Design of Linear-Phase Nonuniform Filter Banks
With Partial Cosine Modulation
Wei Zhong, Guangming Shi, Xuemei Xie, and Xuyang Chen

Abstract—The majority of the existing work on designing nonuniform filter banks (NUFBs) cannot achieve linear-phase (LP) property, because of the high complexity associated with phase distribution. This correspondence proposes an idea of partial cosine modulation to obtain the LP property of NUFBs with rational sampling factors. It makes the efficient modulation technique possible to be used to design LP NUFBs. Except the separately designed lowpass and highpass analysis/synthesis filters, we obtain the bandpass by cosine modulation of several prototypes, worth of the name “partial cosine modulation.” By analyzing the phase issue of significant aliasing terms, we derive the matching conditions to make the required LP NUFB achievable. With these criteria being satisfied, the design problem becomes that of several prototypes as well as the lowpass and highpass filters, leading to a less design effort. By using the proposed method, near-perfect-reconstruction LP NUFBs can be obtained in a simple and efficient way as demonstrated by examples.

Index Terms—Linear-phase, nonuniform filter bank, near-perfect-reconstruction, partial cosine modulation.

I. INTRODUCTION

Nonuniform filter banks (NUFBs) have been widely used in many signal processing applications due to their flexibility in partitioning subbands. In some cases, such as image coding, it is crucial for all filters to have linear-phase (LP) property. This is because LP filters can avoid artifacts in the reconstructed images.

It should be noticed that, among various methods of designing NUFBs [1]–[18], only a few are capable of achieving LP objective. In the indirect structure, the LP requirement was achieved in [9], where certain subbands of an LP uniform bank are recombined by synthesis filters of transmultiplexers. The NUFB so obtained has a long system delay due to the two-stage architecture. In contrast, the direct structure [10]–[18], which has only one stage, is more popular. In [10]–[14], the modulated NUFBs are constructed from uniform banks through transition filters or linear combination. However, due to the high complexity of phase distribution, there are no results about these designs on LP NUFBs. As to the direct design of individual nonuniform filters, some works have been done in [15]–[18]. Reference [15] presented a NUFB design based on the modulation of several prototypes. Unfortunately, the LP property of the resulting filters cannot be achieved either. Although the design in [16] adopts modulation technique to obtain the LP NUFB, the decimation filters with complex coefficients are required for the rational case to reduce the aliasing distortion. Reference [17] formulates the perfect-reconstruction (PR) constraints in a quadratic form of filter coefficients so as to avoid the nonlinear optimization. But as for the LP NUFBs, it only considers

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the necessary conditions on integer sampling factors implemented by a binary tree. In our previous work [18], a direct design was proposed for LP near-PR (NPR) NUFBs with rational sampling factors. But the individual implementation of all $M$ filters leads to an extremely computational load compared to the modulated bank which only needs the design of several prototypes. Thus, in this context, it is valuable to exploit the efficient design of NUFBs with LP property.

In this correspondence, a method called partial cosine modulation is proposed for designing LP NUFBs with rational sampling factors. It takes advantage of modulation technique while maintaining the LP property of individual filters. In the proposed method, we design the lowpass and highpass analysis/synthesis filters separately, and obtain the bandpass by cosine modulation of several prototypes, worth of the name “partial cosine modulation.” With a rigorous analysis of phase issue, the matching conditions are derived to obtain the required LP NUFB. By using the proposed method, the requirements for the entire LP NUFB design are simply represented as those imposed on several prototypes as well as the lowpass and highpass filters, leading to a less design effort. In addition, the proposed LP NUFB is based on the one-stage direct structure and thus has lower system delay. As illustrated by examples, the proposed method can provide a simple and efficient alternative for designing NPR NUFBs with the expected LP property.

The rest of this correspondence is organized as follows. Section II describes the idea of partial cosine modulation. Section III analyzes the significant aliasing cancellation, followed by the filter design and examples in Section IV and conclusions in Section V.

II. IDEA OF PARTIAL COSINE MODULATION

Fig. 1 shows the direct structure of $M$-channel NUFB with rational sampling factors $p_k/q_k$, $0 \leq k \leq M - 1$. In the modulated NUFB, each pair of the analysis and synthesis filters, $h_k(n)$ and $f_k(n)$, are obtained by the cosine modulation of an LP prototype filter $p_k(n)$,

$$
\begin{align*}
    h_k(n) &= 2p_k(n) \cos \left( \frac{2m_k + 1}{2q_k} \left( n - \frac{N_k}{2} \right) + \theta_k \right), \\
    f_k(n) &= 2p_k(n) \cos \left( \frac{2m_k + 1}{2q_k} \left( n - \frac{N_k}{2} \right) - \theta_k \right), \\
    &= h_k(N_k - n)
\end{align*}
$$

where $0 \leq k \leq M - 1$, $0 \leq n \leq N_k$, $N_k$ is the order of $p_k(n)$, $m_k$ is the position parameter which selects the frequency interval extracted by $h_k(n)$ and $f_k(n)$, and $\theta_k$ is the modulation phase.

In the modulated NUFB proposed in [15], $\theta_k$ is chosen as $\{\pi/4 \text{ or } -\pi/4\}$ subjected to the conditions for canceling different types of aliasing couplings in adjacent channels. The detailed explanation of those types of aliasing couplings will be given in Section III-A. Thus the resulting filters do not possess LP property. Our target is to achieve the LP property of individual filters relying on the efficient modulation technique. This additional phase requirement for the LP objective makes the design problem quite different.

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Fig. 1. Direct structure of $M$-channel NUFB with rational sampling factors.
Next we derive the proposed method of partial cosine modulation. We first consider how to obtain the LP property of \( h_k(n) \) and \( f_k(n) \) modulated from (1). For the convenience of analysis, we write their frequency responses as the sum of positive and negative frequency components, denoted by \( U_k(e^{j\omega}) \) and \( V_k(e^{j\omega}) \):

\[
H_k(e^{j\omega}) = e^{j\theta_k}U_k(e^{j\omega}) + e^{-j\theta_k}V_k(e^{j\omega})
\]

\[
F_k(e^{j\omega}) = e^{-j\theta_k}U_k(e^{j\omega}) + e^{j\theta_k}V_k(e^{j\omega})
\]

and

\[
U_k(e^{j\omega}) = e^{-j\omega N/2}P_{\text{LP}}(\omega - \pi(2mk + 1)/2q_k)
\]

\[
V_k(e^{j\omega}) = e^{-j\omega N/2}P_{\text{LP}}(\omega + \pi(2mk + 1)/2q_k)
\]

where \( P_{\text{LP}}(\omega) \) is the amplitude response of the prototype \( P_k(e^{j\omega}) \). Substituting (3) into (2), we can see that to ensure the LP property of \( H_k(e^{j\omega}) \) and \( F_k(e^{j\omega}) \), \( e^{j\theta_k} \) should be chosen as \( 0 \) or \(-1\) for symmetry and \( \{j\text{ or }-j\} \) for antisymmetry. Correspondingly, \( \theta_k \) has to be selected as \( \{0 \text{ or } \pi\} \) for symmetry and \( \{\pi/2 \text{ or } -\pi/2\} \) for antisymmetry. That is why this method is called partial cosine modulation. In our work, the same prototype is used in channels with the same sampling factors. The analysis and synthesis filters hold the time-reverse relation, i.e., \( f_k(n) = h_k(N_k - n) \).

Unlike the conventional non-LP modulated NUBF [15], the additional LP requirement makes the proposed method sacrifice the identity of designing all filters by modulation. Thus, the issue on the matching of individual LP filters needs to be considered. In what follows, we will analyze the significant aliasing cancellation, deriving the matching conditions of all the LP filters involved that make the proposed LP NUBF achievable.

### III. Cancellation of Significant Aliasing Distortion

#### A. Four Types of Aliasing Couplings

From the direct structure of NUBFs shown in Fig. 1, we can obtain the expression of aliasing term \( A_{k,i,k}^{(\text{low})}(z) \) as

\[
A_{k,i,k}^{(\text{low})}(z) = \sum_{i_0=0}^{p_k-1} \sum_{k_0=0}^{M-1} H_k \left( z^{1/p_k} W_{p_k}^{i_0} W_{p_k}^{i_0} \right) F_k \left( z^{1/p_k} W_{p_k}^{i_0} \right)
\]

where \( W_{p_k}^{i_0} = e^{-j2\pi i_0/p_k} \). Substituting (2) into (4) with \( z = e^{j\omega} \), we get

\[
T(e^{j\omega}) = \sum_{i_0=0}^{p_k-1} \sum_{k_0=0}^{M-1} U_k \left( e^{j\omega/p_k} W_{p_k}^{i_0} \right) + \sum_{i_0=0}^{p_k-1} \sum_{k_0=0}^{M-1} V_k \left( e^{-j\omega/p_k} W_{p_k}^{i_0} \right)
\]

\[
\sum_{i_0=0}^{p_k-1} U_k \left( e^{j\omega/p_k} W_{p_k}^{i_0} \right) \text{ and } \sum_{i_0=0}^{p_k-1} V_k \left( e^{-j\omega/p_k} W_{p_k}^{i_0} \right)
\]

so that \( k = 0 \) and \( M = 1 \), (5) can be simplified into

\[
T(e^{j\omega}) \approx \sum_{i_0=0}^{p_k-1} \sum_{k_0=0}^{M-1} U_k \left( e^{j\omega/p_k} W_{p_k}^{i_0} \right) + \sum_{i_0=0}^{p_k-1} \sum_{k_0=0}^{M-1} V_k \left( e^{-j\omega/p_k} W_{p_k}^{i_0} \right)
\]

\[
+ \sum_{i_0=0}^{p_k-1} \sum_{k_0=0}^{M-1} U_k \left( e^{j\omega/p_k} W_{p_k}^{i_0} \right) + \sum_{i_0=0}^{p_k-1} \sum_{k_0=0}^{M-1} V_k \left( e^{-j\omega/p_k} W_{p_k}^{i_0} \right)
\]

Under the analysis of (2) on the LP requirement of selecting \( \theta_k \) and \( \theta_{M-1} \) being \( \{0 \text{ or } \pi\} \) for symmetry and \( \{\pi/2 \text{ or } -\pi/2\} \) for antisymmetry, the first two cross-terms in (6),

\[
\sum_{i_0=0}^{p_k-1} \sum_{k_0=0}^{M-1} U_k \left( e^{j\omega/p_k} W_{p_k}^{i_0} \right) V_k \left( e^{-j\omega/p_k} W_{p_k}^{i_0} \right)
\]

and

\[
\sum_{i_0=0}^{p_k-1} \sum_{k_0=0}^{M-1} U_k \left( e^{j\omega/p_k} W_{p_k}^{i_0} \right) V_k \left( e^{-j\omega/p_k} W_{p_k}^{i_0} \right)
\]

create significant distortions around the frequencies \( \omega = 0 \) and \( \omega = \pi \) in \( \pi/2 < \omega < \pi \).

Therefore, in order to cancel these amplitude distortions while maintaining the LP property of individual filters, we have to design the lowpass filter \( h_0(n) \) and highpass filter \( h_{M-1}(n) \) separately, whereas obtaining the bandpass \( h_k(n) \), \( 1 \leq k \leq M - 2 \), by cosine modulation of \( p_k(n) \) with \( \theta_k \) selected being \( \{0 \text{ or } \pi\} \) for symmetry and \( \{\pi/2 \text{ or } -\pi/2\} \) for antisymmetry. That is why this method is called partial cosine modulation. In our work, the same prototype is used in channels with the same sampling factors. The analysis and synthesis filters hold the time-reverse relation, i.e., \( f_k(n) = h_k(N_k - n) \).

Unlike the conventional non-LP modulated NUBF [15], the additional LP requirement makes the proposed method sacrifice the identity of designing all filters by modulation. Thus, the issue on the matching of individual LP filters needs to be considered. In what follows, we will analyze the significant aliasing cancellation, deriving the matching conditions of all the LP filters involved that make the proposed LP NUBF achievable.
It can be seen clearly, one aliasing term in the \( k \)th channel can be coupled with the one in the \((k+1)\)th channel having same central frequencies \( \pm \sum_{\theta} p_i \pi / q_i \), as circled in Fig. 3, forming an aliasing coupling. As to the other aliasing term in the \( k \)th channel, it can be coupled with the one in the \((k-1)\)th channel locating around the same position \( \pm \sum_{\theta} p_i \pi / q_i \). Those couplings have four types in adjacent channels, leading to significant aliasing cancellation [15]:

\[
\text{high-high type, } p_k A_{k}^{(b\text{high})}(z) \downarrow p_k \\
+ p_{k+1} A_{k+1}^{(b\text{high})}(z) \downarrow p_{k+1} = 0 \tag{9a}
\]

\[
\text{high-low type, } p_k A_{k}^{(b\text{high})}(z) \downarrow p_k \\
+ p_{k+1} A_{k+1}^{(b\text{low})}(z) \downarrow p_{k+1} = 0 \tag{9b}
\]

\[
\text{low-high type, } p_k A_{k}^{(b\text{low})}(z) \downarrow p_k \\
+ p_{k+1} A_{k+1}^{(b\text{high})}(z) \downarrow p_{k+1} = 0 \tag{9c}
\]

\[
\text{low-low type, } p_k A_{k}^{(b\text{low})}(z) \downarrow p_k \\
+ p_{k+1} A_{k+1}^{(b\text{low})}(z) \downarrow p_{k+1} = 0 \tag{9d}
\]

where \( 0 \leq k \leq M - 2 \). Further, we notice that in each coupling, if the magnitude responses of the two aliasing terms have the same amount at the same frequency point and their phases are chosen to differ by \( \pi \), the significant aliasing can be cancelled completely.

B. Phase Property of Aliasing Couplings in Adjacent Channels

In the proposed partial modulated bank, \( H_{s}(z) \), \( 1 \leq k \leq M - 2 \), are modulated from the prototypes \( P_{k}(z) \), whereas \( H_{0}(z) \) and \( H_{M+1}(z) \) are designed separately. For the convenience of discussion, we classify those couplings in adjacent channels into two cases: 1) bandpass and 2) lowpass and highpass.

1) Bandpass: For the cancellation of aliasing couplings in adjacent channels from the 1th to \((M - 2)\)th, corresponding to the bandpass \( H_{s}(z) \) modulated from (1), \( 1 \leq k \leq M - 2 \), the conditions on the modulation phase \( \theta_k \) have been given in [15]. Different from [15], in our work to obtain the LP property, we select \( \theta_k \) to be \( \pi / 2 \) or \(-\pi / 2 \) and \( 0 \) or \( \pi \) alternately. Moreover, for whole system, since the lowpass \( H_{0}(z) \) must be symmetric, we have to select \( \theta_1 \) in the modulation of the first bandpass \( H_{1}(z) \) to begin with \( \pi / 2 \) or \(-\pi / 2 \). The matching condition between the separately designed \( H_{0}(z) \) and modulated \( H_{1}(z) \) will be analyzed in the next case.

2) Lowpass and Highpass: This case corresponds to the aliasing cancellation in the lowpass filter and the first bandpass filter, as well as that in the last bandpass filter and the highpass filter.

For the lowpass filter, since \( H_{0}(z) \) involves only one pair \( A_{0}^{(b\text{high})}(z) \) with \( l_0 = \pm (m_0 + 1) = \pm 1 \), only two types of couplings in (9) should be considered in the 0th and 1st channels. They are high-high type in (9a) and high-low type in (9b) with \( k = 0 \). Similarly for the aliasing cancellation in the \((M - 2)\)th and \((M - 1)\) th channels, only two types of couplings should be focused on, high-low type in (9b) and low-low type in (9d) with \( k = M - 2 \).
where \( H_{DL}(\omega) \) is the amplitude response of lowpass filter \( H_0(\epsilon^{j\omega}) \), and \( P_{DL}(\omega) \) is that of the prototype \( P_1(\epsilon^{j\omega}) \) in the 1th channel. The detailed derivations of (11) and (12) are given in Appendix.

From (11) and (12), it can be seen that the phases of the two components are \(-\omega N_0/p_k + \pi N_0/q_0\) and \(-\omega N_1/p_1 + \pi N_0/q_0 - 2\theta_1\). Without considering the sign term \(-2\theta_1\), to make the two phases being same, the analysis filters need to satisfy the following condition on the orders \( N_0 \) and sampling factors \( p_k, 0 \leq k \leq M - 1 \):

\[
N_0/p_0 = N_1/p_1 = \cdots = N_k/p_k = \cdots = N_{M-1}/p_{M-1} = 1.
\]

(13)

Considering the sign term \(-2\theta_1\) in (12), we have to select \( \theta_1 \) to be \( \{\pi/2 \text{ or } -\pi/2\} \) to make the above two phases differ by \( \pi \). Recalling the analysis of (2) that \( \theta_1 \) should be chosen as \( \{0 \text{ or } \pi\} \) for symmetry and \( \{\pi/2 \text{ or } -\pi/2\} \) for antisymmetry, we conclude \( H_1(z) \) must be antisymmetric. Similarly for negative frequency components of two aliasing terms within the coupling illustrated in Fig. (4a), we get the same result.

C. Conditions for Significant Aliasing Cancellation

In the proposed partial modulated LP NUBF, we assume that \( h_k(n) \) and \( f_k(n) \) hold the time-reverse relation and the filter orders fulfill (13). Then the findings up to now can be summarized for the cancellation of significant aliasing distortion:

a) The analysis filters \( h_k(n), 0 \leq k \leq M - 1 \), satisfy an alternate symmetry property, that is, \( h_k(n) \) are symmetric and antisymmetric alternately.

b) The magnitude responses of the two aliasing terms within each coupling need to have the same amount at the same frequency point. That is, the stretched transition bands of \( p_k(n), 1 \leq k \leq M - 2 \), \( h_0(n) \) and \( h_{M-1}(n) \) after downsampled by the corresponding \( p_k \) should have the same shape.

IV. FILTER DESIGN AND EXAMPLES

Under the assumption that the stopband attenuation is sufficiently high and the passband is flat enough, to achieve the NPR property, we need consider only the amplitude distortion at the transition bands of adjacent filters. In the proposed partial modulated bank, the following constraints should be satisfied among the lowpass filter \( H_0(\epsilon^{j\omega}) \), prototypes \( P_k(\epsilon^{j\omega}) \), \( 1 \leq k \leq M - 2 \), and highpass filter \( H_{M-1}(\epsilon^{j\omega}) \):

\[
|H_0(\epsilon^{j\omega}/q_0)|^2 + |P_1(\epsilon^{j\omega}/q_0)|^2 = 1
\]

(14)

and

\[
p_0(\pi/q_0 - \epsilon_0) < \omega < p_0(\pi/q_0 + \epsilon_0)
\]

\[
\varphi_0 = p_0\pi/q_0 + p_1\pi/2q_1
\]

\[
|P_k(\epsilon^{j\omega}/q_k)|^2 + |P_{k+1}(\epsilon^{j\omega}/q_{k+1})|^2 = 1
\]

\[
1 \leq k \leq M - 3
\]

(15)

where \( \epsilon_0 \) takes half of the transition bandwidth, \( 0 \leq k \leq M - 1 \). The above requirements would be met by forcing the stretched transition bands of adjacent filters to follow the cosine roll-off characteristics. Interested readers please refer to [9]. By doing so, the second condition in Section III-C can also be satisfied.

The proposed LP NUBF can be performed by using one of the available filter design tools. In this paper, we employ the Parks-McClellan algorithm for illustration. The Parks-McClellan algorithm for filter design has the optimal solution in the minimax sense with respect to the specified magnitude response of the filter and it can easily be used by the function \( \text{FIRPM in MATLAB} \). By those good properties, we use this algorithm to specify the transition bands of adjacent filters so as to make their stretched versions approximate the cosine roll-off characteristics, thereby simplifying the design of the desired LP NUBF.

The systematic design procedures of the proposed LP NUBF are summarized below. The filters involved are all designed by the Parks-McClellan algorithm with the stretched transition bands having the form of cosine roll-off function.

Choose the filter orders \( N_0/p_0 = \cdots = N_k/p_k = \cdots = N_{M-1}/p_{M-1} = 1 \) (13)

2) Design the symmetric lowpass filter \( H_0(\epsilon^{j\omega}) \) with order \( N_0 \), cutoff frequency \( \omega_{0,\epsilon} = \pi/q_0 \) and transition bandwidth \( 2\epsilon_0 \).

3) Design the prototypes \( P_k(\epsilon^{j\omega}) \), \( 1 \leq k \leq M - 2 \), with order \( N_k \), cutoff frequency \( \omega_{k,\epsilon} = \pi/2q_k \) and transition bandwidth \( 2\epsilon_0 \).

4) Obtain the modulated bandpass filters \( H_k(\epsilon^{j\omega}) \), \( 1 \leq k \leq M - 2 \), using (1), with \( \theta_k \) selected to be \( \{\pi/2 \text{ or } -\pi/2\} \) and \( \{0 \text{ or } \pi\} \) alternately, and \( m_k \) chosen to extract the feasible frequency component of input signal upsampled by \( p_k \).

5) Design the highpass filter \( H_{M-1}(\epsilon^{j\omega}) \) with order \( N_{M-1} \), cutoff frequency \( \omega_{M-1,\epsilon} = (q_{M-1} - 1)\pi/q_{M-1} \) and transition bandwidth \( 2\epsilon_{M-1} \). \( H_{M-1}(\epsilon^{j\omega}) \) is symmetric for odd \( M \), and is antisymmetric for even \( M \).

6) Set \( p_0 = \cdots = p_{M-2} = 0 = \cdots = p_{M-1} \) to make sure that the stretched transition bands of \( H_k(\epsilon^{j\omega}) \), \( P_k(\epsilon^{j\omega}) \), \( 1 \leq k \leq M - 2 \), and \( H_{M-1}(\epsilon^{j\omega}) \) after downsampled by the corresponding \( p_k \) have the same bandwidth.

Now we give two examples to illustrate the performance of the proposed method.

Example 1: In this example, a 2-channel LP NUBF is considered as a comparison to the recombination one shown in the first example of [9]. They have the same sampling factors \([3/4, 1/4]\). Under the similar design specifications, such as transition bandwidth and stopband attenuation, we choose the filter orders to be \( N_0 = 177, N_1 = 59 \). Following the design procedures given above, we design the asymmetric lowpass filter \( H_0(\epsilon^{j\omega}) \) and asymmetric highpass filter \( H_1(\epsilon^{j\omega}) \) to achieve the required LP NUBF.

Further we compare the proposed method with indirect structure [9] in terms of system delay and implementation complexity as summarized in Table I. The implementation complexity is measured by the number of the required multiplications per unit-time (MPU) and additions per unit-time (APU). From Table I we can see that, our proposed LP NUBF is a low-delay system. Compared with indirect structure [9], the system delay of the proposed method is almost only one-third of that of [9] (59 versus 167), at the expense of higher complexity. Such a low-delay system is very crucial for those delay sensitive applications.

Example 2: This example is a 4-channel LP NUBF with sampling factors \([2/7, 2/7, 2/7, 1/7]\). The filter orders are \( N_0 = N_1 = N_2 = 162, N_3 = 81 \). In this example \( p_j/q_j = p_2/q_2 \), we need design only one prototype for modulating two bandpass filters. In the modulation formula (1), we select the position parameters \( m_1 = 5, m_2 = 4 \). And the modulation phases are \( \theta_1 = 0, \theta_2 = 0 \). By the above design procedures, we obtain the desired result shown in Fig. 6. The aliasing and amplitude distortions are \( E_k = 1.484 \times 10^{-3}, E_{pp} = 2.783 \times 10^{-3} \), respectively. And the stopband attenuation is 87.8 dB.
tab responses of analysis filters. (b) Aliasing error. (c) Amplitude distortion.

Fig. 5. The 2-channel LP NUBF with sampling factors [3/4, 1/4]. (a) Magnitude responses of analysis filters. (b) Aliasing error. (c) Amplitude distortion.

TABLE I

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<tr>
<th>Example 1:</th>
<th>Indirect Structure [9]</th>
<th>Proposed Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>[3/4, 1/4]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N_f = 85, N_o = 62</td>
<td></td>
<td></td>
</tr>
<tr>
<td>System Delay</td>
<td>167</td>
<td>59</td>
</tr>
<tr>
<td>MPU</td>
<td>133 [482]</td>
<td>149 [594]</td>
</tr>
<tr>
<td>APU</td>
<td>131 [476]</td>
<td>148 [590]</td>
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<tr>
<th>Example 2:</th>
<th>Direct Design of Individual LP Filters [18]</th>
<th>Proposed Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2/7, 2/7, 2/7, 1/7]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N_f = N_o = N_i = 162</td>
<td></td>
<td></td>
</tr>
<tr>
<td>System Delay</td>
<td>81</td>
<td>81</td>
</tr>
<tr>
<td>MPU</td>
<td>151 [1060]</td>
<td>105 [734]</td>
</tr>
<tr>
<td>APU</td>
<td>150 [1053]</td>
<td>104 [729]</td>
</tr>
</tbody>
</table>

'[*]' Direct implementation without using polyphase decomposition.

For this example, we compare the proposed method with direct design of individual LP filters [18]. Under the same design specifications, we simulate this example by the method given in [18], which is the authors’ previous work. The simulation result shows that the two direct design methods can achieve comparable stopband attenuation, aliasing error, and amplitude distortion. But from the comparison shown in Table I, we can see that the design and implementation complexities of our method are much lower than that of [18]. This is because in our method, two LP bandpass filters are obtained by modulating one prototype, thanks to the efficient modulation technique. It should be noticed that in the proposed LP NUBF, the more the channels having the same sampling factors, the more the filters modulated from several prototypes, and, thus, the less the implementation complexity.

V. CONCLUSION

In this correspondence, particular attention has been paid to the problem of achieving the LP property of NUBFs relying on the modulation technique. As a solution, a method called partial cosine modulation is proposed. With the matching conditions of all LP filters involved satisfied, the design problem of LP NUBFs can be simplified into that of the prototype filters, leading to a less design effort. Two examples and comparison results show that under the similar design specifications, the proposed method has lower system delay than the indirect structure [9] and lower implementation complexity than the direct design of individual LP filters [18].

APPENDIX

DERIVATIONS OF (11) AND (12)

Consider the coupling of high-high type in the 0th and 1th channels shown in (10). It consists of two aliasing terms, \( p_0 A_0^{(b|b)} (z) \downarrow p_0 \) and \( p_1 A_1^{(b|h)} (z) \downarrow p_1 \). We analyze their positive frequency components in the case illustrated in Fig. 4(a).

For the first term \( p_0 A_0^{(b|b)} (z) \downarrow p_0 \), its positive frequency component is \( p_0 H_0 (z : W_W) F_0 (z) \downarrow p_0 \). It locates at \( p_0 \pi / a_0 \) as its central frequency in \([-\pi, \pi]\), shown in the top of Fig. 4(a). We analyze \( H_0 (z : W_W) F_0 (z) \) first. Since the LP lowpass filter \( h_0 (n) \) has to be symmetric, that is, \( h_0 (n) = h_0 (N_0 - n) \). Under the time-reverse relation \( f_k (n) = h_k (N_k - n) \) with \( k = 0 \), we obtain

\[
\hat{f}_0 (n) = h_0 (n) \text{ or } F_0 (z) = H_0 (z).
\]  

(17)

Substituting (17) and the expression of symmetric \( H_0 (e^{j\omega}) \), i.e., \( H_0 (e^{j\omega}) = e^{-j \omega N_0 / 2} H_{0R} (\omega) \), \( H_0 (z : W_W) F_0 (z) \) becomes

\[
H_0 (z : W_W) F_0 (z) \equiv e^{-j \omega N_0 / 2} H_{0R} (\omega) F_0 (e^{j\omega}) \times H_{0I} (\omega - 2\pi / a_0) H_{0I} (\omega).
\]  

(18)

After downsampled by \( p_0 \), (18) becomes

\[
p_0 H_0 (z : W_W) F_0 (z) \downarrow p_0 \equiv \sum_{n=0}^{p_0-1} e^{-j (\omega - 2\pi n / p_0)} \times H_{0R} (\omega - 2\pi / a_0) / p_0
\]  

(19)

With \( i_0 = 0 \), we get its response in \([-\pi, \pi]\) as expressed in (11).
For the second term $p_1 \mathcal{H}_{[\text{high}]}(z) \downarrow p_1$, its positive frequency component is $p_1 H_1(z^{-1}W_{q_1}^{m+1}) F(z) \downarrow p_1$. It locates around the same position as that of the first term, shown in the bottom of Fig. 4(a). We also consider $H_1(z^{-1}W_{q_1}^{m+1}) F(z) \downarrow p_1$. Substituting (2) and (3),

$$H_1(z^{-1}W_{q_1}^{m+1}) F(z) = \mathcal{H}_1(e^{i\omega}),$$

we get its frequency response in $[-\pi, \pi]$ as expressed in (12).

**REFERENCES**


**Radial Function Based Kernel Design for Time-Frequency Distributions**

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**Abstract**—A framework based on the $r$-dimensional Fourier transform of a radically symmetric function is introduced to design kernels for Cohen time-frequency distributions. Under this framework, we derive a kernel formula which generalizes and unifies Margenau–Hill, Born–Jordan, and Bessel distributions, using a realization based on a $r$-dimensional radial delta function. The higher order radial kernels suppress more cross-term energy compared with existing lower order kernels, which is illustrated by the time-frequency analysis of atrial fibrillation from surface electrocardiogram data.

**Index Terms**—Bessel distribution, Born–Jordan distribution, Cohen class, kernel design, Margenau–Hill distribution, multidimensional Fourier transform, time-frequency distributions (TFDs).

**I. INTRODUCTION**

Time-frequency distributions (TFDs) have been used extensively in analyzing nonstationary signals such as speech, radar, and physiological signals. They overcome the drawbacks of conventional Fourier analysis by simultaneously representing time and frequency for a given signal, thus enabling the analysis of the time variation of the spectrum content. A number of TFDs have been proposed in the past with their inherent advantages and drawbacks. The spectrogram, which is widely used as a TFD, has the inherent trade off between time and frequency resolutions. Later, the Wigner–Ville distribution was introduced with excellent time and frequency resolutions, but it suffers from cross-term