et al., 2008; Fort et al., 2013; Adelino et al., 2014), little is known of the mechanism behind the financial growth cycle. Berger and Udell (1998) proposes that the evolution of financial growth stages may be related to the firm’s information opacity, but they didn’t explain what exact information opacity it is and how the information asymmetries affect different potential funding sources. It is not clear why the financial growth cycle evolves in the way we observe and what factors determine the boundaries of each stage. Furthermore, it is unclear how does the evolution of the financing opportunities affect SMEs growth
decisions. In this paper I develop a theory that explains firms financial growth cycle by analyzing the availability of collateral. Collateral is a common feature of credit contracts between firms and lenders and is well known as a commitment device to mitigate asymmetric information problems (Chan and Kanatas, 1985; Bester, 1985; Besanko and Thakor, 1987). Existing literature on the information role of collateral implicitly assumes that firms have unlimited supply of collateral, and the amount of collateral the borrower is willing to commit is solely determined by the quality of its project\(^1\). However, in the reality, SMEs are often short of collateral and the availability of collateral has played a crucial role in small business lending. This concern of availability of collateral for small business has received particular attention during the recent financial crisis. In his speech at the Fed 2010 meeting on restoring credit flow to small business, Ben S. Bernanke said that "the declining value of real estate and other collateral securing their loans poses a particularly severe challenge...it seems clear that some creditworthy businesses—including

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\(^1\)There’s another strand of literature (Bernanke and Gertler, Bernanke and Gertler; Kiyotaki and Moore, Kiyotaki and Moore; Krishnamurthy, Krishnamurthy) focuses on how collateral constraint may amplify economic shocks. In this paper I focus on how availability of collateral may affect the firm’s funding cost and its access to external funding sources.
some whose collateral has lost value but whose cash flows remain strong—have had difficulty obtaining the credit that they need to expand...". Mann (2014) documents that firms post intelligence patents, which are traditionally viewed as intangible assets, as collateral to increase the available amount of collateral.

In this paper I incorporate the idea of collateral shortage by introducing the size of the firm as a natural constraint on the amount of collateral the entrepreneur can post. While large firms have enough collateral to show qualities of their projects, small firms only have limited assets and thus face a collateral shortage problem. This endogenous collateral shortage friction prevents these small firms from showing the qualities of their projects solely by posing collateral, resulting in a high cost of capital and potentially limited access to external funding sources. Using a costly information technology, financial intermediaries (e.g., banks) can mitigate this asymmetric information problem and lower a firm’s funding costs. However, there are situations in which the profits a bank can earn from lending to an infant entrepreneur is not sufficient to compensate for the information collection costs. These nascent firms may have to rely on informal lending channels and are financially constrained.

The collateral shortage also plays an important role in SMEs growth decisions. Given limited collateral, expansion not only increases the firm’s daily operating cash flows, but also enables the entrepreneur to post more assets as collateral, lowering the firm’s future funding costs and potentially broadening its access to external funding sources. The benefit of a lower cost of capital gives firms a strong incentive to expand, but can only be realized when firms are able to find a lender to finance their projects. When firms gain access to formal lending channels, the benefit of lower cost of capital in the future is fully realized and it motivates SMEs to expand until they reach a mature stage where they additional collateral have no value in lowering funding cost. However, when firms are so small that they have to rely on informal lending channels, not every good investment opportunity are able to be financed because those informal lenders can be themselves financially constrained and are unable to offer funding. At informal lending
stage the benefit of lower cost of capital is limited by the probability that a good project may not be financed, making expansion less desirable. When small firms find themselves too small so that the distance they must travel to gain access to formal lending sources is overwhelmingly large, they optimally choose not to expand and pay all cash flows as dividend.

This stay small result for small firms provides an alternative explanation for several empirical observations. In development economics, empirical studies (Banerjee and Duflo, 2005; Mel et al., 2008; Banerjee and Duflo, 2014) find that even though the marginal gross return rates for small enterprises are fairly high, those SMEs often stay small and perform low growth rates. This growth-trap puzzle has received considerable attentions and a popular perception is that this is because those SMEs are severely financially constrained and thus don’t have enough fund to support expansion. This explanation can not explain why those small business don’t reinvest their own profit to expand. My model provides an alternative view that these owner may actually don’t want to expand even when they are able to do so. Financial frictions do not prevent firms from expanding directly, but make growth too costly.

Firm size distribution is closely related to firm growth decision. Cross section evidence (Cooley and Quadrini, 1992; Cabrai and Mata, 2003; Angelini and Andrea, 2008) shows that firm size distribution is heavily skewed toward small firms. Different from existing explanations that are based on specific assumptions on production technologies shocks or exogenous financial constraints or particular market structure, this model provides a fairly general explanation that is based on the endogenous evolution of financial constraints firms face in early stages.

Small businesses are important because they create most new jobs in the economy (Neumark et al., 2011; Fort et al., 2013), and job creation is associated with firm growth decision. The stay small result is consistent with an emerging literature on the implication of firm age on job creation. Based on detailed datasets, Haltiwanger et al. (2013) and Adelino et al. (2014) find that different from popular perception that small firms create
most jobs, within the small firm category the startup and young firms contribute almost all the net creation of jobs. This result implies that many aged small firms don’t perform high growth rates. This paper generates a similar growth pattern. If a firm has survived for years but still remains small, its optimal decision is to continue to stay small and the growth rate would be fairly low.

Besides corporate finance implications, this paper also provides asset pricing implication on risk premium associated with entrepreneurial activities. Access to formal lending channels completely changes the firm’s external funding environment and its expectation of availability of future funding. When the firm approaches the financial intermediaries lending stage, the desire to reach this threshold introduces nonconcavity in the firm value function. The benefit of access to formal lending channel makes entrepreneurs that are risk neutral in terms of dividend flows locally risk loving. This locally risk shifting behavior can potentially explain the low private equity risk premium (Hamilton, 2000; Heaton and Lucas, 2000; Moskowitz and Vissing-Jorgensen, 2002).

My framework provides a closed form solution for both firm value and firm growth rate at different stages. This feature enables one to apply this model to examine how effective firm growth rate as a popular proxy for firm values in empirical studies on SMEs. Data for firm value, especially for small and nonpublic firms, are often difficult to observe. Empirical researchers thus often employ some proxies to help evaluate firm value, and the effectiveness of those measures is important for academic researchers, policy makers, and the public to interpret empirical findings and evaluating policies. In this paper I argue that firm growth rate may not be a good proxy for firm value. The basic idea lies in the fact that firm value and firm growth rate may reflect different characteristics of firm value function. While firm value is solely determined by the level of firm value function, firm growth rate is largely determined by the funding cost it faces. However, funding cost is closely related to the slope of the firms value function. With a steep curve value function, the firm finds the profits that might be generated by a project very attractive and has a strong incentive to take risky projects. Taking that into account, lenders would charge
a high interest rate and leaves the firm a lower project profit. To illustrate this point, I simulate data within the model and run regressions similar to Fisman and Love (2003). I find that the firm growth rate may lead to false empirical implications.

The remainder of the paper proceeds as follows. Section 2 reviews related literature. Section 3 introduces the model and in section 4 the model is solved and the firms optimal growth strategy is identified. Section 5 analyzes how the endogenous firm financial growth cycle affects a firm’s growth strategy. Section 6 studies how the desire to reach next financing stage affect firm’s risk taking behavior. Section 7 evaluates firm growth rates as a proxy for firm value in empirical research and section 8 discusses extensions. Section 9 concludes.

2 Literature Review

This paper contributes to the literature on firm financial growth cycle. Berger and Udell (1998) is the first one documents the evolution of financing environment for firms at different stages and argue that information opaqueness may be the key factor to generate this pattern. Avery et al. (1998) and Robb and Robinson (2012) confirm this pattern with a new and more detailed dataset. A large body of literature tries to explain entrepreneur’s choice of financing method at certain stages (Rajan, 1992; Boot and Thakor, 1994; Peterson and Rajan, 1994). A notable exception which focuses on the life cycle of firm financing is Rajan (2012). The author argues that entrepreneur’s choice between internal and external funding sources is determined by the trade off between differentiating her enterprise to generate higher net present value and standardizing her enterprise to lower external funding cost. My paper is the first one trying to analyze the evolution of external financing environment in the firm life cycle.

Collateral as a committing device to mitigate asymmetric information problem has been widely studied both theoretically (Chan and Kanatas, 1985; Bester, 1985; Besanko and Thakor, 1987) and empirically (Avery et al., 1998; Peterson and Rajan, 1994; Voordeckers and Steijvers, 2006). Different from traditional unlimited available collateral
assumption, this paper introduces firm size as a natural constraint on the availability of collateral. This collateral shortage endogenously determines the evolution of financial growth cycle and firm growth decision.

There’s a large strand of literature focus on economic shocks and collateral constraint (Bernanke and Gertler, Bernanke and Gertler; Kiyotaki and Moore, Kiyotaki and Moore; Krishnamurthy, Krishnamurthy). In this paper I abstract from economic shocks and focus on how firm growth relates to the availability of collateral and its funding cost and access to external funding sources.

This paper is related to both development economics and entrepreneur finance literature that focus on growth-trap for small enterprises. A bunch of empirical studies (Banerjee and Duflo, 2005; Mel et al., 2008; Banerjee and Duflo, 2014) show that entrepreneurs may keep their firms small even with high marginal returns. They view it as an evidence that firms are extremely financially constrained and thus are unable to expand. My paper propose an alternative explanation that entrepreneurs may be able to expand but find it too costly.

Another closely related research topic is firm size distribution. Cabrai and Mata (2003); Angelini and Andrea (2008) find that entrepreneurial firms are centered at the low end of the firm size distribution. Existing models (Hopenhayn, 1992; Cooley and Quadrini, 1992) rely on specific production technology shock, market structure assumptions or exogenous financial constraint to explain this generic pattern. Based on collateral shortage, my paper provides a more generic explanation for firm dynamics with endogenous financial frictions.

There’s an emerging literature on job creation and firm age. Haltiwanger et al. (2013) and Adelino et al. (2014) find that different from popular perception that small firms create more jobs, within the small firm category the startup and young firms contribute almost all the net creation of jobs. My paper offers an explanation that those small and mature firms may optimally choose to stay small and perform low growth rates.

Empirical evidence finds that the premium to entrepreneurial activity is surprisingly
Most hypotheses that have been offered are based on the idea that entrepreneurs have a different set of preferences or beliefs (e.g., risk tolerance or overoptimism). Another interesting explanation is Vereshchagina and Hopenhayn (2009), they assume the entrepreneur has a new production technology which is only available when the entrepreneur’s wealth reaches certain threshold. This discrete choice of production technology introduces nonconcavity in the value function, thus creates a locally risk taking incentive. Similar to Vereshchagina and Hopenhayn (2009), this paper also introduces risk shifting behavior as a result of nonconcavity in the firm value function. Different from their work, my model doesn’t rely on the specific production technology assumption and the nonconcavity in the firm value function comes from the evolution of external financing environment, which is arguably a more general and natural channel.

My paper also contributes to empirical assessments on SMEs (King and Levine, 1993; Rajan and Zingales, 1998; Beck et al., 2008; Fisman and Love, 2003). Because it is difficult to observe SMEs value in the data, firm growth rate often serves as a popular proxy to assess firm values in empirical researches. My paper shows that firm growth rates may not be a good measure for firm values.

### 3 The Model

Consider a continuous time economy with an entrepreneur who runs a firm over an infinite horizon. At any time $t$, the firm operates its current asset and generates cash flow. The market value of its current asset is the presented value of all future cash flows the asset would generate discounted by a constant size-adjusted operation asset return rate $r_a$. The size of the firm is denoted by the market value of its asset $A_t$ and the operating profit it generates per unit of time can be written as $r_a A_t$. Besides daily operation, the firm can also implement projects at any time. Each project requires an investment of $I(A_t^{-})$, where $A_t^{-} \equiv \lim_{s \to t^-} A_s$ since the size of the firm may jump whenever it successfully implements a project. $I(A_t^{-}) = \gamma A_t^{\alpha}$, where $\gamma > 0$ is a constant and $\alpha \in (0,1)$. So
\(I(A_{t^-})\) is strictly increasing and strictly concave in \(A_{t^-}\). The firm needs external funding to finance its projects, this may because its current asset is illiquid or the project is built on its current operation. The firm and potential external financial sources are all risk neutral. The entrepreneur value future payoff at the discount rate \(r > r_a\), while potential lenders share the same discount rate \(r_a\). The higher discount rate for the entrepreneur may come from the fact that she invests her human capital which enables the firm to take projects overtime.

3.1 The Project

The standard project would generate permanent cash flow \(I(A_{t^-})\pi\) when it succeeds and 0 when it fails. The firm can also deviate to a risky project which is high risk \(P_R < P_B\) and high return \(\pi_R > \pi\) such that \(P_R\pi_R < r_L\). Besides the choice of types of projects, the firm also decides when to implement a project. Over each infinitesimal period of time \([t; t + dt]\), the firm privately observes a good signal with probability \(\lambda dt\), and the good signal generating process is described as \(\{N_t\}\). With a good signal the project would be successful with probability \(P_G\), which is normalized to 1 while without the signal the probability of success is \(P_B\). One can interpret the good signal as some private information on good market conditions. I assume that \(P_B\pi < r_L < P_G\pi\), where \(r_L\) is the funding cost for lenders. Thus it is socially optimal to only invest the standard project with a good signal. For simplicity I denote standard project with good signal as type \(G\) project, standard project without good signal as type \(B\) project, and risky project as type \(R\) project.

3.2 The External Financing Environment

All projects need external funding to be implemented. The assumption is natural in the sense that the firm’s current asset maybe illiquid or the project is built on its current operation. In this paper I focus on the asymmetric information problem on qualities of projects and its implications on firm’s financial growth cycle and growth decision. This
friction has no effect on equity financing, so to discuss equity, one needs to introduce some other frictions which makes the model unnecessarily complicated. A lot of empirical studies (Berger and Udell, 1998; Robb and Robinson, 2012) have shown that for SMEs debt is the dominating external funding source. For example, Robb and Robinson (2012) reported that only 5% firms in their data had external equity. It seems that for most SMEs the equity funding is not a feasible option and it is reasonable to focus on debt financing while keep in mind that there’s an unknown important friction that shuts down equity financing channel. By focusing on debt financing my paper doesn’t cover important problems such as innovative firms funded by Venture capital, where the overall quality of the firm becomes an important factor and the equity financing is non-negligible.

When the firm decides to start a project, it approaches all potential lending parties and each of them proposes a loan contract. A loan contract can be described as \( \{A_p, r_p\} \), where \( A_p \) is the collateral requirement and \( r_p \) is the interest payment. To be more specific, in this paper I only consider perpetual debt contract and the firm promises to pay a permanent cash flow rate \( r_p \).\(^2\) Introducing maturity of debts would introduce all remaining debt maturities as additional state variables, making the model difficult to solve. The firm compares all proposed contracts and decides whether to abandon the project or to accept one of the contracts to make the investment. When the project is invested and succeeds, the firm receives the project cash flow net off its funding cost \( I(A_t- r_p) \). When it fails, the firm gets nothing and has to pay the lender the amount of collateral \( 0 \leq A_p \leq A_t- \). Here \( A_p \leq A_t- \) represents a natural constraint on the maximum amount of collateral available to the firm.

There are two types of lenders in the economy: informal lenders and formal lenders. Lenders all face the same funding cost \( r_L \) and differ in terms of their funding capacities and information technologies. Informal lenders can be interpreted as entrepreneur’s family members, friends or trade partners. Being common individuals or small firms, in-

\(^2\)One can also rewrite the model into a mathematically equivalent version in which all projects instantaneously generate a lump sum payment, and the interest rate \( r_p \) becomes a lump sum charge \( R_p \equiv \frac{I(A_t- r_p)}{r_a} \).
formal lenders are not specialized in lending business and are often themselves financially constrained, so they may not be able to fund entrepreneur’s project whenever she needs. In the model I assumed that at any time $t$ informal lending channel is only available to the entrepreneur with a probability $P_I$. The model is tractable even when the informal financing probability is a function of the project size, but here for simplicity I just assume $P_I$ to be a constant. Motivated by empirical findings that firms are severely financial constrained even with informal funding, I assume that:

Assumption 1. $r > r_a + P_I \lambda (\frac{1}{P_B} - 1)$.

The term $(\frac{1}{P_B} - 1)$, as shown latter, is the benefit of a lower cost of capital in the future when the firm expands and has more asset that can be posted as collateral. This parameter condition suggests that when the firm only relies on informal lending, The benefit of lower cost of capital is limited because only some of future good projects would be able to be financed, making expansion benefit too low for the entrepreneur. This assumption highlights the negative effect of firm financial constraint on firm value.

Another important feature for informal lending channel is its information technology to facilitate lending. Here the information technology is modeled as the ability to tell whether the firm is doing a risky project or not. This specification roots from the fact that soft information may help lenders to monitor firm’s operation, but is not a perfect solution to the asymmetric information problem. In this model, since taking risky project fundamentally changes the project’s payoff structure, it has to be associated with taking a different project, or operating the project in a different way. Those are behaviors that are somehow detectable. On the other hand, the private signal can be interpreted as a good market condition or a good timing, which may rely on specialized professional expertise or the entrepreneur’s private observation and are often difficult to be verified. Those soft information is generated from daily personal or business interactions, so the information cost can be treated as a sunk cost for informal lenders.

Formal lenders are deep-pocked, so they are able to finance every project the entrepreneur proposes. Similar to informal lenders, They can monitor the firm and tell
whether it is a risky project but don’t know whether the firm observes a good signal or not. However, monitoring technology is costly for formal lenders. One can interpret formal lenders with information technology as banks and formal lenders without private information as competitive banking or investors in corporate bonds market. When formal lenders collect information, they spend continuous effort cost \(e\) to maintain and update the information overtime. Since informal lending is the most convenient and possibly the fastest way to raise funds, the firm would rely on it whenever it is not dominated by other financing channels in terms of funding cost\(^3\). When the informal lending channel is unavailable, each bank would win the loan contract with probability \(P_c\). The probability \(P_c\) can be interpreted as a measure of banking sector competitiveness.

### 3.3 The Firm

The firm’s strategy includes both cash flow decisions and project decisions. At any time \(t\) the firm decides how much cash flow to reinvest and the remaining cash flow would be paid out as dividends. The cash flow reinvestment decisions overtime are described as \(I_{\text{Cash}}\), subject to \(I_{\text{Cash}} \in [0, r_a A_t^-]\). At any time \(t\), the project implementation decision is described as \(I_p = \{G, B, R\}\), where \(I_p = G\) means only investing standard project when a good signal is available. Associated is the process of type \(i\) projects implementation, denoted as \(\{N, i\}\). For type \(i\) project, at time \(t\) the set of available lending contracts is \(\mathcal{L}_i(A) \equiv \{L(A_{ip}(A_t^-), r_{ip}(A_t^-)), L(0, \pi_i)\}\), where the element \(L(0, \pi_i)\) means lenders decide not to finance this project. I denote the firm’s project investment process and its type \(i\) available loan contract path as \(I_P = \{I_p\}\) and \(\mathcal{L}_i(A) = \{\mathcal{L}_i(A)\}\), respectively. The firm will go bankruptcy when \(A_t\) drops to 0 and the entrepreneur gets her outside option value, which is normalized to 0.\(^4\) Given the set of lending contracts for different types of projects \(\mathcal{L}\), the firm chooses its optimal project investment decision \(I_P\), loan contract choice \((A_F, r_F) \in \mathcal{L}(I_F)\) and cash flow reinvestment policy \(I_{\text{Cash}}\) to maximize its expected

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\(^3\)Another way to model this is to assume an arbitrarily small filing cost for formal funding sources.

\(^4\)In section 7 I consider the case when the outside option is nonzero and may evolve as the size of the firm changes.
discounted value of dividend:

\[ V_F(A) = \max_S E^S \left\{ \int_0^\infty e^{-rt} \left[ r_a A_t - I_t^{Cash} \right] dt | A_0 = A \right\} \]

(1)

Where \( S = (\{I_F, L_F(A_F, r_F), I^{Cash}, IF\}) \) is the firm’s optimal policy, and the stopping time \( \tau \equiv \inf \{s : A_s = 0\} \) describes firm’s bankruptcy time. The cost of project failure is implicitly included in the stopping time \( \tau \) since firm has to pay the collateral \( A_F \) once its project fails.

Firm’s discounted rate, \( r \), is assumed to satisfy the following assumption:

**Assumption 2.** \( r < r_a + \lambda \left( \frac{1}{P_B} - 1 \right) \).

Intuitively, assumption 2 suggests that if the firm can finance its projects whenever it observes a good signal, the value of the firm would be higher than the size of its asset (that is to say, the market value of the firm’s asset). Otherwise the firm would not have incentive to grow.

To make information technology a valuable tool for formal lenders, the information cost \( c \) should be sufficiently low, which is stated in the following technical assumption:

**Assumption 3.**

\[
\frac{c}{(1 - P_I)P_c} \leq \min \left\{ \frac{\lambda \gamma^{\frac{1}{1 - a}} \left[ \frac{\pi_R - \pi}{r_a (P_R^{1 - \kappa_1} - P_B^{1 - \kappa_1})} \right]^{\frac{\alpha}{1 - a}} - \left( P_B^{-\kappa_1} - 1 \right) \frac{\gamma (\pi_R - \pi)}{r_a (P_R^{1 - \kappa_1} - P_B^{1 - \kappa_1})} \right\}^{\frac{1}{1 - a}},
\]

\[
\lambda \gamma^{\frac{1}{1 - a}} \left[ \frac{\pi_R - \pi}{r_a (P_R^{1 - \kappa_1} - 1)} \right]^{\frac{\alpha}{1 - a}} - \left( P_B^{-\kappa_1} - 1 \right) \frac{\gamma (\pi_R - \pi)}{r_a (P_R^{1 - \kappa_1} - 1)} \}
\]

(2)

Where \( K_1 \) is a constant defined in the following section. This assumption, as shown latter, guarantees that formal lenders would not finance any projects without gathering information when the firm is small.
At any time $t$, the sequence of events during the infinitesimal time interval $[t, t + dt]$ can heuristically be described as follows:

Step 1: The firm generates cash flows and pays out dividends.

Step 2: The firm may or may not observe a good signal, and may propose a project to all lending parties.

Step 3: Informal lenders and formal lenders (if they gather information), learn whether the project is risky or not, and every lender offers a loan contract $L_i = \{r_p, A_p\}$.

Step 4: The firm makes the project investment decision, and chooses one loan contract to finance its project.

Step 5: The project payoff is realized and both the firm and the lender get paid according to the loan contract. Formal lenders gathering information decide whether to continue to gather information or not.

4 Solution of the Model

I start the analysis by stating the following conjectures about the equilibrium.

Conjecture 1. In the equilibrium, without observing the good signal, the firm will be indifferent between taking or not taking the standard project.

Conjecture 2. Let $r_p(A)$ be the lowest interest rate to satisfy the signal IC constraint. $r_p(A)$ is decreasing in $A$.

Because only type $G$ project generates positive NPV, lenders only want to finance type $G$ projects. Since no information technology can tell when the firm observes the good signal, this conjecture directly suggests an IC constraint. Given firm’s strategy $S$, lenders need to determine the loan contract such that:
\[
V_F(A_{t-}) \geq P_B[V_F(A_{t-} + \gamma(A_{t-})\frac{\pi - r_P}{r_a}) + (1 - P_B)V(A_{t-} - A_p)]
\]

Firm value when project succeeds

\[
+ (1 - P_B)V(A_{t-} - A_p)
\]

Firm value when project fails

Equation (3) is referred as the signal IC constraint. There are many loan contracts that satisfy this signal IC constraint, but the following lemma shows that the loan contracts in the equilibrium must be unique, and the signal IC constraint is always binding.

**Lemma 1.** In the equilibrium, for firm with asset \( A_{t-} \), \( A_p = A_{t-} \) whenever \( r_p(A_{t-}) \geq r_L \). Otherwise \( A_p = \inf \{ A' | r_p(A') = r_L \} \).

**Proof.** See Appendix.

This lemma basically states that lenders would require as large amount of collateral requirement as possible unless the firm has big enough asset. This is fairly intuitive since posting more collateral is costly when the firm doesn’t observe a good signal.

The model is difficult to solve because the endogenous evolution of external funding environment and the interaction between lenders’ strategies and the firm’s policy \( S \). To get around those problem, I solve this model in four steps.

Step 1. With the conjecture, partially solve firm’s value function without considering lender’s lending strategies.

Step 2. Given the partially solved value function \( V_F \), back out loan contracts in different scenarios.

Step 3. Back out lending strategies for different parties and the evolution of the firm’s external financing environment.

Step 4. Pin down the firm’s investment decision and its value function given lenders’ lending policies.
4.1 Partially Solving Firm’s Value Function

Since only standard projects with good signals have positive NPV. Lenders only want to finance type \( G \) projects. Taking that into account, firm’s Hamilton-Jacobi-Bellman equation is:

\[
 rV_F(A_t-) = \max_S \left\{ V'_F(A_t-)I^{Cash}(A_t-) + r_aA_t- - I^{Cash}(A_t-) + P_P\lambda(V_F(A_t+) + \gamma(A_t-)\frac{\pi - r_P}{r_a} - V_F(A_t-)) \right\}
\]  

(4)

Where \( P_P \) is the probability that a standard project with a good signal to be financed. It would be \( P_I \) when only informal lenders are available and 1 when formal lenders are willing to lend.

Now consider the case \( A_t- \leq \inf\{A'\mid r/L(A') = r_L\} \). Given the conjecture and lemma 1, signal IC constraint must be binding and \( A_p = A_t- \), so the signal IC constraint can be written as:

\[
 V_F(A_t-) = P_B[V_F(A_t- + \gamma(A_t-)\frac{\pi - r_P}{r_a})] + (1 - P_B)V(0);
\]  

(5)

Where \( V(0) \) is entrepreneur’s outside option when the firm bankrupts. Substituting the signal IC constraint into firm’s HJB equation, one get:

\[
 rV_F(A_t-) = \max_S \left\{ V'_F(A_t-)I^{Cash}(A_t-) + r_aA_t- - I^{Cash}(A_t-) + P_P\lambda\left(\frac{1}{P_B} - 1\right)V_F(A_t-) \right\};
\]  

(6)

Notice that equation (6) is independent of lenders’ lending strategies and the firm’s project profit liquidation policy. This comes from the fact that the good signal only changes the probability of success, but not the exact payoff when the project succeeds or fails. No matter how complicate the lending policies and firm’s project profit liq-
uidation decisions might be, with the help of the signal IC constraint, one can always express the firm’s gain from project success as a function of its current firm value and its outside option value.\(^5\) Now one can analyze firm’s value function without considering lending policies and the firm’s project profit liquidation policy. Since entrepreneur is risk neutral, the optimal cash flow reinvestment policy would be corner solutions: \(I^{\text{Cash}}(A_t^-) = r_A A_t^-\) whenever \(V_F'(A_t^-) \geq 1\) and \(I^{\text{Cash}}(A_t^-) = 0\) when \(V_F'(A_t^-) < 1\). So when \(A_t^- \leq \inf \{A' | r_p(A') = r_L\}\), there are four possible scenarios: Reinvesting cash flow and only informal lending; Paying out cash flow and only informal lending; Reinvesting cash flow and with access to formal lending; Paying out cash flow and with access to formal lending. Take the case Reinvesting cash flow and with access to formal lending as example, firm’s HJB function would be

\[
rV_F(A_t^-) = V_F'(A_t^-) r_A A_t^- + \lambda \left( \frac{1}{P_B} - 1 \right) V_F(A_t^-);
\]

Solve this ODE one get:

\[
V_F(A_t) = Z_1 A_t^{K_1}
\]

Where \(Z_1\) is a strictly positive constant and \(K_1 = \frac{r - \lambda(\frac{1}{P_B} - 1)}{r_A} < 1\). Since it is still unclear for what size of asset \(A_t\) formal lenders would be willing to lend and \(V_F'(A_t) \geq 1\), one can not determine the boundary condition and thus can not pin down the constant \(Z_1\). Similarly, one can also partially solve firm’s value function in other scenarios:

Reinvesting cash flow and only informal lending:

\[
V_F(A_t) = Z_0 A_t^{K_0};
\]

Where \(Z_0\) is a strictly positive constant and \(K_0 = \frac{r - P_I \lambda(\frac{1}{P_B} - 1)}{r_A} > 1\).

\(^5\)In the basic model I only consider the case when outside option value is always 0. In section 7 I study a case in which entrepreneur’s outside option value increases as her firm grows.
Paying out cash flow and with access to formal lending:

\[ V_F(A_t) = \frac{r_a}{r - \lambda \left(\frac{1}{P_B} - 1\right)} \cdot A_t = \frac{1}{K_1} A_t; \]  

(10)

By assumption 2, in this scenario \( V_F'(A_t) > 1 \). So with access to formal financing channel, firms would reinvest its daily operation cash flows whenever the interest rate is higher than \( r_L \).

Paying out cash flow and only informal lending:

\[ V_F(A_t) = \frac{r_a}{r - P_I \lambda \left(\frac{1}{P_B} - 1\right)} \cdot A_t = \frac{1}{K_0} A_t; \]  

(11)

So far I partially solve the firm’s value functions in different scenarios whenever \( A_t \leq \inf \{ A' | r_p(A') = r_L \} \). To determine the evolution of different scenarios and value function when firm’s asset is large enough to take the lowest possible interest rate \( r_L \), one needs to solve the optimal lending policies for lending parties.

4.2 Evolution of Financing Environment

I start the analysis by looking at the case when the firm has access to formal lending channel. Given the partial solution for firm’s value function and the binding signal IC constraint, one get:

\[ Z_1 A_t^{K_1} = P_B Z_1 [A_t^{-\alpha} I(A_t^{-\alpha}) \cdot \frac{\pi - r_P}{r_a}]^{K_1}; \]  

(12)

So the implied interest rate is \( r_{PS}(A_t^{-\alpha}) = \pi - r \frac{A_t^{1-\alpha}}{\gamma} (P_B^{-1} - 1) \). It is straight forward to see that the interest rate is decreasing in firm’s size.

Other than the signal IC constraint, formal lenders may need to take firm’s risk shifting behavior into account. If they don’t exert effort to gather information, then their loan contracts need to satisfy another IC constraint:
The figure shows the interest rates implied by the signal and risk shifting IC constraints. The annual based parameter choice is reported in table 1. See Appendix.

\[ Z_1 A_{t-}^{K_1} = P_R[Z_1(A_{t-} + \gamma (A_{t-}) \frac{\pi_R - \tau P}{r_a})^{K_1}]; \] (13)

This IC constraint is referred as the risk shifting IC constraint. The implied interest rate is \( r_{PR}(A_{t-}) = \pi_R - r \frac{A_{t-}^{1-\alpha}}{P_R^{1-\gamma}}(P_R^{1-\gamma} - 1) \). Notice that \( r_{PR}'(A_{t-}) < r_{PB}'(A_{t-}) < 0 \), and \( r_{PR} (A_{t-}) \geq r_{PS} (A_{t-}) \) whenever \( A_{t-} \leq \left[ \frac{\gamma \frac{\pi_R - \tau P}{r_a} \frac{1}{P_R^{1-\gamma} - P_B^{1-\gamma}} \right]^{\frac{1}{1-\alpha}} \), which can be shown in figure 2.

As discussed in the static case, collateral requirement is more costly for firms taking risky projects. When firms become large, they find taking risky projects so costly that the risk shifting IC constraint is implied by the signal IC constraint. Since information technology is costly, formal lenders would prefer not to gather information and only rely on collateral requirement. When \( A_{t-} \geq \left[ \frac{\gamma \frac{\pi_R - \tau P}{r_a} \frac{1}{P_R^{1-\gamma} - P_B^{1-\gamma}} \right]^{\frac{1}{1-\alpha}} \equiv A_C \), the firm would get access to formal lending channel without information technology. In other words, firm can borrow from competitive banking sector or can issue corporate bonds in the financial
market.

When \( A_{t^-} < A_C \), collecting information enables formal lenders to relax the risk shifting IC constraint, lowering the interest rate they offer. Collecting information would be profitable whenever \( I(A_{t^-}) \frac{r_P - r_L}{r_a} \geq \frac{c}{\lambda(1-P_I)P_c} \). It is easy to verify that \( I(A_{t^-}) \frac{r_P - r_L}{r_a} \) is a strictly concave function starting from 0 and from assumption 3 one can conclude \( I(A_C) \frac{r_P - r_L}{r_a} > \frac{c}{\lambda(1-P_I)P_c} \). Formal lenders find it profitable to lend with information technology whenever \( A_B \leq A_{t^-} \leq A_C \), where \( A_B = \inf \{ A : I(A) \frac{r_P - r_L}{r_a} = \frac{c}{\lambda(1-P_I)P_c} \} \). So for formal lenders, they would start to gather information whenever \( A_t \geq A_B \) and stop doing that whenever \( A_t \geq A_C \).

When \( A_{t^-} < A_B \), formal lenders may gather information and charge a interest rate higher than \( r_{PS}(A^-) \), or do not exert effort and just charge \( r_{PR}(A^-) \). The following lemma ensures that those strategies are undesirable on the equilibrium path.

**Lemma 2.** Given assumption 3, \( I(A_B) \frac{r_{PS}(A_B) - r_L}{r_a} \leq \frac{c}{\lambda(1-P_I)P_c} \) and \( r_{PR}(A_B) \geq \pi \), so formal lenders won’t lend to the firm whenever \( A_{t^-} < A_B \).

**Proof.** See Appendix.

To summarize, formal lenders would not lend to the firm when its size is below \( A_B \) because lending profit can not compensate their costly information technology. When the size of the firm is sufficiently large(\( A > A_C \)), collateral can fully show qualities of projects and formal lenders would solely rely on public information to lend, which means that the firm gets access to corporate bonds market.

### 4.3 Firm’s Investment Policies

Given the evolution of firm’s financial growth cycle, one can back out firm value function, associated dividend policies. Similar to the analysis above, I start with the final stage when firm gains access to formal lending channels.

To pin down the constant \( Z_1 \), one only needs to find the proper upper boundary condition. Since whenever \( r_{RS} > r_L \), equation (11) shows that the firm would always
want to reinvest, at the upper bound the firm would face the lowest interest rate $r_L$. A natural candidate is when the firm reaches the lowest possible interest rate $r_{PS}(A) = r_L$, denote the size of the firm when it reaches upper bound as $\overline{A}$, which is determined by $\therefore$. Since the firm would pay out all cash flows as dividends, the value of the firm is:

$$P_B(V_F(\overline{A}) + I(\overline{A})\frac{\pi - r_L}{r_a}) = V_F(\overline{A});$$ (14)

However, the value of the firm with size $\overline{A}$ is difficult to solve and thus there’s no reduced form solution for $Z_1$. However, since it just functions as a scale factor, $Z_1$ would not change the firm dynamic. In latter section I study a case when the firm can choose to liquidate some of its assets, and $Z_1$ can be solved explicitly. So the firm’s value function is $V_F(A) = Z_1 A^{K_1}$ when $A \in [A_B, \overline{A}]$. The firm would reinvest all its cash flows to expand. When $A > \overline{A}$, the firm would pay out all cash flows as dividends, and its value function is $V_F(A) = \frac{r_a A + \lambda I'(A)\pi - r_L}{r}$. Figure 5 illustrates the firm’s value function and the evolution of its external financing environment.

When $A \in [0, A_B]$, the firm would solely rely on informal lending. The following lemma plays an important role to pin down the firm’s value function:

**Lemma 3.** The firm’s value function $V_F(A)$ is strictly increasing and continuous.

**Proof.** For any $A_1 > A_2$, the firm with size $A_1$ can always copy project choice and cash flow reinvestment strategy from the firm with size $A_2$, and commit all its asset as collateral but accepts the same interest rate as firm with size $A_2$, it would receive strictly more cash flows and has higher evaluation. For any $A > 0, \forall \epsilon \in (0, A)$, consider $A' = A(\frac{V_F(A) - \epsilon}{V_F(A)})^{\frac{1}{r_a}}$. The firm can always choose not to do any project and only reinvest its cash flow until it reaches $A$, and its value would be $V(A) - \epsilon$. Then for all $A_t \in [A', A])$, $V_F(A_t) \geq V_F(A) - \epsilon$. □

There’s another case in which the firm may still want to expand since larger size would introduce larger projects in the future. In that case, the upper bound would be determined by $\frac{r_a + \lambda'(A)\pi - r_L}{r} = 1$, which is essentially the same. For simplicity I only focus on the basic case.
Given lemma 3, the boundary condition is given by $V_F(A_B) = Z_1 A^{K_1}$, where $A_B$ is determined by:

$$\frac{c}{\lambda (1 - P_I) P_c} = \gamma A_B^a \frac{\pi - r_p}{r_a} - ((P_B)^{-\frac{1}{K_1}} - 1) A_B; \quad (15)$$

Which stated that at $A_B$ the expected lending profit just compensates the information cost. Suppose the firm without access to bank loans experiences both cash flows reinvestment and cash flows dividend stages, there should exist at least one point $A_{Ic}$ such that $V_F(A_{Ic}) = Z_0 A_{Ic}^{K_0} = \frac{1}{K_0} A_{Ic}$, then it would be the case that $V'_F(A_{Ic}) = Z_0 K_0 A_{Ic}^{K_0-1} = \frac{1}{K_0} A_{Ic} K_0 A_{Ic}^{-1} = 1$. Since $Z_0 A_{Ic}^{K_0}$ is a strictly convex function, and it only holds when the firm is willing to reinvest its cash flows, it must be the case that $A_{Ic}$ must be the starting point of a stage of cash flows reinvestment.\(^7\) Since $A_{Ic}$ can not be the ending point of a cash flows reinvestment stage, it must be the case that there exists a unique $A_{Ic} < A_B$ such that the firm would reinvest its cash flows when $A_{Ic} \leq A_t \leq A_B$ and would pay out all cash flows as dividend when $A < A_{Ic}$. Then the firm’s value function can be pinned down by

$$V_F(A_B) = Z_1 (A_B)^{K_1} = Z_0 (A_B)^{K_0}; \quad (16)$$

and

$$Z_0 A_{Ic}^{K_0} = \frac{1}{K_0} A_{Ic}; \quad (17)$$

Given $A_B$, one can solve $Z_0 = Z_1 (A_B)^{K_1-K_0}$ and $A_{Ic} = (Z_0 K_0)^{-\frac{1}{K_0-1}}$. Figure 6 illustrate the firm’s value function and the dynamic of its cash flow reinvestment policy.

The firm’s value function is $V_F(A_t) = \frac{1}{K_0} A_t$ when $A_t < A_{Ic}$, and all daily operating cash flows would be paid as dividends. When $A_{Ic} \leq A_t \leq A_B$, firm’s value is $V_F(A_t) = K_0 (A_t)^{K_0}$, and it would reinvest all daily operating cash flows to expand.

\(^7\)Similarly, one can verify that $Z_0 K_0 A_{Ic}^{K_0-1} > Z_1 K_1 A_{Ic}^{K_1-1} > 1$, so $A_{Ic} < A_B$. 

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4.4 Lending Policies

With the firm’s value function and reinvestment strategy, one can now solve optimal lending policies for different lending parties.

As discussed above, formal lenders’ information collection decision can be described as \( \{A_B, A_C\} \), where \( A_B \) is the threshold size of the firm for formal lenders to start gathering information, and \( A_C \) is the end point. When \( A_{t-} \geq A_C \), firm can get funding through competitive banking or corporate bonds market. when \( A_{t-} < A_B \), only informal lending channel is available and firm is financially constrained in the sense that not every standard project with a good signal can be financed.

All lending parties share the same optimal lending contracts \( \{(A_P, r_P)\} \). As lemma 1 states, when \( A_{t-} \leq A \) they would ask \( A_P = A_{t-} \) and charges the interest rate satisfies signal IC constraint:

\[
V_F(A_{t-}) = P_B V_F(A_{t-} + I(A_{t-}) \frac{\pi - r_P}{r_a});
\]

when \( A_{t-} > A \), they would charge \( r_P = r_L \) and the collateral requirement satisfies:

\[
V_F(A_{t-}) = P_B [I(A_{t-}) \frac{\pi - r_L}{r_a} + V_F(A_{t-})] + (1 - P_B) V_F(A_{t-} - A_P);
\]

A more detailed description of firm’s financial growth cycle is reported in Table 3.

5 Collateral Constraint and Firm Growth Decision

In the equilibrium, the evolution of financial environment stems from the fact that the firm only has limited asset to be collateral. This collateral constraint limits the firm’s ability to fully signal qualities of its projects, resulting high cost of capital. Expansion not only generates more daily cash flow overtime, but also enables firm to be able to post more asset as collateral, reducing its future lending costs. To evaluate this effect, a benchmark case would be at what discount rate \( r_{\text{benchmark}} \) would the firm be indifferent
between reinvesting or paying dividend, that is to say, $V_F(A) = A$.

When the firm has access to formal lending channels, the HJB equation, conditional on $V_F(A) = A$, becomes

$$r_{benchmark}A = r_a A + \lambda \left( \frac{1}{P_B} - 1 \right) A;$$

Hence:

$$r_{benchmark} = \underbrace{r_a}_{\text{Cash flows}} + \underbrace{\lambda \left( \frac{P_G}{P_B} - 1 \right)}_{\text{Lower cost of capital}};$$

Equation 21 implies that the benefit of cash flow reinvestment can be decomposed into two parts. More assets not only generate more cash flows over time, but also enable the firm to post more collateral when it wants to take new projects, lowering its future cost of capital. By assumption 2, $r < r_{benchmark}$. When all type $G$ project would be taken, the benefit of lower cost of capital is fully realized. The firm finds accumulating assets attractive as long as expansion suggests a lower cost of capital. Thus when the firm gains access to formal lending channels, it would reinvest all cash flows to expand until its funding cost reaches the lowest possible rate $r_L$, that is to say, it reaches the mature stage. Notice that the firm value function $V_F(A)$ is concave at region $[A_B, \bar{A}]$ because as $A$ increases, it approaches the threshold $\bar{A}$ for the lowest possible interest rate $r_L$ and the marginal benefit of a lower cost of capital decreases. At $A = \bar{A}$, the interest rate reaches the lower bound and there would be no more benefit of future lower cost of capital. Taking that into account, the firm optimally chooses to stop cash flow reinvestment.

Similarly, when $A \in (0, A_B]$, the HJB equation becomes:

$$r_{benchmark} = \underbrace{r_a}_{\text{Cash flows}} + \underbrace{P \lambda \left( \frac{P_G}{P_B} - 1 \right)}_{\text{Lower cost of capital}};$$
Similar to equation 21, the benefit of expansion comes from both more cash flow and lower cost of capital. In this case because the firm can only rely on informal lending channel, type $G$ project can be financed only with probability $P_I$. Since the benefit of lower cost of capital can only be realized when the firm’s project is actually get financed, it is limited by the probability $P_I$. By assumption 1, $r > r_{benchmark}$. The firm finds reinvesting asset costly because only a fraction of future projects would be financed, the benefit of lower cost of capital is limited. Expecting gaining access to formal lending at $A_B$, the firm with asset $A \in [A_I, A_B]$ is not far away from reaching the threshold and still wants to grow even though expansion is costly. The firm value function $V_F(A)$ now becomes a convex function because as $A$ increases, it approaches the threshold $A_B$ and the firm would have a stronger incentive to grow in order to get access to bank loans. When the firm are too small, that is to say, $A \in (0, A_I]$, it finds the threshold $A_B$ too far and too costly to reach and optimally chooses not to reinvest its daily operating cash flows.

The endogenous collateral constraint and its effect on firm growth decision suggests that when firms are small and severely financially constrained, it may find expansion too costly and optimally chooses to stay small. This ”stay small” result fits several empirical findings.

### 5.1 Firm’s Financial Growth Cycle and Growth Trap

In development economics, empirical studies(Banerjee and Duflo, 2005; Mel et al., 2008; Banerjee and Duflo, 2014) find that even though the marginal gross return rates for small enterprises are fairly high, those SMEs are often stay small and perform low growth rates. For example, based on a directed lending program in India, Banerjee and Duflo (2014) estimates that the marginal return rates for average small firms in India is 105%. This ”growth trap” puzzle has received considerable attentions and the existing literature explains this by arguing that those SMEs are highly financially constrained and thus don’t have enough fund to support firm expansion.
My paper contributes to the "growth trap" puzzle by offering an alternative explanation. Similar to existing literature, my model also attributes this "stay small" result to the external financial constraint faced by SMEs, but the underlying mechanism is different. Different from the traditional view that financial constraint resulting in insufficient fund to support firm growth. In this paper, the firm can always reinvest its daily operation cash flow and becomes larger. This is realistic because extremely high rates of return implies that firms may generate a significant amount of net cash flow over time, which is a missing part in the traditional explanations. My model introduces an explanation that is consistent with the cash flow part. As discussed earlier, with the financial constraint, the firm is unable to finance all of its future projects, lowering the benefit of lower cost of capital. Thus the firm may find expansion too costly when it is highly financially constrained, and would optimally pay out all cash flow as dividends.

This explanation has a fairly different policy implication. Here "stay small" is not a suboptimal result due to the lack of fund to invest, but an optimal decision by the firm because the benefit of expansion is limited by the financial constraint. While the traditional view suggests that small firms may need some funding to support their expansion, my model emphasizes on loosing firms financial constraint to increase the benefit of lower cost of capital, motivates firms to grow.

5.2 Firm’s Financial Growth Cycle and Firm Size Distribution

This "stay small" result is also related to the academic discussion on firm size distribution. It is well documented that the size distribution of firms is heavily right skewed Cabrai and Mata (2003); Angelini and Andrea (2008). Figure 6(a) illustrates the empirical observation of firm size distribution. The existing literature(Hopenhayn, 1992; Cooley and Quadrini, 1992) tries to explain this with some productivity technology shocks and market structure assumptions. Few Firms receive some positive productivity technology shocks would grow up, and some limitations on market entry and firm imitation implies that those remaining small firms can not catch up those growing firms, some theoreti-
ical models also incorporates financial environments by introducing some fixed cost of financing, making financing too costly for small firms.

This paper highlights the importance of financial environment on firm size distribution. Different from the exogenous fixed cost assumption in the literature, my model introduces endogenous collateral constraint as the main factor to determine the firm’s growth decision. Figure ?? shows the simulated firm size distribution and its evolution. Consistent with empirical observations, the size distribution is heavily skewed towards small firms. While existing theories often rely on specific production technology shock or market structure assumptions to generate the firm size distribution, this financial environment channel provides a more general explanation to firm size distribution, which is consistent with the fact that firm size distributions are quite robust across different industries.

5.3 Firm’s Financial Growth Cycle and Job Creation

One reason for the importance of SMEs in academic researches and policy debates is that it small businesses create a large fraction of jobs(Neumark et al., 2011; Fort et al., 2013). Job creation is also a measure of firm growth. Recently there’s a growing literature(Haltiwanger et al., 2013; Adelino et al., 2014) exploring the implication of firm age. As shown by figure 8(a), based on detailed datasets, they find that different from popular perception that small firm create more jobs, within the small business firms, the startup and young firms contribute almost all the net creation of jobs, suggesting that many small firms don’t perform high growth rates. This cross section evidence matches my model prediction that while some small firms would reinvest and expands. A significant fraction of small businesses would optimally choose to stay small and perform low growth rates over time. Figure 8(b) presents the simulated small firm growth rates based on ages. This implies that if a firm has survived for years but still remains small, its optimal decision is to continue to stay small and the growth rate would be fairly low.

Furthermore, Haltiwanger et al. (2013) documents an ”up or not” dynamic of young
firms. That is, conditional on survival, young firms either grow rapidly or exit. In latter session I consider an extension in which the entrepreneur has the option to liquidate her firm asset. In that case, small entrepreneur will optimally choose to liquidate her firm when its size is too small and thus $V_F(A) \leq A$.

6 Risk Taking by Entrepreneurs

Earlier sections have demonstrated that the firm value function performs nonconcavity. The convex firm value function in informal lending stage suggests that entrepreneurs may be locally risk loving even though it is risk neutral in terms of evaluating its dividend flows.

In the basic model, the daily operation would generate a risk free cash flow. To analyze the locally risk loving behavior, here I consider an extension in which the entrepreneur can also choose to operate her firm in a risky way. Besides the cash flow $r_A A_t$, this risky strategy also gives the firm some growth opportunities that would increase the size of the firm from $A$ to $\eta A$, where $\eta > 1$. However it may also lead firm to default. The probability of growth opportunity to arrive and the firm defaults during the infinitesimal time interval $(t, t + dt]$ is $\lambda_G$ and $\lambda_B$, respectively. I assume that:

Assumption 4. $\lambda_G(\eta - 1) = \lambda_B$;

Assumption 4 suggests that risky operation strategy generates the same expected return than the original operation strategy. If the firm implements the risky operation strategy in informal lending stage and is willing to reinvest, similar to the basic model, conjecture that the new firm value function is $V_F(A_t) = Z_0 A_t^{K_1}$, then the HJB equation becomes:

$$rV_F(A_{t-}) = r_A A_{t-} V'_F(A_{t-}) + \lambda_G(\eta^{K_1} - 1)V_F(A_{t-}) - \lambda_B V_F(A_{t-}) + P_t \lambda \left(\frac{1}{P_B} - 1\right) V_F(A_{t-});$$

(23)
Solves this ODE one get:

\[ V_F(A_t) = Z_0 A_t^{K'}; \]  

(24)

Where \( K' = \frac{r - p_I \lambda (\frac{1}{PB} - 1) - \lambda_G (\eta K_1 - 1)}{r_a}. \) The conjecture is true if and only if \( K' = K_1, \) then \( K' \) is determined by \( \frac{r - p_I \lambda (\frac{1}{PB} - 1) + \lambda_B}{r_a} > K' + \frac{\lambda_G}{r_a} (\eta K' - 1). \) Given assumption 4, it is straightforward to see that \( 1 \leq K' \leq K_0. \) That is to say, with the risky operating strategy, the firm still finds expansion costly, but it is less costly than the original operating strategy. Since \( \lambda_G (\eta K' - 1) > \lambda_B, \) the firm would take the risky operation strategy whenever it wants to reinvest the cash flow and only has access to informal lending channel.

I have shown that the entrepreneur would be locally risk loving even though she might be risk neutral in terms of evaluating dividend flow. Even though the risky operation strategy has a lower expected face value return, the value function convexity makes it attractive. This risk taking behavior has an interesting asset pricing implication. Entrepreneurial activity is risky and poorly diversified, and standard asset pricing models would suggest that the entrepreneurial risk should be compensated by a significant premium in returns. This model, however, implies a low entrepreneurial risk premium.

Empirical evidence finds that the premium to entrepreneurial activity is surprisingly low (Hamilton, 2000; Heaton and Lucas, 2000; Moskowitz and Vissing-Jorgensen, 2002). Most hypotheses that have been offered are based on the idea that entrepreneurs have a different set of preferences or beliefs (e.g., risk tolerance or over optimism). Another interesting explanation is Vereshchagina and Hopenhayn (2009), they assume the entrepreneur has a new production technology which is only available when the entrepreneur’s wealth reaches certain threshold. This discrete choice of production technology introduces non-concavity in the value function, thus creates a locally risk taking incentive. Similar to Vereshchagina and Hopenhayn (2009), this paper also introduces risk shifting behavior as a result of nonconcavity in the firm value function. Different from their work, my model doesn’t rely on the specific production technology assumption and the nonconcavity in the firm value function comes from the evolution of external financing environment, which
is arguably a more general and natural channel.

This risk taking behavior also has some implications on firm size distribution across different industries. Industries may perform heterogeneity in terms of availability and capacity of risk taking behavior. While some industries like high tech are easy to take more risk, it is relatively more difficult to take risk in some other industries like traditional manufacturing and service. Small firms in high tech industry may take more risk and are likely to be either successful or defaults. Figure 8 shows the firm size distributions for different risk taking industries. Taking risk would help very few firms becomes more successful, but at the cost that a large fraction of small firms fail and default.

7 Application: Firm Growth Rate as a Proxy for Firm Value

Given the importance of SMEs, there are a lot of policy debates and academic researches focusing on how certain shock or policy reform would affect SMEs. Due to lack of data, it is difficult to directly measure values of small and mid size firms. Alternatively, empirical studies often employ some proxies such as firm growth rates or firm return rates to measure firm value. It is intuitive to claim that firms are better off if they perform higher growth rates or return rates, and those proxies are popular in a large body of empirical assessments. Thus it is important to know how effective those proxies can reflect firm value. The analytic framework in my paper enables one to investigate whether those popular proxies serve as good measures for firm value. My model shows that growth rates and other related measures(return rates etc.) may not measure firm welfare properly.

The basic intuition is best illustrated by figure 3. While firm value solely depends on the level of value function, firm growth rate is largely determined by the slope of value function. The most important factor in firm growth rate is the financial cost the firm faces. The interest rate, however, depends on the slope of firm’s value function. To be more specific, when firm value function is steep, the potential value added from reinvesting one
dollar project profit is high, making type B and type R projects more attractive. Taking that into account, lenders would ask for a higher interest rate, resulting in a lower project profit. Similarly, when firm value function has a flat curve, the potential benefit from project profit reinvestment is low and the firm is less likely to take type B and type R projects, and lenders would ask a lower interest rate, resulting in a larger project profit. Thus firm value and firm growth rates reflect two different characteristics of firm value function.

To better illustrate this point, I take Fisman and Love (2003) as an example. In the study the authors argue that since trade credit provides an alternative source of funds, industries with higher dependence on trade credit financing would be relatively better in countries with weaker financial institutions. More specifically, they tested the hypothesis with the following regression:

\[
Growth_{ci} = \alpha_i + \zeta_c + \beta Priv_c (Apay/TA)_i + \epsilon_{ci};
\]

\[\text{where } c \text{ denotes country and } i \text{ describes industry. } Apay/TA \text{ is the proxy for trade credit dependence measured by ratio of accounts payable over total assets, and } Priv \text{ is the} \]

\[\text{32}\]
proxy for financial market development measured by ratio of private domestic credit held by monetary authorities and depositary institutions scaled by GDP. They find that $\beta$ is significant negative, suggesting that industries that are more dependent on trade credit will grow relatively faster in countries with less developed institutional finance. Based on this empirical test, they conclude that more available trade credit would make firms better off.

This story is fairly intuitive because more trade credit alleviates firm’s financial constraint and makes it possible for more projects to be financed. Firms exhibit higher rates of growth for two reasons. The first one is that more projects would be implemented and generate more cash flows. The second reason is that since firms know they have more growth opportunities in the future, taking risk to implement projects other than standard project with good signal becomes less attractive and the lender would thus charge a lower interest rate. Both factors contribute to a more flat value function, implying a higher growth rate.

However, this flat shape does not necessarily imply a higher level of firm value. More supply of trade credit makes firms less likely to borrow from financial institutions, lowering expected lending profit. Taking that into account, banks would optimally delay their entry time. In other words, though firms are better off in the short run by having more projects financed by informal lending channel, they may be worse off in the long run because bank lending stage is delayed.

Following the original paper, I use the model to simulate a data set consists of 6 countries and 5 industries. For each industry in each country, I randomly simulate 5000 different firms. Countries are different in terms of bank monitoring cost, while industries have different probability for informal lending. Based on the simulated firm level data, I then run the following two regressions:

$$\text{Growth}_{cij} = C + \alpha P_I + \zeta r_L + \beta (1 - P_I)c + \epsilon_{cij};$$

(26)
\[ \text{Value}_{ci} = C + \alpha P_I + \zeta r_L + \beta (1 - P_I)c + \epsilon_{ci}; \]  

(27)

Where \( P_I \) is the probability for firm to get informal lending and \( c \) is the information technology cost for banks. I use \( P_I \) to proxy the availability of trade credit and \( c \) to measure the development of financial institutions. The main results are reported in Table 1.

Table 1: Firm Growth Rate and Firm Value

<table>
<thead>
<tr>
<th>Sample</th>
<th>Fisman and Love(2003)</th>
<th>Simulated Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Firm Growth Rates</td>
<td>Firm Growth Rates</td>
</tr>
<tr>
<td>( \beta )</td>
<td>-2.01</td>
<td>-0.0852</td>
</tr>
<tr>
<td></td>
<td>(0.597)</td>
<td>(0.0131)</td>
</tr>
<tr>
<td>N</td>
<td>1217</td>
<td>150000</td>
</tr>
<tr>
<td>Adjusted ( R^2 )</td>
<td>0.28</td>
<td>0.04</td>
</tr>
</tbody>
</table>

This table reports the key coefficient \( \beta \) in three different regressions. The first column represents the original result in Fisman and Love (2003), while the other two show the results based on simulated data. The regressions on simulated data have lower \( R^2 \) because the data is firm level data and the firm value function is a nonlinear function of firm size. In the simulation, countries is characterized by different information technology cost ranging from \([3, 6]\) and industries are different in terms of the informal lending probability \( P_I \in [0.1, 0.4] \).

As Table 1 shows, firms with more access to trade credit do exhibit higher growth rate, while its value may be lower. This example suggests that firm growth rates is not necessarily a good proxy for firm value. Empirical assessments with firm growth rates may produce misleading empirical results. Thus one should be cautious to interpret those empirical findings.

8 Extensions and Discussions

8.1 General Project Payoff Structure

In the basic model I assume a simple lottery payoff structure. The model is also robust to more general payoff structure assumptions. To be more specific, let \( F(x), x \in [0, \pi] \) be
the PDF of standard project outcome with a good signal. Similar to the basic model, the PDF of standard project outcome without good signal \( F_B(x) \) satisfies \( F_B(x) = P_B F(x) + (1 - P_B) \). That is to say, without good signal the project would fail with probability \( 1 - P_B \), and would generate exactly the same outcome as the project with a good signal with probability \( P_B \). Taking the case when firm gains access to formal lending channel and wants to reinvest its cash flows as example, now one can rewrite firm’s HJB equation as

\[
rv_F(A_{t-}) = r_a A_{t-} V'_F(A_{t-}) + \lambda [E \int_{r_P}^{\pi} V_F(A_{t-} + I(A_{t-}) \frac{x - r_P}{r_a}) dF(x) - V_F(A_{t-})];
\]

(28)

And the signal IC constraint becomes:

\[
V_F(A_{t-}) = P_B E \int_{r_P}^{\pi} V_F(A_{t-} + I(A_{t-}) \frac{x - r_P}{r_a}) dF(x);
\]

(29)

The expectation terms in both HJB equation and signal IC constraint is complicated to solve directly. However, based on the payoff structure assumption, one can still substitutes the signal IC constraint into the HJB equation and get:

\[
rv_F(A_{t-}) = r_a A_{t-} V'_F(A_{t-}) + \lambda (\frac{1}{P_B} - 1)V_F(A_{t-});
\]

(30)

The new firm’s HJB equation is exactly the corresponding HJB equation in the basic model, so one can still solve firm value function, and the implied interest rate is determined by the new signal IC constraint (29).

### 8.2 Non Constant Outside Option

So far I have assumed that the entrepreneur’s outside option is a constant which is normalized to 0. Anecdotal evidence suggests that operating a larger firm may give entrepreneur more social recognition and access to a broader network, increasing the value of her outside option. Results in this paper are robust to non constant outside
option assumption. Denote $V_0(A)$ as entrepreneur’s outside option value when her firm size is $A$. For simplicity, let $V_0(A) = A^{\alpha_0}$. Similar to the general project payoff structure extension, taking the case when the firm gains access to bank loans and wants to reinvest its cash flows as example, then the firm HJB equation (7) becomes:

$$r V_F(A_{t-}) = V'_F(A_{t-}) r_a A_{t-} + \lambda \left( \frac{1}{P_B} - 1 \right) [V_F(A_{t-}) - V_0(A_{t-})];$$  \hfill (31)

One can still substitute the signal IC constraint into the firm HJB equation and the new ODE can be solved as:

$$V_F(A_t) = Z A_t^{K_1} + \frac{\lambda \left( \frac{1}{P_B} - 1 \right)}{r_a (\alpha_0 - K_1)} A_t^{\alpha_0};$$  \hfill (32)

Now the new firm value function has one more term for the outside option. Similarly, when only informal lending channel is available, the value function becomes $V_F(A_t) = Z A_t^{K_0} + \frac{P_1 \lambda \left( \frac{1}{1 - P_B} - 1 \right)}{r_a (\alpha_0 - K_0)} A_t^{\alpha_0}$. As long as $\alpha_0 \in (K_1, 1]$, the value function would still be convex in informal lending stage and would be concave when the firm can borrow from formal lenders. Thus the non constant outside option assumption doesn’t qualitatively change the firm value function.

### 8.3 Cash Saving

Firms that are financially constrained often try to save some cash to finance their future projects. In the basic model the firm can either reinvest its cash flow or pay it out as dividend. In this extension I allow the firm to save some cash within the firm and can use them to finance the next project. However, the following analysis would show that this doesn’t change the equilibrium and firm cash flow policy.

The main argument is that saving cash is strictly dominated by cash flow reinvestment strategy. Now denote $C_t$ as the total amount of cash saved in the firm, and $dC_t$ is the cash flow that saved within the firm at time $t$. Now without loss of generality, suppose the firm decides to start to save cash at time $t_0$ following some cash saving policy $C^*$. Now
consider the following alternative strategy: reinvest all cash flows that would be saved according to $C^*$, besides that also follow the firm’s cash flow reinvest policy. That is to say, the new cash flow reinvestment $I_{Cash}^t$ satisfies $I_{Cash}^t = I^*_{t}^{c}dC^*_t$, where $I^*_{t}^{c}$ is the firm’s cash flow reinvestment policy when it implements $C^*$. Now consider at some time $t_1$, the firm wants to finance a new project. Similar to the basic model, the binding signal IC constraint implies that for each project the value of the firm would have a jump of $\frac{1}{P_B} - 1)$, independent of the firm’s cash saving strategy. Also, the alternative strategy always pays out $r_a C^*_t$ additional dividend flow and thus dominates the cash saving strategy.

8.4 Asset Liquidation

In the basic model the firm always keep its asset and the size of the firm only decreases when a project fails and the collateral is claimed by the lender. However, firms in the real world may liquidate some of their assets so it is useful to understand how the liquidation option may change the results. In this extension I assume that there exists some outside buyers that can also operate the firm’s asset, generating $r_a A$ cash flows overtime. Those buyers has a discount rate $r_a$, so the firm can liquidate its asset at its face value $A$. Now when the firm gains access to formal lending and is willing to reinvest its cash flow, the signal IC constraint becomes:

$$Z_1 A_t^{K_1} = P_B[Z_1(A_t - \gamma(A_t -)\frac{\pi - r_P}{r_a} - D_t)K_1 + D_t];$$  \hspace{1cm} (33)

Where $D_t$ is the amount of asset the firm wants to liquidate when the project succeeds. Substituting the signal IC constraint into the firm HJB equation, one gets:

$$rV_F(A_t) = V'_F(A_t)r_aA_t + \lambda(\frac{1}{P_B} - 1)V_F(A_t);$$  \hspace{1cm} (34)

Which is the same as in the basic model. Similarly, one can solve the firm value function in different scenarios. The firm would only liquidate its asset whenever $V'_F(A) \leq 1$. So the
firm would liquidate some of its asset when \( A \geq \overline{A} \) and the amount of asset liquidation is \( A - \overline{A} \). Also it would liquidate all of its asset when \( V_F(A) \leq A \), that is to say, when the firm is sufficiently small.

When the firm can liquidate asset, it would liquidate all project profits when \( A \geq \overline{A} \), the one gets the boundary condition:

\[
V_F(A) = \frac{r_a\overline{A} + \lambda I(A) \frac{\pi - r_L}{r}}{r}.
\]  
Equations (35) and (14) thus imply that

\[
\overline{A} = \left[ \frac{P_B}{1 - P_B} \frac{\frac{1}{r_a} - \lambda r_L}{\frac{1}{r} - \frac{1}{P_B}} \right]^{\frac{1}{1 - \alpha}}.
\] (36)

and \( Z_1 \) is:

\[
Z_1 = \gamma \frac{\pi - r_L}{r_a} \frac{P_B}{1 - P_B} \overline{A}^{1 - K_1}.
\] (37)

### 8.5 Risk Averse Entrepreneur

Since entrepreneurial activity is risky and poorly diversified, it is reasonable to consider the case when the entrepreneur is risk averse. In the risk aversion case, the entrepreneur would always pay out non zero cash flow as dividend overtime, and the optimal cash flow reinvestment decision is no longer a corner solution. This feature suggests that one can not categories the firm into several scenarios, making it difficult to solve the model explicitly. However, one can solve the model numerically. Figure 4 illustrates the numerical solution for firm value function when the entrepreneur has CARA utility function.

As shown in figure 4, there’s some subtle difference between the case of risk neutral entrepreneur and the case of risk averse entrepreneur. In the risky neutral case, since marginal utility for additional consumption flow is a constant, the main factor is how additional asset would affect the firm’s external funding environments. Since expansion is
costly before the firm gains access to bank loans, the firm value function at the informal lending stage performs convexity, and the firm would optimally choose to stay small whenever its size is below certain threshold $V_{Ic}$. When the entrepreneur is risk averse, the marginal return for additional consumption flow is high and this gives the entrepreneur incentive to expands its business, however when the firm is not too small, the marginal return is diminishing and the cost of capital effect becomes more important, making the firm optimally choose to stay small. So the firm value function is concave in the informal lending stage. However, similarly to the risk neutral case, the threshold of formal lending introduces different external financing environment, generating a kink in the firm value function. The firm value function thus exhibits nonconcavity around that threshold and the firm is risk loving even when it is risk averse in terms of evaluating its consumption flow.

Figure 4: Firm value with risk averse entrepreneur. The risk aversion parameter is $\beta = 1$. 
9 Conclusion

This paper studies the evolution of the external financial environment for SMEs at different growth stages. Asymmetric information on the quality of projects makes external financing costly for firms. Collateral can mitigate this problem, but its availability is determined by the size of the firm. This natural constraint endogenously generates the financial growth cycle.

The collateral constraint also influences firm’s growth decision. As the firm grows, more collateral becomes available and this extends its access to external financing channels and lowers its cost of capital. Growth decision is affected by how effective more collateral can lower the future cost of capital, and the distance to the next financing stage. The firm may optimally choose to stay small when it is financially constrained and is too far to reach the formal lending stages.

When the firm is reaching the formal lending stage, it has a strong incentive to grow and may become locally risk loving.

This framework also helps in understanding and evaluating some popular empirical proxies for firm value. I show that high growth rates or profit rates are not necessarily associated with high firm value.
References


Appendix A

Proof of Lemma 1

Proof. Since type $G$ projects would be successful for sure, collateral is not costly for firms with type $G$ projects, so from equation (3) they are willing to commit higher collateral requirement to lower the interest rate. Lenders thus would charge highest possible collateral requirement $A_p = A_{t-}$. When the rate reaches the lowest bound $r_L$, lenders just charge $A_p = \inf \{A' | r_p(A') = r_L\}$.

Proof of Lemma 2

Proof. At the ending boundary of banking stage $A_C$, each bank’s profit from lending is $\gamma \frac{P_G \pi - r_L}{r_a} \frac{I(\pi_R - \pi)}{P_R - P_B} \frac{\alpha}{1 - \alpha}$. So when $\frac{c}{1 - P_I} \geq \lambda \gamma \frac{P_G \pi - r_L}{r_a} \frac{I(\pi_R - \pi)}{P_R - P_B} \frac{\alpha}{1 - \alpha}$, banks find informational lending profitable and would do so.

Another possibility is that before bank lending stage, banks may just charge $r_{PR}$. However, when $\frac{c}{1 - P_I} \leq \lambda \gamma \frac{P_G \pi - r_L}{r_a} \frac{\alpha}{1 - \alpha} \frac{\alpha \omega_a(\lambda + r - \alpha \omega_a)}{(P_B \pi - 1)(\lambda + r)(\lambda + r - \alpha \omega_a)}$, it is easy to verify that $r_{PR} \geq \pi$ whenever $A \leq A_{RP}$. Since $A_B < A_{RP}$, interest rates offered by banks are always too high for firms before banking stage.
Appendix B Tables

Table 2: Parameter Choice

<table>
<thead>
<tr>
<th>Variables</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.13</td>
</tr>
<tr>
<td>$r_L$</td>
<td>0.02</td>
</tr>
<tr>
<td>$r_a$</td>
<td>0.04</td>
</tr>
<tr>
<td>$P_b$</td>
<td>0.5</td>
</tr>
<tr>
<td>$P_f$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0.8</td>
</tr>
<tr>
<td>$\pi_r$</td>
<td>3</td>
</tr>
<tr>
<td>$P_i$</td>
<td>0.4</td>
</tr>
<tr>
<td>$P_c$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.1</td>
</tr>
<tr>
<td>$c$</td>
<td>5</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>100</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.35</td>
</tr>
</tbody>
</table>

This table reports the value choice of variables used in the quantitative simulations.

Table 3: Firm Financial Growth Cycle

<table>
<thead>
<tr>
<th>Stage</th>
<th>Size</th>
<th>Formal Lending</th>
<th>Cash Flow</th>
<th>$V_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Growth by Project</td>
<td>$[0, A_{lc}]$</td>
<td>No</td>
<td>Dividend</td>
<td>$\frac{1}{K_0}A$</td>
</tr>
<tr>
<td>Growth(Informal)</td>
<td>$[A_{lc}, A_B]$</td>
<td>No</td>
<td>Reinvestment</td>
<td>$Z_0A^{K_0}$</td>
</tr>
<tr>
<td>Growth(Banks)</td>
<td>$[A_B, A_C]$</td>
<td>Bank Loans</td>
<td>Reinvestment</td>
<td>$Z_1A^{K_1}$</td>
</tr>
<tr>
<td>Growth(Financial Market)</td>
<td>$[A_C, A]$</td>
<td>Financial Market</td>
<td>Reinvestment</td>
<td>$Z_1A^{K_1}$</td>
</tr>
<tr>
<td>Mature</td>
<td>$[A, \infty)$</td>
<td>Financial Market</td>
<td>Dividend</td>
<td>$\frac{r_aA + \lambda I(A)(\pi - r_L)/r_a}{r}$</td>
</tr>
</tbody>
</table>
Appendix C Figures
Figure 5: Firm Value and Financial Growth Stages

The figure reports the evolution of the firm’s external financing environment. The informal lending stage refers to the scenario when the firm has to rely on informal lending channels, while the firm is able to borrow from banks when it is at the formal lending stage. The financial market stage is characterized as the stage when the firm can borrow when banks don’t need to gather information.
The figure illustrates the firm’s growth decision dynamic. When additional asset can reduce its cost of capital, the firm would reinvest all its cash flows in the formal lending stage. In the informal lending stage, when the firm is not far away from the threshold to get access to formal lending, the firm would still reinvest. When the firm is too small, it would choose to pay out its cash flow as dividend.
Figure 6(a) reports the empirical firm size distribution in Angelini and Andrea (2008). Based on firm level data from an Italy bank, Angelini and Andrea (2008) documents the firm size distribution in different firm age categories. Figure 6(b) shows the corresponding simulated firm size distribution. The parameter in the simulation is reported in table 2. Following empirical findings, in the simulation the initial firm size distribution is a truncated normal distribution with mean $A_i$ and variance 2, and the domain is $[0.5A_i, A_{max}]$. 

Figure 7: Firm Size Distribution and Firm ages
Figure 8: Firm Size Distribution and Firm Risk Taking

Figure 8 reports the simulated firm size distribution for firms with different operation strategies at age 25. The risky operation strategy is characterized by parameters $\eta = 1.5$, $\lambda_G = 0.1$ and $\lambda_B = 0.1$. 
Figure 9: Firm Size Distribution and Firm ages

Figure 8(a) reports the empirical job creation by firm ages in Haltiwanger et al. (2013). Figure 8(b) shows the corresponding simulated firm growth rates. The parameter choice is reported in table 2. In the simulation the initial firm growth is treated as a project success.