Optimal solutions are given for the two following problems: the condition for a degree 4 polynomial to have only positive values and the condition for an ellipse to be inside the unit circle.

By complexity of a solution, we mean the size and the degree of the polynomials appearing in it, as well as the length of the logical formula. Rather surprisingly, all these complexities may be simultaneously optimised in both examples.

1. Positive Polynomials

Let \( P(x) := ax^4 + bx^3 + cx^2 + dx + e \); the problem is to solve

\[ (\forall x) P(x) \geq 0. \]  \( \text{(1)} \)

Suppose that \( a > 0 \). The change of variable \( x' = x - b/4a \) gives the polynomial

\[ Q(x) := x'^4 + px'^2 + qx' + r, \]

and our problem is equivalent to

\[ (\forall x) Q(x) \geq 0. \]  \( \text{(2)} \)

The first remark is the following:

**Lemma 1.** Let \( Q(x) \) be a monic polynomial of degree \( d \); the sign of its discriminant is

\[ s = (-1)^{d^2-n/2}, \]

where \( r \) is the number of its real roots. Thus \( r \equiv d^2 + s + 3 \quad \text{mod} \quad 4 \) alinea: This is the proof; See Weiss (1963, p. 166).

The second remark is that if \( Q(x) \) has four real roots, they cannot be all of the same sign (their sum is 0). Thus, if \( r > 0 \), the product of the roots is positive and the curves
\[ y = x^4 + px^2 + r \] and \[ y = -qx \] have two intersection points with \( x > 0 \) and two with \( x < 0 \). The symmetry of \( y = x^4 + px^2 + r \) shows easily that if \( r > 0 \) and if this curve has 0 or four common points with \( y = -qx \) for some \( q \), the same is true for all \( q' \) with \( |q'| \leq |q| \).

Property (2) implies that \( \text{discriminant} (Q) \geq 0 \), and putting \( q' = 0 \) in the last remark we get the equivalence of (2) with

\[
\text{discriminant} (Q) \geq 0 \quad \text{and} \quad (\forall x) x^4 + px^2 + r \geq 0
\]

(this implies \( r \geq 0 \); the limit cases \( r = 0 \) or \( \text{discriminant} (Q) = 0 \) have to be handled separately). Thus we get the following:

**Proposition 1.** The condition

\[ (\forall x) x^4 + px^2 + qx + r \geq 0 \]

is equivalent to

\[ \text{discriminant} (x^4 + px^2 + qx + r) \geq 0 \quad \text{and} \quad (p \geq 0 \text{ or } r \geq p^2/4). \]

**Theorem 1.** The condition

\[ (\forall x) ax^4 + bx^3 + cx^2 + dx + e \geq 0 \]

is equivalent to

\[ (a > 0 \text{ and } D > 0 \text{ and } (8ac - 3b^2 \geq 0 \text{ or } R \geq 0)) \]

or \( (a = 0 \text{ and } b = 0 \text{ and } c > 0 \text{ and } d^2 - 4ce \leq 0) \)

or \( (a = 0 \text{ and } b = 0 \text{ and } c = 0 \text{ and } d = 0 \text{ and } e \geq 0) \)

where

\[ R = 64a^3e - 16a^2bd - 16a^2c^2 + 16ab^2c - 3b^4 \]

and

\[ D = 256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144ab^2ce^2 - 27b^4e^2 \]
\[ + 144a^2cd^2e - 6ab^2d^2e - 80abc^2de + 18b^3cde + 16ac^4e \]
\[ - 4b^2c^3e - 27a^2d^4 + 18abc^3d - 4b^3d^3 - 4ac^3d^2 + b^2c^3d^2. \]

It suffices to substitute in the conditions of Proposition 1 the values of \( p, q, r \) obtained by the change of variables of the beginning; the cases with \( a = 0 \) are easy.

**Remark 1.** If we substitute \( (e, d, c, b, a) \) for \( (a, b, c, d, e) \) we obtain clearly an equivalent solution.

**Proposition 2.** The condition \( (\forall x) P(x) > 0 \) is equivalent with

\[ (e > 0 \text{ and } D > 0 \text{ and } (8ce - 3d^2 \geq 0 \text{ or } R' \geq 0)) \]

or \( (e > 0 \text{ and } D = 0 \text{ and } 8ce - 3d^2 > 0 \text{ and } R' = 0) \)

where

\[ R' = 64e^3a - 16e^2db - 16e^2c^2 + 16ed^2c - 3d^4. \]

We have used Remark 1 to avoid the cases \( a = 0 \) of Theorem 1; if \( (\forall x) P(x) \geq 0 \) and \( (\exists x) P(x) = 0 \), the last \( x \) has to be a double root and \( D = 0 \). However, if \( P(x) \) has two imaginary double roots we have \( D = 0 \), but \( (\forall x) P(x) > 0 \). The second condition of the proposition takes this case into account; it corresponds to the condition \( p < 0 \) and \( p^2 - 4r = 0 \), the nullity of the discriminant implying \( q = 0 \).
REMARK 2. The Sturm theorem between $-\infty$ and $+\infty$ gives similar conditions with $R$ replaced by

$$1/8 \det \begin{bmatrix} b & 0 & 0 & 4a & 0 \\ 2c & b & 4a & 3b & 0 \\ 3d & 2c & 3b & 2c & 0 \\ 4e & 3d & 2c & d & 0 \end{bmatrix} = 16a^2ce - 6ab^2e - 18a^2d^2 + 14abcd - 3b^2d - 4ac^3 + b^2c^2$$

which is slightly more complicated than $R$. Moreover, when $D$ and this determinant are null we have to add a condition which separates the case of two double roots from a triple root. This solution was given by Arnon (1985). Thus the polynomial and the logical formula appearing in this solution are both more complicated than in our.

REMARK 3. In the five-dimensional space of the parameters $(a, b, c, d, e)$, the set of solutions of (1) is of dimension 5 and its boundary is of dimension 4; this boundary contains an open subset of the hypersurface $D = 0$. Thus the polynomial $D$ or some multiple of it must appear in every solution. The fact that the polynomials $R$ and $8ac - 3b^2$ are as simple as possible is clear in Proposition 1, where they are $p$ and $p^2 - 4r$. Thus, the solution of Theorem 1 is the simplest with respect to the complexity of the polynomials which appear in it.

2. The Ellipse Problem

This problem, also called the Kahan problem, consists in writing down conditions such that the ellipse $E(x, y) = 0$ with

$$E(x, y) := \frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} - 1$$

be inside the circle $C(x, y) = 0$ with

$$C(x, y) := x^2 + y^2 - 1.$$  

This problem can be stated as

$$(\forall x)(\forall y) E(x, y) = 0 \Rightarrow C(x, y) \leq 0.  \quad (3)$$

A first approach consists in a parametrisation of the ellipse

$$x = c + \frac{2at}{1 + t^2}, \quad y = d + b \frac{1 - t^2}{1 + t^2},$$

which gives the equivalent condition

$$(\forall t)(1 + t^2)^2 C \left( c + \frac{2at}{1 + t^2}, d + b \frac{1 - t^2}{1 + t^2} \right) \leq 0.$$  

The numerator of the left-hand side of this inequality is a polynomial in $t$ of degree 4. Thus the formula of Theorem 2 gives a solution of the problem. Unfortunately, this solution is not the simplest. However, applying Lemma 1, we get the polynomial $T$ in $a, b, c, d$ such that $T = 0$ is equivalent to “the ellipse and the circle are tangent” and $T > 0$ is equivalent to “the ellipse and the circle have 0 or four simple common points”. As above for $D$, an open subset of the hypersurface $T = 0$ is a piece of the boundary of the set.
of solutions and the polynomial $T$ must appear in every solution. This irreducible polynomial is

$$
T := a^4d^4 + ((2a^2b^2 + 2a^4)c^2 + (-4a^4 + 2a^2)b^2 + 2a^6 - 4a^4)d^6
+ ((b^4 + 4a^2b^2 + a^4)c^4 + ((-6a^2 - 2)b^4 + (2a^4 + 2a^2)b^2 - 2a^6 - 6a^4)c^2
+ (6a^4 - 6a^2 + 1)b^4 + (-6a^6 + 10a^4 - 6a^2)b^2 + a^8 - 6a^6 + 6a^4)d^4
+ ((2b^4 + 2a^2b^2)c^6 + (-2b^6 + (2a^2 - 6)b^4 + (-6a^4 + 2a^2)b^2 - 2a^4)c^4
+ ((6a^4 + 4)b^6 + (-10a^4 - 6a^2 + 6)b^4 + (6a^6 - 6a^4 - 10a^2)b^2 + 4a^6
+ 6a^4)c^2 + (-4a^4 + 6a^2 - 2)b^6 + (6a^6 - 8a^4 + 4a^2 - 2)b^4
+ (-2a^8 + 4a^6 - 8a^4 + 6a^2)b^2 - 2a^4 + 6a^6 - 4a^4)d^2 + b^4c^8
+ (2b^6 + (-4a^2 - 4)b^4 + 2a^2b^2)c^6 + (b^8 + (-6a^2 - 6)b^6
+ (6a^4 + 10a^2 + 6)b^4 + (-6a^4 - 6a^2)b^2 + a^4)c^4 + ((-2a^2 - 2)b^8
+ (6a^4 + 4a^2 + 6)b^6 + (-4a^6 - 8a^4 - 8a^2 - 4)b^4 + (6a^6 + 4a^4 + 6a^2)b^2
- 2a^6 - 2a^4)c^2 + (a^4 - 2a^2 + 1)b^8 + (-2a^6 + 2a^4 + 2a^2 - 2)b^6
+ (a^8 + 2a^6 - 6a^4 + 2a^2 + 1)b^4 + (-2a^8 + 2a^6 + 2a^4 - 2a^2)b^2 + a^8 - 2a^6 + a^4.
$$

In view of a simple solution of our problem, we need the following.

**Lemma 2.** Suppose $a > b$.

(a) If the ellipse is inside the circle, the same is true if $c$ is replaced by zero.

(b) If, when $c$ is replaced by zero, the new ellipse is inside the circle, then the ellipse and the circle have at most two real intersection points.

Suppose by symmetry that $c > 0$.

(a) If $c$ decreases, the half of the ellipse such that $x > c$ remains clearly inside the circle.

(b) When $c$ increases from zero, the number of intersection points changes only when the curves are tangent. We can compute the corresponding points: Let $x = \sin u$, $y = \cos u$ be a parametrisation of the circle, $x = a \sin t$, $y = d + b \cos t$ a parametrisation of the ellipse. The points which become tangent after a horizontal translation satisfy

$$
\cos u = d + b \cos t \quad \text{and} \quad \frac{\cos u}{\sin u} = \frac{a \cos t}{b \sin t}
$$

we get by eliminating $u$:

$$
b^2(a^2 - b^2)(\cos t)^4 + 2db(a^2 - b^2)(\cos t)^2
+ (d^2(a^2 - b^2) + b^4 - a^2)(\cos t)^2 + 2db^3 \cos t + d^2b^2.
$$

This polynomial in $\cos t$ has exactly one root in each interval $]-\infty, -1[ \cup [-1, 0[ \cup [0, 1[ \cup [1, +\infty[$. This gives four values for $t$ and two positive and two negative values of $c$ for which the curves are tangent. The first positive value corresponds to the moment where the number of intersection points passes from 0 to two. The second is the value of $c$ when the ellipse becomes outside of the circle. There is no possibility for passing from two to four intersection points.

**Corollary.** If $a > b$, the ellipse is inside the circle, iff $T > 0$, the ellipse is inside the circle when $c$ is replaced by 0 and the ellipse has a point strictly inside the circle.
It remains to solve the case where \( c = 0 \). For that we suppose \( d \geq 0 \). Leaving \( a \) and \( b \) fixed, let \( d \) be varying. If the ellipse is inside the circle, the same is true when \( d \) decreases. When \( d \) increases, we find a value \( d_0 \) such that the ellipse is not inside the circle for \( d > d_0 \), but is in it for \( d < d_0 \). If the minimum curvature of the ellipse is less than 1, \( d_0 \) corresponds to the value of \( d \) for which the curves are tangent at \( x = 0, y = 1 \). In the other case, the value of \( d_0 \) corresponds to the case where the curves are bitangent. Thus we get

**PROPOSITION 3.** If \( a > b \) and \( c = 0 \), the ellipse is inside the circle if and only if

\[
b^2 + d^2 \leq 1 \quad \text{and} \quad (a^2 \leq b \text{ or } d^2 a^2 \leq (1 - a^2)(a^2 - b^2)).
\]

**THEOREM 2.** In the general case, the ellipse is inside the circle if and only if

\[
(a > b \quad \text{and} \quad T \geq 0 \quad \text{and} \quad c^2 + (b + |d|)^2 - 1 \leq 0
\]

\[
\text{and} \quad (a^2 \leq b \text{ or } a^2 d^2 \leq (1 - a^2)(a^2 - b^2))
\]

\[
\text{or} \quad (a = b \quad \text{and} \quad c^2 + d^2 \leq (1 - a)^2 \quad \text{and} \quad a \leq 1)
\]

\[
\text{or} \quad (a < b \quad \text{and} \quad T \geq 0 \quad \text{and} \quad d^2 + (a + |c|)^2 - 1 \leq 0
\]

\[
\text{and} \quad (b^2 \leq a \text{ or } b^2 c^2 \leq (1 - b^2)(b^2 - a^2)).
\]

This follows immediately from all the preceding results.

**REMARK 4.** In all our proofs we have used some convexity argument. It seems that it is necessary for obtaining simple results in quantifier elimination. How can we put such convexity argument in a general procedure?

**REMARK 5.** A formulation which is logically simpler but where more polynomials have to be evaluated in each test is

\[
T \geq 0 \quad \text{and} \quad c^2 + (b + |d|)^2 \leq 1 \quad \text{and} \quad d^2 + (a + |c|)^2 \leq 1
\]

\[
\text{and} \quad ((b^2 \leq a \quad \text{and} \quad a^2 \leq b))
\]

\[
\text{or} \quad (a > b \quad \text{and} \quad a^2 d^2 \leq (1 - a^2)(a^2 - b^2))
\]

\[
\text{or} \quad (b < a \quad \text{and} \quad b^2 c^2 \leq (1 - b^2)(b^2 - a^2)).
\]

**REMARK 6.** Clearly the above calculations are not purely hand made. The polynomials \( T \), \( D \) and \( R \) were computed with MACSYMA which was also used for various experimentations.

**REMARK 7.** Among the polynomials which appear in Lauer’s solution are

\[
d^2(U - c^2(a^2 + b^2)) \quad \text{with} \quad U = a^2 d^2 - (1 - a^2)(a^2 - b^2)
\]

\[
-V + d^2(a^2 + b^2) \quad \text{with} \quad V = b^2 c^2 - (1 - b^2)(b^2 - a^2)
\]

\[
d^2 T,
\]

which have to be compared with those appearing in our solution.
References


