

The pricing of discretely sampled Asian and lookback options: a change of numeraire approach

Jesper Andreasen

This paper considers the pricing of discretely sampled Asian and lookback options with floating and fixed strikes. In the modelling framework of Black and Scholes (1973), it is shown that a change of numeraire of the martingale measure can be used to reduce the dimension of these path-dependent option pricing problems to one in addition time. This means that the pricing problems can be solved by numerically solving one-dimensional partial differential equations. The author demonstrates how a Crank–Nicolson scheme can be applied to the numerical solution. Finally, the methodology is extended to the case when the underlying stock exhibits discontinuous returns, and it is shown that in this case the Asian and lookback option pricing problems can be solved by numerically solving one-dimensional partial integrodifferential equations.

1. INTRODUCTION

Exotic options that have payoffs that depend on the arithmetic average, the maximum, or the minimum of the underlying stock over a certain time period have become increasing popular hedging and speculation instruments over recent years. Parallel to that, a growing body of literature has considered the pricing and hedging of such derivatives. Within the Black and Scholes (1973) model, closed-form solutions have been obtained for lookback option prices by Goldman, Sosin, and Gatto (1979), Goldman, Sosin, and Shepp (1979), and Conze and Viswanathan (1991). No closed-form solution has yet been derived for Asian option prices, but various transforms and approximations have been obtained; see, for example, Geman and Yor (1993) and Rogers and Shi (1995).

The closed-form solutions for the lookback options are based on the assumption that the maximum is taken over the whole continuous path of the underlying. But, for most traded lookback options, the maximum is not based on daily highs of the underlying over the whole life of the option. The maximum is rather based on daily closing prices over either the whole life of the option or only over a discrete number of trading days. For such contract specifications, the assumption of continuous observations seems as a poor approximation. The same goes for the average options. The approximations obtained for the options depending on the arithmetic average are also based on the assumption that the average is sampled over continuous intervals, typically the whole life of the option. However, all traded Asian options depend on averages sampled over a discrete and often a low number of trading days. The consequence is that in practice one has to resolve to Monte Carlo simulations in order to price these types of contracts.

In this paper we suggest a simple and computationally efficient alternative to Monte Carlo simulations for four types of path-dependent options: the Asian option, the average strike option, the lookback option with fixed strike, and the lookback option with floating strike. The idea is to make use of change of numeraire techniques to obtain the option prices as functions of time and a *one*-dimensional Markovian state variable only. The technique has previously been applied to the pricing of lookback options with floating strike by Babbs (1992) and Wilmott, Dewynne, and Howison (1993), but, as indicated, this paper extends the methodology to the pricing of three other types of path-dependent option.

Owing to the discrete observations, the state variables involved here exhibit jumps at the observation points with probability 1. However, in between two observation points the state variable evolves continuously, so it is possible to describe the option price as the solution to a standard partial differential equation (PDE) in such a region. Letting the first PDE generate the terminal boundary condition of the second, and so forth, we obtain a sequence of PDEs that can be solved numerically by finite-difference techniques. In the paper we employ Crank–Nicolson schemes for the numerical solution of the pricing problems. We could in fact also set up binomial or trinomial trees for the numerical solution, but we choose not to for two reasons. First, the nonstandard dynamics of the state variable yields problems with the stability of such trees. That is, one has to take unreasonable small time steps in order to insure the stability of the numerical solution. Second, the nature of the pricing problems are similar to barrier option pricing problems. Trees give rise to odd–even problems for such pricing problems; see, for example, Boyle and Lau (1994).

As mentioned, the fixed strike option-pricing problems for both the lookback and the Asian option can be converted into barrier option-pricing problems. This is rather surprising given the nature of the original pricing problems. However, the determining state variables that we identify here have a ‘barrier’ type of behavior, in the sense that if they go through a certain level, typically in the money, their dynamics become more tractable and it is possible to derive the risk-adjusted expectation of the terminal payoff in closed form.

For the floating strike options treated in this paper, it is also possible to apply the PDE technique to the pricing of options with an American feature.

We provide numerical examples that illustrate the speed and the accuracy of our procedures. Our benchmark is Monte Carlo simulations with a large number of samples combined with a control variate technique. In most cases, the finite-difference solutions get within penny accuracy compared to the Monte Carlo solutions in less than one second of CPU time.

Finally, we show how the technique can be applied to the case when the underlying exhibits discontinuous dynamics. Our model is in this case a ‘risk-neutralized’ version of the Merton (1976) model, where the jumps are triggered by a Poisson process and the jumps in return are displaced lognormal distributed. In this case, the sequence of PDEs is replaced by a sequence of partial integrodifferential equations (PIDEs) that also can be solved by finite-difference techniques.

The paper is organized as follows. The second section of the paper shortly describes the modelling framework and the main trick applied in this paper: the change of

numeraire for the martingale measure. We then have a section for each of the options considered here, in respective order these are: the Asian (fixed) strike option, the average strike option, the fixed strike lookback, and the floating strike lookback option. Each section contains numerical examples of the accuracy and the speed of our solution procedure. The final section of the paper shows how our technique also can be applied to noncontinuous dynamics of the underlying stock.

2. THE MODEL AND CHANGE OF NUMERAIRE

For simplicity we start by considering the standard Black–Scholes economy with two assets: a dividend paying stock and a money-market account. We will later extend the model to cover the case when the underlying exhibits discontinuous dynamics. We assume the existence of an equivalent martingale measure \mathcal{Q} under which all discounted security prices (including accumulated dividends) are martingales. This assumption implies absence of arbitrage.

Under \mathcal{Q} , the stock is assumed to evolve according to the stochastic differential equation

$$\frac{dS(t)}{S(t)} = (r - q) dt + \sigma dW(t), \quad (1)$$

where r is the constant continuously compounded interest rate, q is the constant continuous dividend yield, σ is the instantaneous volatility of the stock return, and W is a standard \mathcal{Q} Brownian motion.

If one considers the pricing of currency or commodity options, q denotes the foreign interest rate and minus the proportional cost-of-carry, respectively. The money-market account evolves according to

$$\frac{dB(t)}{B(t)} = r dt, \quad B(0) = 1.$$

Suppose that a security promises a payment of $\$H$ at time T , where H is a random variable that can be represented by some well-behaved functional taken on the stock price up to time T . Then the fair price at time t of this claim can be represented as

$$F(t) = E_t[e^{-r(T-t)}H], \quad (2)$$

where $E_t[\cdot]$ denotes expectation taken under the measure \mathcal{Q} conditional on the information at time t .

One might also solve the security valuation problem by applying a change of numeraire resulting in the alternative valuation equation

$$F(t) = S(t)E_t\left[e^{-q(T-t)}\frac{H}{S(T)}\right], \quad (3)$$

where $E'_t[\cdot]$ denotes conditional expectation under \mathcal{Q}' , which is defined by

$$d\mathcal{Q}' = \frac{S(T)}{S(t)e^{(r-q)(T-t)}} d\mathcal{Q} \tag{4}$$

on $[t, T]$. By the Girsanov theorem it follows that, under \mathcal{Q}' ,

$$W'(t) = W(t) - \sigma t, \tag{5}$$

and so

$$\frac{dS(t)}{S(t)} = (r - q + \sigma^2) dt + \sigma dW'(t).$$

When H depends on the whole path of the underlying up to the terminal date T , we should in principle keep track of the whole path of the underlying up to current time t , in order to calculate the expectation in the valuation equation (2) or the expectation in the alternative valuation equation (3). However, if we are able to come up with a *Markov* process x , with evolution

$$dx(t) = \mu(t, x(t)) dt + v(t, x(t)) dW'(t),$$

so that

$$\frac{H}{S(T)} = \zeta(x(T))$$

for some function $\zeta(\cdot)$, then it is not necessary to keep track of the whole path of the underlying. Because of the Markov property of x , the expectation in (3) can be evaluated by keeping track only of the current value of x . Hence, the deflated option price $f \equiv F/S$ will be a function of $(t, x(t))$ only and f can be represented as the solution to the one-dimensional PDE

$$qf = \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}v^2 \frac{\partial^2 f}{\partial x^2},$$

subject to the terminal boundary condition $f(T, x) = \zeta(x)$. The PDE can be solved numerically by finite-difference techniques, which, as we will demonstrate, is much faster than solving the expectation by Monte Carlo methods.

We will now show that such a Markov representation is indeed possible for the Asian options with fixed and floating strikes and for the lookback options with fixed and floating strikes.

3. THE ASIAN OPTION WITH FIXED STRIKE

Let

$$0 = t_0 \leq t_1 < \dots < t_n \leq t_{n+1} = T,$$

and define

$$A(t) = \sum_{1 \leq i \leq n: t_i \leq t} S(t_i)$$

$$m(t) = \sup\{1 \leq i \leq n : t_i \leq t\}.$$

The Asian option with fixed strike promises the holder the time- T payment¹

$$\left(\frac{1}{n}A(T) - K\right)^+,$$

where K is some fixed amount—the strike price.

The object now is to evaluate the time- t fair price of the option,

$$F(t) = S(t)\mathbb{E}_t^Q\left[e^{-q(T-t)}\left(\frac{1}{n}A(T) - K\right)^+\right]. \quad (6)$$

Let

$$x(t) = \frac{\frac{1}{n}A(t) - K}{S(t)}. \quad (7)$$

When we hit an observation point t_i , the process x will jump by $1/n$. To see this, note that, for $1 \leq i \leq n$, we have²

$$\begin{aligned} x(t_i+) &= \frac{\frac{1}{n}\sum_{j=1}^i S(t_j) - K}{S(t_i)} \\ &= \frac{\frac{1}{n}(\sum_{j=1}^{i-1} S(t_j) + S(t_i)) - K}{S(t_i)} \\ &= \frac{\frac{1}{n}\sum_{j=1}^{i-1} S(t_j) - K}{S(t_i)} + \frac{1}{n} \\ &= x(t_i-) + \frac{1}{n}. \end{aligned}$$

At times between observations, the process x evolves continuously, because only the denominator in (7) changes as time evolves. Hence, using Itô's lemma, we have³

$$\begin{aligned} dx(t) &= -(r - q)x(t-)dt - \sigma x(t-)dW'(t) + \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{t=t_i} \\ &= -(r - q)x(t-)dt - \sigma x(t-)dW'(t) + \frac{1}{n}dm(t). \end{aligned}$$

We see that, under \mathcal{Q}' , the evolution of x depends only on x itself, so x is a Markov process under \mathcal{Q}' . This implies that evaluation of the expectation in (6) requires knowledge only of the current position of x , and we may write $F(t)/S(t) = f(t, x(t))$.

Further, we observe that if $x(t) \geq 0$ then $x(u) \geq 0$ for all $u \geq t$ with probability 1. This implies that, for all $x(t) \geq 0$,

$$\begin{aligned} f(t, x(t)) &= \mathbb{E}'[e^{-q(T-t)}x(T)^+ \mid x(t)] \\ &= \mathbb{E}'[e^{-q(T-t)}x(T) \mid x(t)] \\ &= e^{-r(T-t)}x(t) + \frac{1}{n}\sum_{i:t < t_i \leq T} e^{-r(T-t_i)-q(t_i-t)} \\ &\equiv g(t, x(t)). \end{aligned} \quad (8)$$

The third equality is given in the Appendix.

¹ We define $z^+ = \max(0, z)$.

² We define $z(t-) = \lim_{\epsilon \rightarrow 0} z(t - |\epsilon|)$ and $z(t+) = \lim_{\epsilon \rightarrow 0} z(t + |\epsilon|)$.

³ We let $\mathbf{1}_A$ denote the indicator function on the set A .

We note that if $x(t) < 0$ then the process x can only pass through the level $x = 0$ at the future sampling points $\{t_i\}_{m(t) < i \leq n}$. Suppose x passes the level $x = 0$ at some point t_i ($i \leq n$). We then have

$$f(t_i, x(t_i)) = g(t_i, x(t_i)) = g(t_i, x(t_i^-) + 1/n).$$

In the case that the level $x = 0$ is not passed for any $\{t_i\}_{i=1, \dots, n}$, the holder of option will receive nothing. To formalize this, let us define τ to be the first passage time among the observation points $\{t_i\}_{i=1, \dots, n}$ of the level $x = 0$ for the process x , i.e.

$$\begin{aligned} \tau &= \inf\{\{t_i\}_{i=1, \dots, n} : x(t_i) \geq 0\} \\ &= \inf\{\{t_i\}_{i=1, \dots, n} : x(t_i^-) \geq -1/n\}. \end{aligned}$$

We can then write

$$f(t, x(t)) = E[e^{-r(\tau-t)}g(\tau, x(\tau)) \mid x(t)] \tag{9}$$

$$= E[e^{-r(\tau-t)}g(\tau, x(\tau^-) + 1/n) \mid x(t)]. \tag{10}$$

Solving for f is then a first passage time problem for a Markovian process. This demonstrates the parallel to an ‘up-and-in’ barrier option: the stock price deflated option price, f , is the risk-adjusted expectation of the discounted value of a payoff at the first passage time to a certain level. This problem can be formulated as a PDE problem that can be solved numerically, as we will show in the section below.

Before we do so, we note that

$$F(t) \Big|_{K=K'} = S(t)f\left(t, \frac{\frac{1}{n}A(t) - K'}{S(t)}\right),$$

which gives us the possibility of solving for more than one option price once the function f is identified.

It should be mentioned that this technique does not enable us to solve for the American style fixed strike Asian option price. The reason is that the state variable x does not supply sufficient information to determine the (deflated) intrinsic value of the Asian option at any time point before maturity of the option. This follows from the definition of x in equation (7). Consequently, to solve the American style problem, we would have to additionally keep track of the level of the underlying stock, i.e. we would be back to a two-dimensional formulation of the fixed strike Asian option pricing problem. However, as we shall see in the next section, if the strike is floating, the American exercise problem can be handled using the numeraire technique.

3.1 Numerical Solution and Numerical Results

One can now set up a system of trinomial or binomial trees that discretize the random evolution of x . Except from the observation points $\{t_i\}_{i=1, \dots, n}$, the state variable x evolves as a geometric Brownian motion, so a Cox, Ross, and Rubinstein (1979) binomial tree applied to the x process could be used on the regions between observation points. However, the jumps at the observation points are constant and additive. This works poorly with a standard binomial tree, which is normally specified to be log-linear. So we choose not to use this approach.

As yet another alternative, one might use the fact that $x(t_i)$ conditional on $x(t_{i-1})$ is lognormal under \mathcal{Q} . Discretizing the state space in the x dimension will therefore make it possible to solve for the option prices using numerical integration at each point t_i and recursing backwards to current time. This might not be more computationally efficient than finite-difference solution of PDEs, since (implicit) finite-difference approximation as the hardest part involves numerical inversion of linear tridiagonal systems and this can be performed in linear computer time. We therefore choose to concentrate on the finite-difference techniques.

We now need to identify the PDE system for the numerical solution of the Asian option price or, rather, f . This can be done directly in (t, x) . We prefer, though, to get rid of the discontinuous dynamics by introducing

$$y(t) = x(t) - \frac{m(t)}{n}.$$

We now have

$$dy(t) = -(r - q)\left(y(t) + \frac{m(t)}{n}\right) - \sigma\left(y(t) + \frac{m(t)}{n}\right) dW'(t). \tag{11}$$

Since $e^{-qt}f(t)$ is a \mathcal{Q} martingale, Itô's lemma and the martingale representation theorem together imply that f is the solution to the PDE

$$qf = \frac{\partial f}{\partial t} - (r - q)\left(y + \frac{m(t)}{n}\right) \frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2\left(y + \frac{m(t)}{n}\right)^2 \frac{\partial^2 f}{\partial y^2} \tag{12}$$

on $\{(t, y) : t_{i-1} < t < t_i, y < -(i - 1)/n; i = 1, \dots, n\}$, subject to the boundary conditions

$$f(t_i^-, y) = \begin{cases} f(t_i^+, y) & \text{if } y < -i/n, \\ g(t_i, y + i/n) & \text{if } -(i - 1)/n > y \geq -i/n, \end{cases} \tag{13}$$

and

$$f(t, y) = 0 \quad \text{if } y < -1 \tag{14}$$

for $t \geq t_n$.

Between two observation points, i.e. on each of the intervals $[t_{i-1}, t_i]$, the PDE (12) can be solved numerically using a Crank–Nicolson scheme. The idea is to approximate the differentials in (12) by central differences. To do this, we rewrite the PDE (12) as

$$0 = \left[-q + \frac{\partial}{\partial t} + \mu(y) \frac{\partial}{\partial x} + v(y) \frac{\partial^2}{\partial x^2} \right] f, \tag{15}$$

where

$$\mu(y) = -(r - q)\left(y + \frac{m(t)}{n}\right) \quad \text{and} \quad v(y) = \frac{1}{2}\sigma^2\left(y + \frac{m(t)}{n}\right)^2.$$

We suppress the notional dependence of μ and v on t , because $m(t)$ is constant on each subinterval $[t_{i-1}, t_i]$.

If we approximate the differentials in (15) by central differences in the point $(t + \frac{1}{2}\Delta t, x)$, we get the (Crank–Nicolson) partial *difference* equation

$$\left[\frac{1}{2}q + \frac{1}{\Delta t} - \frac{1}{2}\mu(y)\delta_y - \frac{1}{2}v(y)\delta_{yy} \right] f(t, y) = \left[-\frac{1}{2}q + \frac{1}{\Delta t} + \frac{1}{2}\mu(y)\delta_y + \frac{1}{2}v(y)\delta_{yy} \right] f(t + \Delta t, y), \tag{16}$$

where δ_y and δ_{yy} are difference operators defined by

$$\delta_y h(y) = \frac{1}{2\Delta y} [h(y + \Delta y) - h(y - \Delta y)],$$

$$\delta_{yy} h(y) = \frac{1}{(\Delta y)^2} [h(y + \Delta y) - 2h(y) + h(y - \Delta y)].$$

For the interval $[t_{i-1}, t_i]$, we limit our state space to the discrete grid

$$\{(s_k, y_l)\}_{k=0, \dots, K; l=0, \dots, L},$$

with

$$s_k = t_{i-1} \frac{K-k}{K} + t_i \frac{k}{K} \quad \text{and} \quad y_l = y_{\min} \frac{L-l}{L} + y_{\max} \frac{l}{L}.$$

Here we have $\Delta t = (t_i - t_{i-1})/K$ and $\Delta y = (y_{\max} - y_{\min})/L$. The upper bound of the grid is dictated to be the upper limit of the domain of f , so $y_{\max} = -(i-1)/n$; the lower bound has to satisfy $y_{\min} < -1$. Typically $y_{\min} = -2$ can be chosen for maturities less than one year. By supplying the artificial boundary conditions $\delta_{yy} f = 0$ at the boundaries y_{\min} and y_{\max} of the grid, we can now state the partial difference equation (16) as a sequence of matrix equations

$$\mathbf{A}f(s_k) = \mathbf{B}f(s_{k+1}), \tag{17}$$

where f is the vector

$$f(s_k) = [f(s_k, y_0), \dots, f(s_k, y_L)]^\top$$

and \mathbf{A} and \mathbf{B} are $(L+1)$ -dimensional tridiagonal matrices with the l th rows given by

$$\mathbf{A}_l = \left(0, \dots, 0, \frac{\mu(y_l)}{2\Delta y} - \frac{v(y_l)}{2(\Delta y)^2}, \frac{q}{2} + \frac{1}{\Delta t} + \frac{v(y_l)}{(\Delta y)^2}, -\frac{\mu(y_l)}{2\Delta y} - \frac{v(y_l)}{2(\Delta y)^2}, 0, \dots, 0 \right),$$

$$\mathbf{B}_l = \left(0, \dots, 0, -\frac{\mu(y_l)}{2\Delta y} + \frac{v(y_l)}{2(\Delta y)^2}, -\frac{q}{2} + \frac{1}{\Delta t} - \frac{v(y_l)}{(\Delta y)^2}, +\frac{\mu(y_l)}{2\Delta y} + \frac{v(y_l)}{2(\Delta y)^2}, 0, \dots, 0 \right),$$

for $l = 1, \dots, L-1$, and

$$\mathbf{A}_0 = \left(\frac{q}{2} + \frac{1}{\Delta t}, 0, 0, \dots, 0 \right),$$

$$\mathbf{A}_L = \left(0, \dots, 0, \frac{\mu(y_L)}{\Delta y}, \frac{q}{2} + \frac{1}{\Delta t} - \frac{\mu(y_L)}{\Delta y} \right),$$

$$\mathbf{B}_0 = \left(-\frac{q}{2} + \frac{1}{\Delta t}, 0, 0, \dots, 0 \right),$$

$$\mathbf{B}_L = \left(0, \dots, 0, -\frac{\mu(y_L)}{\Delta y}, -\frac{q}{2} + \frac{1}{\Delta t} + \frac{\mu(y_L)}{\Delta y} \right).$$

When solving (12), subject to (13) and (14), numerically, we start at time t_n- . By the boundary conditions (13) and (14), we get the values of $f(t_n-)$. We then numerically solve back to time $t_{n-1}+$ by recursively solving the matrix system (17). At time $t_{n-1}+$, the numerical solution together with the function $g(t_{n-1}, \cdot)$ acts as terminal boundary condition for the numerical solution on (t_{n-2}, t_{n-1}) . We now continue like this back to current time, where we get the current value of f and thereby the option price.

Note that the state space of the process y changes as time progresses. At each time, we have $y \leq -(i - 1)/n$ when $t_{i-1} < t < t_i$. But the state space is constant for all t between two observation points, and running backwards in time, the new added regions have boundary conditions specified by the known function $g(\cdot, \cdot)$.

The fact that the matrices A and B are tridiagonal means that the computational effort of solving equation (17) is of order $O(L)$. This in turn implies that the computational burden of the total scheme is of order $O(K \cdot L)$. If we choose $K = O(L)$, this is similar to the computational complexity of a binomial tree as the one of Cox, Ross, and Rubinstein (1979).

The solution technique applied here is a Crank–Nicolson scheme. We refer the reader to Mitchell and Griffiths (1980) for a detailed description of the properties of the Crank–Nicolson scheme, but among its nice properties are that it is uniformly stable and that its local precision is of order $(\Delta t)^2 + (\Delta y)^2$, which is maximal for standard finite-difference schemes for partial differential equations of the parabolic type.

Table 1 compares option prices for various strikes generated by the finite-difference algorithm with different grid sizes to option prices obtained by Monte Carlo simulations. For reference we also report the CPU times for generating the option prices using the two different techniques.

We see that the finite-difference algorithm for this option is surprisingly accurate, and that the prices change very little as the grid size is changed. The maximum relative error compared with the Monte Carlo procedure is approximately 0.4%. Here it is important to note that the Monte Carlo price is not an absolute figure. It might vary a *little* from simulation to simulation; as mentioned in the heading to Table 1, the standard deviation of the option prices is approximately 3×10^{-3} .

For the reported CPU times, here and in the following, it should be noted that all programming was done in C and the hardware used was a Hewlett-Packard 9000 Unix system.

TABLE 1. The parameters are: $r = 0.05$, $q = 0.0$, $\sigma = 0.2$, $T = 1.0$, $t = 0.0$, $n = 10$, $S(0) = 100.0$, $t_i = 0.1i$. MC refers to Monte Carlo solution and FD refers to finite-difference solution. The different I 's refer to the number of time steps. We used $I/10$ number of steps per jump size $1/n$ in the y direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard deviation of the Monte Carlo estimated prices is estimated as 3×10^{-3} .
Reported CPU times are for all 9 strikes.

K	Asian option prices			
	MC	FD ($I = 500$)	FD ($I = 100$)	FD ($I = 50$)
90.0	12.98	12.99	12.99	12.98
92.5	11.05	11.05	11.05	11.05
95.0	9.27	9.27	9.27	9.27
97.5	7.67	7.66	7.66	7.66
100.0	6.24	6.23	6.23	6.23
102.5	5.01	5.00	5.00	5.00
105.0	3.96	3.95	3.95	3.95
107.5	3.08	3.07	3.07	3.07
110.0	2.36	2.35	2.35	2.36
CPU	46.0 s	0.65 s	0.06 s	0.04 s

Let us briefly describe the Monte Carlo technique. We apply a control variate technique to our Monte Carlo simulations in order to decrease the number of simulations needed. That is, we simulate a collection of paths $\{(S(t_1), \dots, S(t_n))(\omega)\}_\omega$ under \mathcal{Q} , and consider the regression equation

$$e^{-r(T-t)} \left(\frac{1}{n} A(T) - K \right)^+ (\omega) = E_t \left[e^{-r(T-t)} \left(\frac{1}{n} A(T) - K \right)^+ \right] + \left[\sum_{i=1}^n a_i (S(t_i) - S(0)) e^{(r-q)(t_i-t)} \right] (\omega),$$

where a_1, \dots, a_n are constants. Note that the regressors under the sum have zero \mathcal{Q} mean. We run an ordinary least-squares regression on this and simultaneously estimate the coefficients $\{a_i\}$ and the \mathcal{Q} mean of the payoff, i.e. the fair price of the option. This also gives us an estimate of the standard deviation of the estimate of the parameters, i.e. an estimate of the standard deviation of the Monte Carlo option prices. The properties of the procedure are described in detail in Davidson and Mackinnon (1993). One can also include different powers of the stock price minus its moments as control variates. We choose not to, because the stock prices alone give sufficient precision for our purpose and because the presence of additional parameters to be estimated makes the Monte Carlo procedure more computationally demanding.

4. THE AVERAGE STRIKE OPTION

With the definitions in the previous section, the terminal time- T payoff of the average strike (put) option can be written as

$$\left(\frac{1}{n} A(T) - \alpha S(T) \right)^+,$$

where α is a constant.

Using the valuation equation (3), we find that the time- t price of the option is given by

$$F(t) = S(t) E_t' \left[e^{-q(T-t)} \left(\frac{1}{n} \frac{A(T)}{S(T)} - \alpha \right)^+ \right].$$

For $t \geq t_1$, define $x(t)$ by

$$x(t) = \frac{A(t)}{S(t)}. \tag{18}$$

Applying the same argument as in the previous section, we get, for $t \geq t_1$,

$$\begin{aligned} dx(t) &= -(r - q)x(t-)dt - \sigma x(t-)dW'(t) + dm(t), \\ x(t_1) &= 1, \end{aligned}$$

This is a Markov process with domain on $x > 0$. The object is now to evaluate the initial-value problem

$$F(t)/S(t) \equiv f(t, x(t)) = E' \left[e^{-q(T-t)} \left(\frac{1}{n} x(T) - \alpha \right)^+ \mid x(t) \right]. \tag{19}$$

Owing to the Markovian property of x , this can be done by solving a sequence of PDEs, as we shall more formally describe in the following section.

Suppose that we want to evaluate an average strike option with an American feature, i.e. the option might be exercised at some time t in the interval $[t_1, T]$ with resulting payout

$$\left(\frac{1}{m(t)} A(t) - \alpha S(t) \right)^+.$$

Finding the fair price of such a contract is a stopping time problem, in the sense that we are supposed to find the exercise time that maximizes the value of the option. To formalize this, let \mathcal{T} be the set of stopping times on the interval $[t_1, T]$ with respect to the filtration generated by the stock price. Then the average strike option with the American feature has the fair value

$$\begin{aligned} F(t) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}_t \left[e^{-r(\tau-t)} \left(\frac{1}{m(\tau)} A(\tau) - \alpha S(\tau) \right)^+ \right] \\ &= S(t) \sup_{\tau \in \mathcal{T}} \mathbb{E}_t \left[e^{-q(\tau-t)} \left(\frac{1}{m(\tau)} \frac{A(\tau)}{S(\tau)} - \alpha \right)^+ \right] \\ &= S(t) \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[e^{-q(\tau-t)} \left(\frac{1}{m(\tau)} x(\tau) - \alpha \right)^+ \mid x(t) \right]. \end{aligned} \tag{20}$$

This defines a Markovian stopping time problem for $f = F/S$ that can be treated in a free-boundary formulation, as we shall illustrate in the following section.

Both in the American and the European style case we have, for $t \geq t_1$,

$$F(t) = S(t) f(t, x(t)).$$

Applying the alternative valuation equation (3) to this quantity, we get, for t_1 ,

$$F(t) = S(t) e^{-q(t_1-t)} f(t_1, 1).$$

It should be mentioned that the above results for the average strike option for the continuous observation case have previously been obtained through PDE techniques by Ingersoll (1987) and Wilmott, Dewynne, and Howison (1993).

4.1 Numerical Solution and Results

As for the fixed strike case, we introduce

$$y(t) = x(t) - m(t),$$

and we have

$$dy(t) = -(r - q)(y(t) + m(t)) dt - \sigma(y(t) + m(t)) dW'(t).$$

On $t_{i-1} < t < t_i$, with $i > 1$, $e^{-qt} f$ is a \mathcal{Q}^i martingale and therefore the solution to

$$qf = \frac{\partial f}{\partial t} - (r - q)(y + m(t)) \frac{\partial f}{\partial y} + \frac{1}{2} \sigma^2 (y + m(t))^2 \frac{\partial^2 f}{\partial y^2}$$

on $y > -i$, subject to the boundary conditions

$$\begin{aligned}
 f(t_{i-}, y) &= f(t_{i+}, y), \\
 f(t_{n+1+}, y) &= f(T, y) = \left(\frac{1}{n}y + 1 - \alpha\right)^+.
 \end{aligned}
 \tag{22}$$

The American style average strike option can be handled by adding the free-boundary condition

$$f(t, y) \geq \left(\frac{1}{m(t)}y + 1 - \alpha\right)^+.
 \tag{23}$$

We apply a linear grid to this problem, supply the same ‘artificial’ boundary conditions as for the fixed strike, and again use the Crank–Nicolson scheme as in (16). This means that (21) can be solved as a sequence of tridiagonal matrix equations as in (17).

Table 2 gives prices generated by the finite-difference algorithm and compares these quantities to numbers generated by Monte Carlo simulations. As for the fixed strike Asian, the precision of the finite-difference solution is remarkable. Even though the grid size changes by a factor 10, the relative price changes are less than 0.6% for all strikes. The maximum relative deviation to the Monte Carlo solution is about 1.2%. But it should again be emphasized that the Monte Carlo solution need not be more accurate than the finite-difference solutions and serves only as a benchmark. The longer computer times here compared with the Asian options is due to the fact that here a finite-difference algorithm has to be run for each α , whereas for the Asian options we need only solve one finite-difference grid to obtain the prices for all strikes.

TABLE 2. The parameters are: $r = 0.05$, $q = 0.0$, $\sigma = 0.2$, $T = 1.0$, $t = 0.0$, $n = 10$, $S(0) = 100.0$, $t_j = 0.1i$. MC refers to Monte Carlo solution, and FD refers to finite-difference solution. The different l 's refer to the number of time steps. We used $l/10$ number of steps per jump size $1/n$ in the y direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard error on the estimated Monte Carlo option prices is approximately 3.0×10^{-3} . Reported CPU times are for all 9 strikes.

α	Average strike option prices			
	MC	FD ($I = 500$)	FD ($I = 100$)	FD ($I = 50$)
0.900	8.98	8.98	8.98	8.99
0.925	7.18	7.18	7.17	7.18
0.950	5.61	5.60	5.60	5.58
0.975	4.28	4.27	4.26	4.26
1.000	3.18	3.18	3.18	3.18
1.025	2.31	2.31	2.30	2.30
1.050	1.65	1.64	1.64	1.63
1.075	1.14	1.14	1.14	1.14
1.100	0.78	0.77	0.77	0.77
CPU	48.0 s	1.94 s	0.12 s	0.05 s

5. THE LOOKBACK OPTION WITH FIXED STRIKE

For $t \geq 0$, define

$$\bar{S}(t) = \sup_{1 \leq i \leq m(t)} S(t_i),$$

with the convention $\bar{S}(t) = 0$ for $t \leq t_1$.

The fixed strike lookback option promises the time- T payment

$$(\bar{S}(t) - K)^+.$$

The solution of this pricing problem is a two-step procedure. First, we solve the option price at time t when $\bar{S}(t) \geq K$. We then solve for the case $\bar{S}(t) < K$ by observing that in this case the option might be viewed as a first passage problem of S to the level K where the reward is equal to the value of the option at $\bar{S}(t) \geq K$.

Suppose $\bar{S}(t) \geq K$. We then have

$$\begin{aligned} F(t) &= E_t[e^{-r(T-t)}(\bar{S}(T) - K)^+] \\ &= E_t[e^{-r(T-t)}(\bar{S}(T) - K)] \\ &= S(t)E_t\left[e^{-q(T-t)}\frac{\bar{S}(T)}{S(T)}\right] - e^{-r(T-t)}K. \end{aligned} \tag{24}$$

Define

$$x(t) = \frac{\bar{S}(t)}{S(t)} \tag{25}$$

for $t \geq t_1$.

For $1 \leq i \leq n$, we have

$$x(t_i+) = \begin{cases} 1 & \text{if } x(t_i-) \leq 1, \\ x(t_i-) & \text{if } x(t_i-) > 1. \end{cases}$$

Elsewhere the evolution of x is continuous, and for $t \geq t_1$ we have

$$\begin{aligned} dx(t) &= -(r - q)x(t-)dt - \sigma x(t-)dW'(t) + (1 - x(t-))^+ dm(t), \\ x(t_1) &= 1. \end{aligned}$$

So x is a Markov process with domain on $x > 0$.

Define

$$\begin{aligned} f(t) &= E_t\left[e^{-q(T-t)}\frac{\bar{S}(T)}{S(T)}\right] \\ &= E_t[e^{-q(T-t)}x(T)] \\ &= E[e^{-q(T-t)}x(T) \mid x(t)], \end{aligned} \tag{26}$$

where the last equality follows from the Markovian property of x .

The quantity f can be written as $f(t) = f(t, x(t))$ and can be found by numerically solving the PDE related to the initial-value problem (26). We will show how this is done in the section below.

This establishes the option price at $t \geq t_1$ for $\bar{S}(t) \geq K$ explicitly as

$$F(t) = S(t)f(t, x(t)) - e^{-r(T-t)}K. \tag{27}$$

Suppose we are sitting at time $t \geq t_1$ with $\bar{S}(t) < K$. The first time $t_i > t$ ($i \leq n$), with S above K , we get a reward of

$$\begin{aligned} F(t_i) &= S(t_i)f(t_i, x(t_i)) - e^{-r(T-t_i)}K \\ &= S(t_i)f(t_i, 1) - e^{-r(T-t_i)}K. \end{aligned} \tag{28}$$

The second equality is valid because, in the above, t_i is the first time $\bar{S}(t)$ goes above K . If a level of K or above is not hit at any of the sampling times t_i ($i = 1, \dots, n$), the holder of the option receives nothing.

Equation (28) implies that, for $t \geq 0$ with $\bar{S}(t) < K$, we may write the option price as

$$\begin{aligned} F(t) &= E_t[e^{-r(\tau-t)}(S(\tau)f(\tau, 1) - e^{-r(T-\tau)}K)\mathbf{1}_{\tau \leq t_n}] \\ &= E[e^{-r(\tau-t)}(S(\tau)f(\tau, 1) - e^{-r(T-\tau)}K)\mathbf{1}_{\tau \leq t_n} \mid S(t)], \end{aligned} \tag{29}$$

where

$$\tau = \inf_{i=1, \dots, n} \{t_i : S(t_i) \geq K\},$$

with the convention $\inf \emptyset = \infty$.

This shows the parallel to an up-and-in barrier option. When f is known, F can be found by numerically solving the first passage time problem (29). We illustrate how this is done in the section below. Finding the option price is therefore a two-step procedure. First we solve for $\{f(u, x)\}$ for all (u, x) with $u \geq \max(t, t_1)$. This is done by numerically solving a initial-value problem from T down to t . If $\bar{S}(t) \geq K$, then the option price is given by (27). Otherwise we keep $\{f(t_i, 1)\}_{1 \leq i \leq n: t_i > t}$ and solve the first passage time problem (29).

5.1 Numerical Solution and Results

The accuracy of the numerical solution of partial differential equations is generally improved if the variables are transformed so that the diffusion term is constant. We therefore perform a log transformation. Let $y = \ln x$ and consider⁴

$$dy(t) = -(r - q + \frac{1}{2}\sigma^2) dt - \sigma dW'(t) + y(t-)^- dm(t).$$

Since $e^{-qt}f(t)$ is a \mathcal{Q}' martingale and y is Markovian, the solution to the initial-value problem (26) can be found as the solution to the following system of PDEs. On $t_{i-1} < t < t_i$ ($i > 1$), f solves

$$qf = \frac{\partial f}{\partial t} - (r - q + \frac{1}{2}\sigma^2) \frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2}, \tag{30}$$

⁴ The notation $(\cdot)^-$ is defined by $z^- = -\min(0, z)$.

subject to the boundary conditions

$$\begin{aligned}
 f(t_i^-, y) &= \begin{cases} f(t_i^+, 0) & \text{if } y < 0, \\ f(t_i^+, y) & \text{if } y \geq 0, \end{cases} \\
 f(t_{n+1}^+, y) &= f(T, y) = e^y.
 \end{aligned} \tag{31}$$

Now redefine y and let $y(t) = \ln(S(t)/K)$. The first passage time problem (29) can be handled by noting that, for $\bar{S}(t) < K$, $g \equiv F/K$ is the solution to

$$rg = \frac{\partial g}{\partial t} + (r - q - \frac{1}{2}\sigma^2) \frac{\partial g}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial y^2} \tag{32}$$

on $\{(t, y) : t_{i-1} < t < t_i \ (i = 1, \dots, n); y < 0\}$, subject to the boundary conditions

$$\begin{aligned}
 g(t_i^-, y) &= \begin{cases} g(t_i^+, y) & \text{if } y < 0, \\ e^y f(t_i, 0) - e^{-r(T-t_i)} & \text{if } y \geq 0, \end{cases} \\
 g(t_n^+, y) &= 0 \quad \text{for } y < 0.
 \end{aligned} \tag{33}$$

The $f(t, \cdot)$ in (33) should be interpreted as function of y , as in (30). This means that we can treat f and g in the same grid and simultaneously solve for f and g , at each time step, in that respective order. At current time t , options of different strikes are generated by $F(t) = Kg(t, S(t)/K)$.

As for the Asian options, we apply the Crank–Nicolson scheme (16) to the numerical solution of this problem, where we supply the ‘artificial’ boundary conditions⁵

$$\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial y} = \frac{\partial^2 g}{\partial y^2} - \frac{\partial g}{\partial y} = 0$$

at the upper and the lower bound of the grid. We arrange the grid so that the level $y = 0$ is on the grid and the points $\{t_i\}$ are among the time points of the grid.

Table 3 shows option prices generated by finite difference and compares these with Monte Carlo solutions. Comparing the finite-difference solution on the 500×500 grid with the Monte Carlo solution shows a maximal relative error of approximately 0.1%, which is clearly within any reasonable demands for precision. But the finite-difference solutions for the two smaller grids do not show sufficient precision. This must be attributed to the two-step procedure involved here; numerical errors might be accumulated in the two steps. The conclusion is that this type of option requires a finer mesh than the options considered in the previous sections.

6. THE LOOKBACK OPTION WITH FLOATING STRIKE

With the definitions of the previous sections the time- T payoff of a floating strike

⁵ These conditions are equivalent to the condition $\partial^2 f / \partial x^2 = \partial^2 F / \partial S^2 = 0$.

TABLE 3. The parameters are: $r = 0.05$, $q = 0.0$, $\sigma = 0.2$, $T = 1.0$, $t = 0.0$, $n = 10$, $S(0) = 100.0$, $t_i = 0.1i$. MC refers to Monte Carlo solution, and FD refers to finite-difference solution. The different I 's refer to the number of time steps *and* also to the number of steps in the γ direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard deviations of the Monte Carlo prices are approximately 3.0×10^{-3} . Reported CPU times are for all 9 strikes.

K	Fixed strike lookback option prices			
	MC	FD ($I = 500$)	FD ($I = 100$)	FD ($I = 50$)
90.0	24.41	24.39	24.27	23.81
92.5	22.07	22.06	21.93	21.47
95.0	19.78	19.77	19.64	19.18
97.5	17.57	17.56	17.43	16.96
100.0	15.48	15.47	15.34	14.87
102.5	13.53	13.52	13.39	12.95
105.0	11.75	11.74	11.62	11.22
107.5	10.14	10.14	10.03	9.67
110.0	8.70	8.71	8.62	8.30
CPU	46.0 s	0.68 s	0.06 s	0.03 s

lookback option can be expressed as

$$(\bar{S}(T) - \alpha S(T))^+.$$

Of the options considered in this paper, this is the easiest option to evaluate numerically. For $t \geq t_1$, the fair price is given by

$$\begin{aligned} F(t) &= E_t' \left[e^{-q(T-t)} \left(\frac{\bar{S}(T)}{S(T)} - \alpha \right)^+ \right] \\ &= E_t' \left[e^{-q(T-t)} (x(T) - \alpha)^+ \mid x(t) \right], \end{aligned}$$

where x is defined as in (25).

This is also observed by Babbs (1992), who treats the American style case in ways similar to what is outlined below. However, Babbs uses a binomial tree for the numerical solution. Wilmott, Dewynne, and Howison (1993) derive the result by manipulation of the fundamental PDE.

Letting $f = F/S$, f solves a Markovian initial boundary problem equivalent to (26). In the section below we supply the PDE with boundary conditions associated with this problem.

If we want to consider a floating strike lookback option with an American feature, note that the fair price of such a contract can be represented as

$$\begin{aligned} F(t) &= \sup_{\tau \in \mathcal{T}} E_t' \left[e^{-r(\tau-t)} (\bar{S}(\tau) - \alpha S(\tau))^+ \right] \\ &= S(t) \sup_{\tau \in \mathcal{T}} E_t' \left[e^{-q(\tau-t)} \left(\frac{\bar{S}(\tau)}{S(\tau)} - \alpha \right)^+ \right] \\ &= S(t) \sup_{\tau \in \mathcal{T}} E_t' \left[e^{-q(\tau-t)} (x(\tau) - \alpha)^+ \mid x(t) \right], \end{aligned} \tag{34}$$

where \mathcal{T} is the set of stopping times on $[t_1, T]$ adapted to the filtration generated by S . As in (20), this is a Markovian stopping time problem that can be reformulated as a free-boundary problem for $f = F/S$. We formulate this as a PDE problem in the section below. In both the European and the American style floating lookback option, we have

$$F(t) = \begin{cases} S(t)e^{-q(t_1-t)}f(t_1, 1) & \text{if } t < t_1, \\ S(t)f(t, x(t)) & \text{if } t \geq t_1. \end{cases}$$

6.1 Numerical Solution and Results

As for the fixed strike lookback, we choose to log-transform the state variable and define $y = \ln x$. We now find that f solves the PDE

$$qf = \frac{\partial f}{\partial t} - (r - q + \frac{1}{2}\sigma^2) \frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2}$$

when $t_{i-1} < t < t_i$, with $i > 1$, subject to the boundary conditions

$$\left. \begin{aligned} f(t_{i-}, y) &= \begin{cases} f(t_i+, 0) & y < 0, \\ f(t_i+, y) & y \geq 0, \end{cases} \\ f(t_{n+1-}, y) &= f(T, y) = (e^y - \alpha)^+. \end{aligned} \right\} \tag{35}$$

If we consider an American style option, we have to add the free-boundary condition

$$f(t, y) \geq (e^y - \alpha)^+. \tag{36}$$

We apply the same ‘artificial’ boundary conditions as in the previous section and again we use the Crank–Nicolson scheme for the numerical solution.

TABLE 4. The parameters are: $r = 0.05$, $q = 0.0$, $\sigma = 0.2$, $T = 1.0$, $t = 0.0$, $n = 10$, $S(0) = 100.0$, $t_i = 0.1i$. MC refers to Monte Carlo solution, and FD refers to finite-difference solution. The different I 's refer to the number of time steps *and* also to the number of steps in the y direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard deviation of the Monte Carlo option prices is approximately 3×10^{-3} . Reported CPU times are for all 9 strikes.

α	Floating strike lookback option prices			
	MC	FD ($I = 500$)	FD ($I = 100$)	FD ($I = 50$)
1.000	10.01	10.00	9.96	9.86
1.025	8.27	8.26	8.23	8.14
1.050	6.77	6.76	6.74	6.68
1.075	5.51	5.50	5.47	5.43
1.100	4.46	4.45	4.42	4.39
1.125	3.59	3.58	3.56	3.53
1.150	2.88	2.87	2.85	2.81
1.175	2.30	2.29	2.28	2.25
1.200	1.83	1.82	1.81	1.79
CPU	46.0s	2.43 s	0.14 s	0.06 s

Table 4 gives option prices generated using the finite-difference solution and compares these with option prices found by Monte Carlo simulations.

We choose only to show prices for values of α greater than 1. This is because all options with $\alpha \leq 1$ are all ‘in the money’ with probability 1, because of the sampling of the maximum that we use here (we have $t_n = T$). This means that, for all $\alpha < 1$, the option contract has a value that equals the value of the contract with $\alpha = 1$ plus $S(0)(1 - \alpha)e^{-qT}$.

Comparing the finite-difference solutions with the Monte Carlo solutions we find that the maximal relative error is about 0.5% for the 500×500 grid, 1% for the 100×100 grid, and approximately 2% for the 50×50 grid. This is acceptable, but the example shows that one has to use a higher degree of precision for the lookback than for the Asian options.

7. DISCONTINUOUS RETURNS OF THE UNDERLYING

In this section we extend the model of the stock price to allow for discontinuous dynamics and show that the technique used in the previous sections can also be applied to this type of stock price behavior.

Under \mathcal{Q} , the stock is assumed to evolve according to the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = (r - q - k\lambda) dt + \sigma dW(t) + I(t) dN(t), \tag{37}$$

where r, q, σ, W are defined as in Section 2, N is a Poisson process with intensity λ and $\{I(t)\}_{t \geq 0}$ is a sequence of independent and identically distributed random variables with distribution given by

$$\ln(1 + I(t)) \stackrel{\mathcal{Q}}{\approx} N(\gamma - \frac{1}{2}\delta^2, \delta^2)$$

and \mathcal{Q} mean

$$k = E[I(t)] = e^\gamma - 1.$$

The processes W, I, N are assumed to be independent.

We note that the economy is now incomplete, i.e. there exists no perfect hedging strategy in the stock and the bond that replicates the payoff of derivatives, and the measure \mathcal{Q} is nonunique. However, this does not influence derivative pricing once a martingale measure \mathcal{Q} is fixed as above by simply assuming the \mathcal{Q} dynamics for the stock given by (37). Of course, the relation between the objective probability measure and the martingale measure matters if we are considering portfolio and hedging decisions, but that is beyond the scope of this paper, so we will avoid this discussion for the remainder of the paper.

Defining \mathcal{Q}' as in (4), the Girsanov theorem implies that⁶

$$\frac{dS(t)}{S(t-)} = (r - q - k\lambda + \sigma^2) dt + \sigma dW'(t) + I'(t) dN'(t),$$

⁶ Note that the \mathcal{Q}' measure is uniquely related to \mathcal{Q} , so that once \mathcal{Q} is fixed so is \mathcal{Q}' .

where W' is a \mathcal{Q}' Brownian motion, as given in (5), $\{I'(t)\}_{t \geq 0}$ is a sequence of independent identically distributed random variables with distribution given by

$$\ln(1 + I'(t)) \stackrel{\mathcal{Q}'}{\sim} N(\gamma + \frac{1}{2}\delta^2, \delta^2),$$

and N' is a \mathcal{Q}' Poisson process with intensity

$$\lambda' = \lambda(1 + k) = \lambda e^\gamma.$$

The processes W' , I' , N' are also independent under \mathcal{Q}' . In this type of economy, the valuation equations (2) and (3) are still valid.

7.1 Path-Dependent Options under Jumps

The tricks applied for the pricing of the options that we considered in the previous sections naturally extend to the case when the underlying exhibits jumps. To see this, let us consider the options one by one.

The Asian option with fixed strike has the value given by (6), and if we define x as in (6) we now have

$$dx(t) = -(r - q - k\lambda)x(t-)dt - \sigma x(t-)dW'(t) - x(t-)\frac{I'(t)}{1 + I'(t)}dN'(t) + \frac{1}{n}dm(t). \quad (38)$$

This is clearly a Markov process with the property that if $x(t) \geq 0$ then $x(u) \geq 0$ for all $u \geq t$ with probability 1. This implies that if $x(t) \geq 0$ then the deflated option price is given by

$$F(t)/S(t) \equiv f(t) = E'_t[x(T)] = g(t, x(t)), \quad (39)$$

where $g(\cdot, \cdot)$ is defined as in (8). The last equality is shown in the Appendix.

Now, if $x(t) < 0$, the process x can still only pass the level $x = 0$ at the points $\{t_i\}_{i=1, \dots, n}$, which again implies that, for $x(t) < 0$, we may write the deflated price f of the option as the solution to a first passage time problem, as we did in (9). We will return to how this is solved numerically in the section below. Once f is obtained, the option price is given by

$$F(t) = S(t)f(t, x(t)).$$

If, for the average strike option considered in Section 4, we define x as in (18), then

$$dx(t) = -(r - q - k\lambda)x(t-)dt - \sigma x(t-)dW'(t) - x(t-)\frac{I'(t)}{1 + I'(t)}dN'(t) + dm(t),$$

$$x(t_1) = 1,$$

for $t \geq t_1$. This is a Markov process with domain on $x > 0$. The solution to the deflated option price is now given as the solution to the Markovian initial-value problem

$$f(t, x(t)) = E' \left[e^{-q(T-t)} \left(\frac{1}{n}x(T) - \alpha \right)^+ \mid x(t) \right].$$

We show how to handle this numerically in the following section. For the average strike option with an American feature, we obtain the same type of Markovian stopping time

problem as in (20). This can be given a free-boundary formulation that we will consider in the next section. Given f , we have

$$F(t) = \begin{cases} S(t)f(t, x(t)) & \text{if } t \geq t_1, \\ S(t)e^{-q(t_1-t)}f(t, 1) & \text{if } t < t_1. \end{cases}$$

The lookback option with fixed strike can also be handled by the technique applied in Section 5. The key observations are the same. We first note that, for $\bar{S}(t) \geq K$, the option price can be written as in (27). Defining $x = \bar{S}/S$, Itô's lemma implies that, for $t \geq t_1$,

$$\left. \begin{aligned} dx(t) &= -(r - q)x(t-)dt - \sigma x(t-)dW'(t) \\ &\quad - x(t-)\frac{I'(t)}{1 + I'(t)}dN'(t) + (1 - x(t-))^+dm(t), \\ x(t_1) &= 1. \end{aligned} \right\} \quad (40)$$

This is clearly a Markov process with domain on $x > 0$. So

$$f(t) \equiv E'_t[e^{-q(T-t)}x(T)]$$

is the solution to a Markov initial-value problem. On the other hand, if $\bar{S}(t) < K$, we can write the option price as the solution to a Markovian first passage time problem as in (29), because S is still a Markovian process. So, once f is obtained for the points $\{t_i\}_{i=1,\dots,n}$, the problem can be handled by numerically solving the first passage time problem. We will return below to how this is done. To summarize, again we have a two-step procedure: if $\bar{S}(t) \geq K$ then the option price is given by

$$F(t) = S(t)f(t, x(t)) - Ke^{-r(T-t)}; \quad (41)$$

otherwise the option price is given by the solution to a Markovian first passage time problem like (29).

Consider now the floating strike lookback option. We have seen that if $x = \bar{S}/S$ then x has the Markovian evolution (40). So the European style option price is given as the solution to the Markovian initial-value problem

$$F(t)/S(t) \equiv f(t, x(t)) = E' \left[e^{-q(T-t)} \left(\frac{1}{n}x(T) - \alpha \right)^+ \mid x(t) \right] \quad \text{for } t \geq t_1,$$

and

$$F(t) = S(t)e^{-q(t_1-t)}f(t_1, 1) \quad \text{for } t < t_1.$$

We will return to how this can be handled numerically in the section below. The American style option is handled as in (34). That is, we have to solve a Markovian optimal stopping time problem. In the following section, we do this by reformulating the problem as a free-boundary problem.

7.2 Numerical Solution under Jumps

The Markovian nature of the reformulated pricing problems that we have seen in the previous section means that the pricing can be done by solving *partial integrodifferential equations* (PIDEs). The term ‘integro’ is added because the PIDEs not only involve

partial derivatives but also integrals, since the processes considered here have discontinuities of random sizes at random times. The numerical solution of such equations can still be obtained on finite grids by applying finite-difference techniques, but we need to supply additional ‘artificial’ boundary conditions in order to make this machinery work. This is because the integrals in the PIDEs typically include terms outside the boundaries of a reasonably sized grid. We will in the following derive the PIDEs that need to be solved numerically and supply our choices of ‘artificial’ boundary conditions.

In the following we will let y_{\min} and y_{\max} denote the lower and upper boundaries, respectively. These quantities are in some cases dependent on the interval (t_{i-1}, t_i) that we are considering, but for brevity we will ignore this.

With $y(t) = x(t) - m(t)/n$ the PIDE analog to the PDE (12) for the fixed strike Asian option can be written as

$$(q + \lambda')f = \frac{\partial f}{\partial t} - (r - q - k\lambda)\left(y + \frac{m(t)}{n}\right) \frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2\left(y + \frac{m(t)}{n}\right)^2 \frac{\partial^2 f}{\partial y^2} + \lambda'E'_{I'}\left[f\left(t, \frac{y + m(t)/n}{1 + I'} - \frac{m(t)}{n}\right) \mathbf{1}_{y_{\min} \leq \frac{y + m(t)/n}{1 + I'} - \frac{m(t)}{n} \leq y_{\max}}\right] + h(t, y). \quad (42)$$

The operator $E'_{I'}[\cdot]$ is defined for any function $v(\cdot)$ by

$$E'_{I'}[v(I')] = \int_{-1}^{\infty} v(\xi)\psi(\xi) d\xi, \quad (43)$$

where $\psi(\cdot)$ is the density for I' under \mathcal{Q}' ,

$$\psi(\xi) = \frac{1}{\sqrt{2\pi}\delta(1 + \xi)} \exp\left[-\frac{1}{2}\left(\frac{\ln(1 + \xi) - \gamma}{\delta} - \frac{1}{2}\delta\right)^2\right].$$

The function $h(\cdot, \cdot)$ is in turn defined as

$$h(t, y) = \lambda'E'_{I'}\left[f\left(t, \frac{y + m(t)/n}{1 + I'} - \frac{m(t)}{n}\right) \left(\mathbf{1}_{\frac{y + m(t)/n}{1 + I'} - \frac{m(t)}{n} < y_{\min}} + \mathbf{1}_{\frac{y + m(t)/n}{1 + I'} - \frac{m(t)}{n} > y_{\max}}\right)\right].$$

The PIDE (42) is to be solved, subject to the boundary conditions (13) and (14), on the set

$$\{(t, y) : t_{i-1} < t < t_i, y < -(i - 1)/n, i = 1, \dots, n\}.$$

Before we can solve this numerically, we need to make a reasonable approximation for $h(\cdot, \cdot)$. We set

$$f(t, y) = 0, \quad y < y_{\min}.$$

As we typically will set $y_{\max} = -(i - 1)/n$ and y cannot cross through the level $-(i - 1)/n$ from below, we consequently get the very simple approximation

$$h(t, y) = 0.$$

Substituting this into (42) and using the additional ‘artificial’ boundary condition $\partial^2 f/\partial y^2 = 0$ at the lower and the upper bounds, we can now numerically solve for the Asian option price using the finite-difference scheme described in Andreasen and

Gruenewald (1996). Without affecting stability, the speed of the procedure might be increased by taking an explicit approximation for the integral and an implicit approximation for the partial derivatives.

For the floating strike option we introduce $y(t) = x(t) - m(t)$ as in Section 4, and we obtain the PIDE analog to the PDE (21),

$$(q + \lambda')f = \frac{\partial f}{\partial t} - (r - q - k\lambda)(y + m(t))\frac{\partial^2 f}{\partial y^2} + \frac{1}{2}\sigma^2(y + m(t))^2\frac{\partial^2 f}{\partial y^2} + \lambda'E'_I \left[f\left(t, \frac{y + m(t)}{1 + I'} - m(t)\right) \mathbf{1}_{y_{\min} < \frac{y+m(t)}{1+I'} - m(t) < y_{\max}} \right] + h(t, y), \quad (44)$$

which is valid on

$$\{(t, y) : t_{i-1} < t < t_i, y > -(i - 1), i = 2, \dots, n + 1\}$$

and has to be solved subject to the boundary conditions (22) and when the option is American style additionally subject to the free-boundary condition (23).

For $y > y_{\max}$, we set

$$\begin{aligned} f(t, y) &= E' \left[e^{-q(T-t)} \left(\frac{1}{n}y(T) + 1 - \alpha \right) \middle| y \right] \\ &= \frac{y + m(t)}{n} e^{-q(T-t)} + \frac{1}{n} \sum_{i \leq n: t_i > t} e^{-r(T-t_i) - q(t_i-t)} - \alpha e^{-q(T-t)} \end{aligned}$$

in the European case, and for the American style option we let

$$f(t, y) = \frac{1}{m(t)}y + 1 - \alpha.$$

For y_{\min} , we set

$$f(t, y) = 0$$

in both cases. This results in the following approximations for $h(\cdot, \cdot)$. For the European case, we get

$$\begin{aligned} h(t, y) &= \lambda \left(y + \frac{m(t)}{n} \right) \frac{e^{-r(T-t)}}{n} \Phi \left(\frac{\ln \frac{y+m(t)}{y_{\max}+m(t)} - \gamma}{\delta} + \frac{1}{2}\delta \right) \\ &\quad + \lambda' \left(\frac{1}{n} \sum_{i \leq n: t_i > t} e^{-r(T-t_i) - q(t_i-t)} - \alpha e^{-q(T-t)} \right) \Phi \left(\frac{\ln \frac{y+m(t)/n}{y_{\max}+m(t)/n} - \gamma}{\delta} - \frac{1}{2}\delta \right). \end{aligned}$$

When the option is American, we have the approximation

$$h(t, y) = \lambda \left(\frac{y}{m(t)} + 1 \right) \Phi \left(\frac{\ln \frac{y+m(t)/n}{y_{\max}+m(t)/n} - \gamma}{\delta} + \frac{1}{2}\delta \right) - \lambda' \alpha \Phi \left(\frac{\ln \frac{y+m(t)/n}{y_{\max}+m(t)/n} - \gamma}{\delta} - \frac{1}{2}\delta \right).$$

Substituting these equations into the PIDE (44), we can now numerically solve the average strike option prices using the algorithm described in Andreasen and Gruenewald (1996).

Consider the lookback option with fixed strike. With $y = \ln(\bar{S}/S)$, we find that f ,

defined as in (26), solves the PIDE

$$(q + \lambda')f = \frac{\partial f}{\partial t} - (r - q - k\lambda + \frac{1}{2}\sigma^2) \frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2} + E'_I[f(t, y - \ln(1 + I'))\mathbf{1}_{y_{\min} < y - \ln(1+I') < y_{\max}}] + h(t, y) \quad (45)$$

on the set

$$\{(t, y) : t_{i-1} < t < t_i, i = 2, \dots, n + 1\},$$

subject to the boundary conditions (31).

Let

$$f(t, y) = \begin{cases} f(t, y_{\min}) & \text{if } y < y_{\min}. \\ e^{-r(T-t)+y} & \text{if } y > y_{\max}. \end{cases}$$

The last condition is obtained by taking the discounted conditional \mathcal{Q}' expectation of $x(T)$ as if there were no jumps in x at the observation points $\{t_i\}_{i=1, \dots, n}$. The function $h(\cdot, \cdot)$ is then approximated by

$$h(t, y) = \lambda e^{-r(T-t)+y} \Phi\left(\frac{y - y_{\max} + \gamma}{\delta} + \frac{1}{2}\delta\right) + \lambda' f(t, y_{\min}) \Phi\left(\frac{y_{\min} - y - \gamma}{\delta} + \frac{1}{2}\delta\right).$$

The PIDE (45) can now be solved numerically on a grid. This gives us the solution for the option price when $\bar{S}(t) \geq K$. If this is not the case, we proceed by noting that, for $\bar{S}(t) < K$ with the definitions $y = \ln(S/K)$ and $g = F/K$, the PIDE equivalent to the PDE (32) is

$$(r + \lambda)g = \frac{\partial g}{\partial t} + (r - q - k\lambda - \frac{1}{2}\sigma^2) \frac{\partial g}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial y^2} + \lambda E_I[g(t, y + \ln(1 + I))\mathbf{1}_{y_{\min} < y + \ln(1+I) < y_{\max}}] + h(t, y) \quad (46)$$

on

$$\{(t, y) : t_{i-1} < t < t_i, i = 1, \dots, n\},$$

subject to the boundary conditions (33).

The operator $E_I[\cdot]$ is defined as in (43), with the modification that the \mathcal{Q}' density is now replaced by the \mathcal{Q} density. By analogy with the previous cases, the function $h(\cdot, \cdot)$ is defined as

$$h(t, y) = E_I[g(t, y + \ln(1 + I))(\mathbf{1}_{y + \ln(1+I) < y_{\min}} + \mathbf{1}_{y + \ln(1+I) > y_{\max}})].$$

For $y < y_{\min}$ we set

$$g(t, y) = 0,$$

and for $y > y_{\max}$ we let

$$\begin{aligned} g(t, y(t)) &= E_t[e^{-r(t_{m(t)+1}-t)}g(t_{m(t)+1}, y(t_{m(t)+1}))] \\ &= E_t[e^{-r(t_{m(t)+1}-t)}e^{y(t_{m(t)+1})}f(t_{m(t)+1}, 1) - e^{-r(T-t)}] \\ &= e^{-q(t_{m(t)+1}-t)+y(t)}f(t_{m(t)+1}, 1) - e^{-r(T-t)}. \end{aligned}$$

From this, we obtain

$$h(t, y) = \lambda' e^{-q(t_{m(t)+1}-t)+y} \Phi\left(\frac{y - y_{\max} + \gamma}{\delta} + \frac{1}{2}\delta\right) - \lambda e^{-r(T-t)} \Phi\left(\frac{y - y_{\max} + \gamma}{\delta} - \frac{1}{2}\delta\right).$$

Using this we can numerically solve the PIDE (46) with the finite-difference machinery.

Finally, let us consider the lookback option with floating strike. Defining $y = \ln(\bar{S}/S)$ and $f = F/S$, we find that f is the solution to (45) on

$$\{(t, y) : t_{i-1} < t < t_i, i = 2, \dots, n + 1\},$$

subject to the boundary conditions (35) and if the option is American style also subject to the free-boundary condition (36). What is left is to supply an approximation of $h(\cdot, \cdot)$ for this option. For $y < y_{\min}$, we set

$$f(t, y) = f(t, y_{\min})$$

for both the American and European style cases.

For $y > y_{\max}$, we set

$$f(t, y) = e^{-r(T-t)+y} - e^{-q(T-t)}\alpha$$

for the European case. This corresponds to the discounted \mathcal{Q}' expected terminal payoff if we ignore the fact that the option could go out of the money and the (possible) jumps at the observation points $\{t_i\}_{i=1,\dots,n}$. For the American style option, we set

$$f(t, y) = e^y - \alpha$$

for $y > y_{\max}$.

In doing so, we get the following approximation for h when the option is European:

$$\begin{aligned} h(t, y) &= \lambda e^{-r(T-t)+y} \Phi\left(\frac{y - y_{\max} + \gamma}{\delta} + \frac{1}{2}\delta\right) \\ &\quad - \lambda' e^{-q(T-t)} \alpha \Phi\left(\frac{y - y_{\max} + \gamma}{\delta} - \frac{1}{2}\delta\right) \\ &\quad + \lambda' f(t, y_{\min}) \Phi\left(\frac{y_{\min} - y - \gamma}{\delta} + \frac{1}{2}\delta\right). \end{aligned}$$

For the American style option, we get the approximation

$$\begin{aligned} h(t, y) &= \lambda e^y \Phi\left(\frac{y - y_{\max} + \gamma}{\delta} + \frac{1}{2}\delta\right) \\ &\quad - \lambda' \alpha \Phi\left(\frac{y - y_{\max} + \gamma}{\delta} - \frac{1}{2}\delta\right) \\ &\quad + \lambda' f(t, y_{\min}) \Phi\left(\frac{y_{\min} - y - \gamma}{\delta} + \frac{1}{2}\delta\right). \end{aligned}$$

With this, we can numerically solve for the price of the lookback option with floating strike.

8. CONCLUSION

This paper has described an approach to the numerical pricing of discretely observed path-dependent options that is highly competitive in terms of accuracy and speed compared with Monte Carlo simulations. We have illustrated this by numerical examples for four types of path-dependent option.

A second advantage of this pricing technique compared with Monte Carlo techniques is the ability to price the floating strike American style options. This cannot be done by standard Monte Carlo methods.

In the Black–Scholes and the jump framework, the technique applies to most types of European options on the average and the maximum (or minimum). Among the types of option that have not been considered in this paper but can be priced using our approach are combinations of maximum, minimum, and average and digital options on the average and/or the maximum.

APPENDIX

Derivation of the Equations (8) and (39)

Let x be defined as in (7) and let $v(\cdot)$ be a deterministic function. Using Itô’s lemma and (38), we obtain

$$d[v(t)x(t)] = (v'(t)x(t-) - (r - q - k\lambda)v(t)x(t-)) dt - \sigma x(t-)v(t) dW'(t) - v(t)x(t-) \frac{I'(t)}{1 + I'(t)} dN'(t) + v(t) \frac{1}{n} dm(t).$$

Inserting

$$v(t) = e^{(r-q)t},$$

we obtain

$$d[v(t)x(t)] = -\sigma x(t-)v(t) dW'(t) - v(t)x(t-) \left(\frac{I'(t)}{1 + I'(t)} dN'(t) - k\lambda dt \right) + v(t) \frac{1}{n} dm(t). \tag{47}$$

We know that the process

$$\left\{ \int_t^s x(u-)v(u) \left(\sigma dW'(u) + \frac{I'(u)}{1 + I'(u)} dN'(u) - k\lambda du \right) \right\}_{s \geq t}$$

is a \mathcal{Q}' martingale, so integrating (47) and taking the \mathcal{Q}' expectation yields

$$\begin{aligned} e^{(r-q)T} E'_t[x(T)] &= e^{(r-q)t} x(t) + \frac{1}{n} \int_t^T e^{(r-q)u} dm(u) \\ &= e^{(r-q)t} x(t) + \frac{1}{n} \sum_{i: t < t_i \leq t_n} e^{(r-q)t_i}. \end{aligned}$$

Finally, we obtain

$$e^{-q(T-t)}\mathbb{E}_t^Q[x(T)] = e^{-r(T-t)}x(t) + \frac{1}{n} \sum_{i:t < t_i \leq t_n} e^{-r(T-t_i)-q(t_i-t)}.$$

For $\lambda = 0$ we have equation (8), and for general λ we have equation (39).

REFERENCES

- Andreasen, J., and Gruenewald, B. (1996). American option pricing in the jump-diffusion model. Working paper, Aarhus University, Denmark.
- Babbs, S. (1992). Binomial valuation of lookback options. Working paper, Midland Montagu Capital Markets, London.
- Black, F., and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, **81**, 637–654.
- Boyle, P., and Lau, S., (1994). Bumping up against the barrier with the binomial method. *Journal of Derivatives*, **1**, 6–14.
- Conze, A., and Viswanathan, R. (1991). Path-dependent options: The case of lookback options. *Journal of Finance*, **46**, 1893–1907.
- Cox, J., Ross, S., and Rubinstein M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, **7**, 229–263.
- Davidson, R., and Mackinnon, J. (1993). *Estimation and Inferences in Econometrics*. Oxford University Press.
- Geman, H., and Yor, M. (1993). Bessel processes, Asian options and perpetuities. *Mathematical Finance*, **34**, 349–375.
- Goldman, M., Sosin, H., and Gatto, M. A. (1979). Path-dependent options: Buy at the low, sell at the high. *Journal of Finance*, **34**, 1111–1128.
- Goldman, M., Sosin, H., and Shepp, L. (1979). On contingent claims that insure ex-post optimal stock market timing. *Journal of Finance*, **34**, 401–414.
- Ingersoll, J. (1987). *Theory of Financial Decision Making*. Rowman and Littlefield.
- Merton, R. (1976). Option pricing when the underlying stock returns are discontinuous. *Journal of Financial Economics*, **5**, 125–144.
- Mitchell, A., and Griffiths, D. (1980). *The Finite Difference Method in Partial Differential Equations*. Wiley.
- Rogers, L., and Shi, Z. (1995). The value of an Asian option. *Journal of Applied Probability*, **32**, 1077–1088.
- Wilmott, P., Dewynne, J., and Howison, S. (1993). *Option Pricing: Mathematical Models and Computation*. Oxford Financial Press.