A quantitative sharpening of Moriwaki's arithmetic Bogomolov inequality

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A. Moriwaki proved the following arithmetic analogue of the Bogomolov unstability theorem. If a torsion-free hermitian coherent sheaf on an arithmetic surface has negative discriminant then it admits an arithmetically destabilising subsheaf. In the geometric situation it is known that such a subsheaf can be found subject to an additional numerical constraint and here we prove the arithmetic analogue. We then apply this result to slightly simplify a part of C. Soulé's proof of a vanishing theorem on arithmetic surfaces.

1 Introduction and statement of result

Let K be a number field with ring of integers $\mathcal{O}_K$ and $X/\mathrm{Spec}\,(\mathcal{O}_K)$ an arithmetic surface, i.e. a regular, integral, purely two-dimensional scheme, proper and flat over $\mathrm{Spec}\,(\mathcal{O}_K)$ and with smooth and geometrically connected generic fibre. Attached to a hermitian coherent sheaf on X are the usual characteristic classes with values in the arithmetic Chow-groups $\widehat{CH}^i(X)$ (cf. [GS1], 2.5), and in particular the discriminant of $\overline{E}$

$$\Delta(\overline{E}) := (1-r)\hat{c}_1(\overline{E})^2 + 2r\hat{c}_2(\overline{E}) \in \widehat{CH}^2(X)$$

where $r := \mathrm{rk}(E)$. The arithmetic degree map

$$\widehat{\deg} : \widehat{CH}^2(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is an isomorphism [GS2] and we will use the same symbol to to denote an element in $\widehat{CH}^2(X)_{\mathbb{R}}$ and its arithmetic degree in $\mathbb{R}$, see [GS2], 1.1 for the definition of arithmetic Chow-groups with real coefficients $\widehat{CH}^*(X)_{\mathbb{R}}$. Following [Mo2] we define the positive cone of X to be

$$\hat{C}_{++}(X) := \{x \in \widehat{CH}^1(X)_{\mathbb{R}} \,|\, x^2 > 0 \text{ and } \deg_K(x) > 0\} \,.$$

Given a torsion-free hermitian coherent sheaf $\overline{E}$ of rank $r \geq 1$ on X and a subsheaf $E' \subseteq E$ we endow E' with the metric induced from $\overline{E}$ and consider the difference of slopes

$$\xi_{\overline{E}',\overline{E}} := \frac{\hat{c}_1(\overline{E}')}{\mathrm{rk}(E')} - \frac{\hat{c}_1(\overline{E})}{r} \in \widehat{CH}^1(X)_{\mathbb{R}}.$$

Recall that a subsheaf $E' \subseteq E$ is *saturated* if the quotient E/E' is torsion-free. Our main result is the following.

Theorem 1 *Let $\overline{E}$ be a torsion-free hermitian coherent sheaf of rank $r \geq 2$ on the arithmetic surface X, satisfying*

$$\Delta(\overline{E}) < 0 \,.$$

Then there is a non-zero saturated subsheaf $\overline{E}' \subseteq \overline{E}$ such that $\xi_{\overline{E}',\overline{E}} \in \hat{C}_{++}(X)$ and

$$\xi^2_{\overline{E}',\overline{E}} \geq \frac{-\Delta}{r^2(r-1)} \,. \tag{1}$$

Remark 2 *The existence of an $\overline{E}' \subseteq \overline{E}$ with $\xi_{\overline{E}',\overline{E}} \in \hat{C}_{++}(X)$ is the main result of [Mo2] and means that $\overline{E}' \subseteq \overline{E}$ is arithmetically destabilising with respect to any polarisation of X, c.f.* loc. cit. *for more details on this. The new contribution here is the inequality (1) which is the exact arithmetic analogue of a known geometric result, c.f. for example [HL], Theorem 7.3.4.*

Remark 3 *A special case of Theorem 1 appears in disguised form in the proof of [So], Theorem 2: Given a sufficiently positive hermitian line bundle $\overline{L}$*

on the arithmetic surface X and some non-torsion element $e \in \mathrm{H}^1(X, L^{-1}) \simeq \mathrm{Ext}^1(L, \mathcal{O}_X)$, *C. Soulé establishes a lower bound for*

$$||e||^2 := \sup_{\sigma : K \hookrightarrow \mathbb{C}} ||\sigma(e)||^2_{L^2}$$

by considering the extension determined by e

$$\mathcal{E} : 0 \longrightarrow \overline{\mathcal{O}_X} \longrightarrow \overline{E} \longrightarrow \overline{L} \longrightarrow 0$$

and suitably metrised as to have $\hat{c}_1(\overline{E}) = \overline{L}$ *and* $2\hat{c}_2(\overline{E}) = \sum_\sigma ||\sigma(e)||^2_{L^2}$, *hence* $\Delta(\overline{E}) = -\overline{L}^2 + 2\sum_\sigma ||\sigma(e)||^2_{L^2}$ *(where we write* $\overline{L} = \hat{c}_1(\overline{L})$ *following the notation of* loc. cit.*).*
If $E_{\overline{\mathbb{Q}}}$ *is semi-stable the arithmetic Bogomolov inequality concludes the proof. Otherwise, the main point is to show the existence of of an arithmetic divisor* $\overline{D}$ *satisfying*

$$\deg_K(\overline{D}) \leq \deg_K(\overline{L})/2 \text{ and} \tag{2}$$

$$2(\overline{L} - \overline{D})\overline{D} \leq [K : \mathbb{Q}] \cdot ||e||^2, \tag{3}$$

c.f. (28) *and* (32) *of* loc. cit. *where these inequalities are established by some direct argument. We wish to point out that the existence of some* $\overline{D}$ *satisfying (2) and (3) is a special case of Theorem 1. In fact, let* $\overline{E}' \subseteq \overline{E}$ *be as in Theorem 1 and define* $\overline{D} := \overline{L} - \hat{c}_1(\overline{E'})$. *We then compute*

$$\xi_{\overline{E}',\overline{E}} = \frac{\overline{L}}{2} - \overline{D}$$

and $\xi_{\overline{E}',\overline{E}} \in \hat{C}_{++}(X)$ *implies (2). Furthermore, the inequality (1) in the present case reads*

$$\xi^2_{\overline{E}',\overline{E}} = \frac{\overline{L}^2}{4} + \overline{D}^2 - \overline{L}\,\overline{D} \geq \frac{-\Delta}{4} = \frac{\overline{L}^2}{4} - \frac{1}{2}\sum_\sigma ||\sigma(e)||^2_{L^2} \text{ , i.e.}$$

$$2(\overline{L} - \overline{D})\overline{D} \leq \sum_\sigma ||\sigma(e)||^2_{L^2},$$

hence the trivial estimate $[K:\mathbb{Q}]\cdot ||e||^2 \geq \sum_\sigma ||\sigma(e)||^2_{L^2}$ *gives (3).*

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2 Proof of Theorem 1

We collect some lemmas first. We call a short exact sequence

$$\mathcal{E} : 0 \longrightarrow \overline{E}' \longrightarrow \overline{E} \longrightarrow \overline{E}'' \longrightarrow 0$$

of hermitian coherent sheaves on X *isometric* if the metrics on E' and E'' are induced from the one on E. This implies that $\hat{c}_1(\overline{E}) = \hat{c}_1(\overline{E}') + \hat{c}_1(\overline{E}'')$ (i.e. $\tilde{c}_1(\mathcal{E}) = 0$). We also have

$$\hat{c}_2(\overline{E}) = \hat{c}_2(\overline{E}' \oplus \overline{E}'') - a(\tilde{c}_2(\mathcal{E})) \quad \text{in } \widehat{CH}^2(X)\ ,$$

where

$$a : \tilde{A}^{1,1}(X_\mathbb{R}) \longrightarrow \widehat{CH}^2(X)$$

is the usual map [SABK], chapter III.

Lemma 4 *If*

$$\mathcal{E} : 0 \longrightarrow \overline{E}' \longrightarrow \overline{E} \longrightarrow \overline{E}'' \longrightarrow 0$$

is an isometric short exact sequence of hermitian coherent sheaves on X *with ranks* $r', r, r'' \geq 1$ *and discriminants* $\Delta', \Delta, \Delta''$*, then*

$$\frac{\Delta'}{r'} + \frac{\Delta''}{r''} - \frac{\Delta}{r} = \frac{rr'}{r''}\xi^2_{\overline{E}',\overline{E}} + 2a(\tilde{c}_2(\mathcal{E})) \quad \textit{in } \widehat{CH}^2(X)_\mathbb{R}\ .$$

Proof We omit the computation using the formulas for $\hat{c}_i(\overline{E})$ recalled above which shows that the left hand side of the stated equality equals

$$\hat{c}_1(\overline{E})^2\left(\frac{r-1}{r} + \frac{1-r'}{r'}\right) + \hat{c}_1(\overline{E}'')^2\left(\frac{r-1}{r} + \frac{1-r''}{r''}\right) +$$

$$+\hat{c}_1(\overline{E}')\hat{c}_1(\overline{E}'')\left(\frac{2(r-1)}{r}-2\right)+2a(\tilde{c}(\mathcal{E})).$$

Similarly one writes $\xi^2_{\overline{E}',\overline{E}}$ as a rational linear combination of $\hat{c}_1(\overline{E})^2, \hat{c}_1(\overline{E}'')^2$ and $\hat{c}_1(\overline{E}')\hat{c}_1(\overline{E}'')$ and comparing the results, the stated formula drops out.
□

Lemma 5 *For $\mathcal{E}$ as in Lemma 4 and $\overline{G}'' \subseteq \overline{E}''$ a saturated subsheaf of rank $s \geq 1$ carrying the induced metric, put*

$$\overline{G} := \ker(E \longrightarrow E'' \longrightarrow E''/G'') \subseteq \overline{E}$$

with the induced metric. Then

$$\xi_{\overline{G},\overline{E}} = \frac{r'(r''-s)}{(r'+s)r''}\xi_{\overline{E}',\overline{E}} + \frac{s}{r'+s}\xi_{\overline{G}'',\overline{E}''} \quad \textit{in } \widehat{CH}^1(X)_{\mathbb{R}} \,.$$

Observe that the coefficients in the last expression are non-negative rational numbers.

Proof We have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \overline{E/G} & \xrightarrow{\simeq} & \overline{E''/G''} & & \\
 & & & & \uparrow & & \uparrow & & \\
\mathcal{E}: 0 & \longrightarrow & \overline{E}' & \longrightarrow & \overline{E} & \longrightarrow & \overline{E}'' & \longrightarrow & 0 \\
 & & \simeq\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \overline{H} & \longrightarrow & \overline{G} & \longrightarrow & \overline{G}'' & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Here, we have endowed $E/G, E''/G''$ and H with the metrics induced from $\overline{E}$,$\overline{E}''$ and $\overline{G}$, hence all rows and columns are isometric by definition. A minor point to note is that with this choice of metrics the two indicated isomorphisms are isometric, indeed this only means that taking sub- (resp. quotient-)metrics is transitive. One has

$$\xi_{\overline{E}',\overline{E}} = \frac{r''\hat{c}_1(\overline{E}') - r'\hat{c}_1(\overline{E}'')}{r'r}$$

and analogously for any isometric exact sequence in place of $\mathcal{E}$. Using this and the diagram one writes both sides of the stated equality as a $\mathbb{Q}$-linear combination of $\hat{c}_1(\overline{E}'), \hat{c}_1(\overline{G}'')$ and $\hat{c}_1(\overline{E''/G''})$ to obtain the same result, namely

$$\frac{r''-s}{(r'+s)r}\hat{c}_1(\overline{E}') + \frac{r''-s}{(r'+s)r}\hat{c}_1(\overline{G}'') - \frac{1}{r}\hat{c}_1(\overline{E''/G''}).$$

□

Finally, we will need the following observation about the intersection theory on X where, for $x \in \hat{C}_{++}(X)$, we write $|x| := (x^2)^{1/2} \in \mathbb{R}^+$.

Lemma 6 *The subset $\hat{C}_{++}(X) \subseteq \widehat{CH}^1(X)_{\mathbb{R}}$ is an open cone, i.e. $x, y \in \hat{C}_{++}(X)$ and $\lambda \in \mathbb{R}^+$ implies that $x+y, \lambda x \in \hat{C}_{++}(X)$. For $x, y \in \hat{C}_{++}$ we have $|x+y| \geq |x| + |y|$.*

Proof This is [Mo2], (1.1.2.2) except for the final assertion which is obvious if $x \in \mathbb{R}y$ and we can thus assume that $V := \mathbb{R}x + \mathbb{R}y \subseteq \widehat{CH}^1(X)_{\mathbb{R}}$ is two-dimensional. We claim that the restriction of the intersection-pairing makes V a real quadratic space of type $(1, -1)$. As we have $x \in V$ and $x^2 > 0$ we only have to exhibit some $v \in V$ with $v^2 < 0$. To achieve this let $h \in \widehat{CH}^1(X)_{\mathbb{R}}$ be the first arithmetic Chern class of some sufficiently positive hermitian line bundle on X such that the arithmetic Hodge index theorem holds for the Lefschetz operator defined by h, c.f. [GS2], Theorem 2.1, ii). Then $a := xh$ (resp. $b := yh$) are non-zero real numbers for otherwise we would have $x^2 < 0$ (resp. $y^2 < 0$). Thus $v := \frac{x}{a} - \frac{y}{b} \in V$ satisfies $v \neq 0$ and $vh = 0$, hence $v^2 < 0$.
Fix a basis $e, f \in V$ with $e^2 = 1, f^2 = -1$ and write

$$x = \alpha e + \beta f \text{ and}$$

$$y = \gamma e + \delta f.$$

To show that $|x+y| \geq |x| + |y|$ we can assume, changing both the signs of x and y if necessary, that $\alpha > 0$. We then claim that $\gamma > 0$. For otherwise there would be $\lambda_1, \lambda_2 \in \mathbb{R}^+$ such that $v := \lambda_1 x + \lambda_2 y$ would have e- coordinate equal to zero, hence $v^2 \leq 0$ contradicting the fact that either $-v$ or v lies in $\hat{C}_{++}(X)$ (depending on whether or not we changed the signs of x and y above).
From $x^2 = \alpha^2 - \beta^2$, $y^2 = \gamma^2 - \delta^2 > 0$ we obtain $\alpha = |\alpha| \geq |\beta|$ and $\gamma = |\gamma| \geq |\delta|$ and then $\alpha\gamma \geq |\beta\delta| \geq \beta\delta$, i.e.

$$xy = \alpha\gamma - \beta\delta \geq 0. \tag{4}$$

To conclude, we use the following chain of equivalent statements

$$\begin{gathered}
|x+y| \geq |x| + |y| \Leftrightarrow \\
(x+y)^2 - (|x| + |y|)^2 \geq 0 \Leftrightarrow \\
2xy - 2|x||y| \geq 0 \Leftrightarrow \\
xy \geq |x||y| \overset{(4)}{\Leftrightarrow} \\
(xy)^2 \geq |x|^2|y|^2 \Leftrightarrow \\
(\alpha\gamma - \beta\delta)^2 \geq (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) \Leftrightarrow \\
\alpha^2\gamma^2 + \beta^2\delta^2 - 2\alpha\beta\gamma\delta \geq \alpha^2\gamma^2 - \alpha^2\delta^2 - \beta^2\gamma^2 + \beta^2\delta^2 \Leftrightarrow \\
2\alpha\beta\gamma\delta \leq \alpha^2\delta^2 + \beta^2\gamma^2 \Leftrightarrow \\
0 \leq (\alpha\delta - \beta\gamma)^2.
\end{gathered}$$

□

Proof of Theorem 1. We first remark that for a torsion-free hermitian coherent sheaf $\overline{F}$ of rank one on X we always have $\Delta(\overline{F}) \geq 0$. In fact,

$$F \simeq \mathcal{L} \otimes \mathcal{I}_Z$$

for some line-bundle $\mathcal{L}$ and $\mathcal{I}_Z$ the ideal sheaf of some closed subscheme $Z \subseteq X$ of codimension 2. This becomes an isometry for the trivial metric on $\mathcal{I}_Z$ and a suitable metric on $\mathcal{L}$ (since $\mathcal{I}_Z$ is trivial on the generic fibre of X). Then

$$\Delta(\overline{F}) = 2\hat{c}_2(\overline{\mathcal{L}} \otimes \mathcal{I}_Z) = 2\hat{c}_2(\mathcal{I}_Z) = 2\,\mathrm{length}(Z) \geq 0\ .$$

By the main result of [Mo2], there is $0 \neq \overline{E}' \subseteq \overline{E}$ saturated such that $\xi_{\overline{E}',\overline{E}} \in \hat{C}_{++}(X)$. We can assume that, as E' varies through these subsheaves, the real numbers $\xi^2_{\overline{E}',\overline{E}}$ remain bounded for otherwise there is nothing to prove. So we can choose $0 \neq \overline{E}' \subseteq \overline{E}$ saturated with $\xi_{\overline{E}',\overline{E}} \in \hat{C}_{++}(X)$ and $\xi^2_{\overline{E}',\overline{E}}$ maximal subject to these conditions. Put $E'' := E/E'$ and consider the isometric exact sequence

$$\mathcal{E} : 0 \longrightarrow \overline{E}' \longrightarrow \overline{E} \longrightarrow \overline{E}'' \longrightarrow 0$$

with discriminants $\Delta', \Delta, \Delta''$ and ranks r', r, r''. We claim that $\Delta' \geq 0$. This is clear in case $r = 2$ from the remark made at the beginning of the proof. In case $r \geq 3$ we assume that $\Delta' < 0$ and we let $\overline{G} \subseteq \overline{E}'$ be a saturated subsheaf with $\xi_{\overline{G},\overline{E}'} \in \hat{C}_{++}$. Then $\overline{G} \subseteq \overline{E}$ is saturated and using lemma 6 we get

$$|\xi_{\overline{G},\overline{E}}| = |\xi_{\overline{G},\overline{E}'} + \xi_{\overline{E}',\overline{E}}| \geq |\xi_{\overline{G},\overline{E}'}| + |\xi_{\overline{E}',\overline{E}}| > |\xi_{\overline{E}',\overline{E}}|$$

contradicting the maximality of $|\xi_{\overline{E}',\overline{E}}|$. So we have indeed $\Delta' \geq 0$. Assume now, contrary to our assertion, that

$$\frac{\Delta}{r} < -r(r-1)\xi^2_{\overline{E}',\overline{E}}\ . \tag{5}$$

Then from Lemma 4, $\Delta' \geq 0$, (5) and $\tilde{c}_2(\mathcal{E}) \leq 0$ ([Mo1], 7.2) we get

$$\begin{aligned} \frac{\Delta''}{r''} &\leq \frac{\Delta}{r} + \frac{rr'}{r''}\xi^2_{\overline{E}',\overline{E}} < \left(-r(r-1) + \frac{rr'}{r''}\right)\xi^2_{\overline{E}',\overline{E}} \\ &= -r^2\frac{r''-1}{r''}\xi^2_{\overline{E}',\overline{E}} \leq 0\ , \end{aligned}$$

hence $\Delta'' < 0$. By induction, there is $0 \neq \overline{G}'' \subseteq \overline{E}''$ saturated with $\xi_{\overline{G}'',\overline{E}''} \in \hat{C}_{++}(X)$ and

$$\xi^2_{\overline{G}'',\overline{E}''} \geq \frac{-\Delta''}{r''^2(r''-1)} > \frac{r^2}{r''^2}\xi^2_{\overline{E}',\overline{E}}\ . \tag{6}$$

Clearly $\overline{G} := \ker(E \to E''/G'') \subseteq \overline{E}$ is saturated and from Lemma 5, the positivity of the coefficients appearing there and lemma 6 we get

$$\begin{aligned} |\xi_{\overline{G},\overline{E}}| &\geq \frac{r'(r''-s)}{(r'+s)r''}|\xi_{\overline{E}',\overline{E}}| + \frac{s}{r'+s}|\xi_{\overline{G}'',\overline{E}''}| \\ &\overset{(6)}{>} \frac{r'(r''-s)}{(r'+s)r''}|\xi_{\overline{E}',\overline{E}}| + \frac{s}{r'+s}\frac{r}{r''}|\xi_{\overline{E}',\overline{E}}| \\ &= \left(\frac{r'(r''-s)+rs}{r''(r'+s)}\right)|\xi_{\overline{E}',\overline{E}}| = |\xi_{\overline{E}',\overline{E}}| \ . \end{aligned}$$

This again contradicts the maximality of $|\xi_{\overline{E}',\overline{E}}|$ and concludes the proof. □

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