Valency of Distance-regular Antipodal Graphs with Diameter 4

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Let $G$ be a non-bipartite strongly regular graph on $n$ vertices of valency $k$. We prove that if $G$ has a distance-regular antipodal cover of diameter 4, then $k \leq \lfloor (2n + 1)/5 \rfloor$, unless $G$ is the complement of a triangular graph $T(7)$, the folded Johnson graph $J(8, 4)$ or the folded halved 8-cube. However, for these three graphs the bound $k \leq \lfloor (n - 1)/2 \rfloor$ holds. This result implies that only one of a complementary pair of strongly regular graphs can be the antipodal quotient of an antipodal distance-regular graph.

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1. INTRODUCTION

Let $H$ be an antipodal distance-regular graph of diameter 4 or 5. Its antipodal quotient is a connected strongly regular graph of diameter 2 according to Brouwer et al. [2, Proposition 4.2.2]. Let us denote it by $G$ and its parameters with $(n, k, \lambda, \mu)$, i.e., $G$ is a connected graph on $n$ vertices with valency $k$, any two adjacent (resp. non-adjacent) vertices having $\lambda$ (resp. $\mu$) common neighbours. We also say that $H$ is a distance-regular antipodal cover of $G$ or that $G$ is a folded graph of $H$. Let $r$ be the antipodal class size of $H$. The intersection array of $H$ is $\{k, k - \lambda - 1, (r - 1)\mu/r, 1; 1, \mu/r, k - \lambda - 1, k\}$ by Brouwer et al. [2, Proposition 4.2.2.], so $\mu \neq 0$. If $\mu = k$, then $G$ is a complete bipartite graph according to Jurišić [7, Proposition 2.7], in which case many examples of their distance regular antipodal covers are known, see, for example, [4, 5]. We study the case $\mu < k$. It is well known that if $H$ has diameter 5, then $k \leq \lfloor (n - 1)/2 \rfloor$, see [1]. But if the diameter of $H$ is 4, then the best known bound has been weaker: $k \leq \lfloor r(n - 1)/(2r - 1) \rfloor$. There were no known examples attaining this bound, so there was a conjecture that $k \leq \lfloor (n - 1)/2 \rfloor$ also for this case, see [6, p. 56] or [8]. In this paper we prove that this conjecture is indeed true. In our main theorem we prove that even sharper bound holds:

THEOREM 1.1. Let $H$ be an antipodal distance-regular graph with diameter 4 and $G$ its antipodal quotient. If $(n, k, \lambda, \mu)$ are the parameters of $G$ and $\mu < k$, then $k \leq \lfloor 2(n + 1)/5 \rfloor$, unless $G$ is the complement of a triangular graph $T(7)$, the folded Johnson graph $J(8, 4)$ or folded halved 8-cube.

This result implies that only one of a complementary pair of strongly regular graphs can be the antipodal quotient of an antipodal distance-regular graph. See [1], where they determine almost all antipodal distance-regular covering graphs of classical distance-regular graphs if diameter $d \geq 2$, cf. [11]. Another application of our result is an alternative proof of the fact that the Krein parameter $q_{\mu \lambda}^2 \neq 0$ in $H$ as mentioned in [8], cf. [3].

After some preliminaries in Section 2 we prove the main result in Section 3. The reader can find all notions, which are not explicitly defined in this paper, in [2] or [8].

2. PRELIMINARIES

Let us review some basic feasibility conditions of a strongly regular graph which has a distance-regular antipodal cover. Let $G$ be a strongly-regular graph and $v$ its vertex. By counting the edges between neighbours and non-neighbours of $v$, we obtain the following

\begin{align*}
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expression for the number of vertices of the graph $G$:

$$n = 1 + k + \frac{k(k - \lambda - 1)}{\mu},$$

(1)

hence $k(k - \lambda - 1)/\mu$ must be an integer.

Let $H$ be a distance-regular antipodal cover of $G$ with diameter 4. It has five distinct eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$ with multiplicities $1 = m_0, \ldots, m_4$ and $\theta_2 > 0 > \theta_3$. Eigenvalues of $G$ are $k, \theta_2$ and $\theta_4$ with multiplicities $1, m_2$ and $m_4$, see [2, p. 142]. Eigenvalues $\theta_2$ and $\theta_4$ are the roots of $x^2 - (\lambda - \mu)x - (k - \mu) = 0$, thus $\theta_2\theta_4 = k - \mu$ and $\theta_2 + \theta_4 = \lambda - \mu$.

The multiplicities are $m_2 = (\theta_4 + 1)k(k - \theta_3)/(\mu(\theta_4 - \theta_2))$ and $m_4 = n - m - 1$. If $m_2 \neq m_4$, then $\theta_2$ and $\theta_4$ are integral, see [2, Theorem 1.3.1]. The other case, i.e., when $m_2 = m_4$, is usually called the half-case and the corresponding graph $G$ is called a conference graph. It is known that conference graphs do not have distance-regular antipodal covers, except for the pentagon, which is covered by the decagon that has diameter $5$, see [2, p. 180]. Therefore, we will consider only non-conference strongly regular graphs, i.e., the case $m_2 \neq m_4$.

Furthermore, by integrality of multiplicities, we will consider only non-conference strongly regular graphs, i.e., the case $m_2 \neq m_4$.

Let $X$ be a set with $n$ elements. The Johnson graph, denoted by $J(n, k)$, has as vertex set all subsets of $X$ with $k$ elements, and two vertices $a, b$ adjacent whenever $a \cap b$ has cardinality $k - 1$. The graphs $J(n, 2)$ are known as the triangular graphs and are also denoted by $T(n)$. The direct product of $d$ cliques of size $2$ is called the $d$-cube.

3. THE INEQUALITY

In this section we prove the main result. First we need three lemmas.

**Lemma 3.1.** Let $H$ be an antipodal tight distance-regular graph with diameter $4$ and $G$ its antipodal quotient. If $(n, k, \lambda, \mu)$ are parameters of $G$, then $\lambda < k/4$, unless $G$ is the complement of triangular graph $T(7)$, the folded Johnson graph $J(8, 4)$, the folded halved $8$-cube, strongly regular graphs with parameters $(126, 45, 12, 18)$ or $(378, 117, 36, 36)$, or possible strongly regular graphs with parameters $(250, 81, 24, 27)$, $(638, 189, 60, 54)$ or $(900, 261, 84, 72)$. 


Proof. Let us suppose, that \( \lambda \geq k/4 \), i.e., \( 4pq + 4p \geq pq^2 + pq + q^2 \). Thus \( p(q - 4)(q + 1) \leq -q^2 < 0 \), so \( q \in \{2, 3\} \).

Case \( q = 2 \). From divisibility conditions for \( p + q \) and \( p + q^2 \) we obtain \( p \in \{1, 2, 4\} \). For \( p = 1 \) we get the complement of triangular graph \( T(7) \), which is the folded Conway–Smith graph. For \( p = 2 \) we have the folded Johnson graph \( J(8, 4) \) and for \( p = 4 \) we get the folded halved 8-cube.

Case \( q = 3 \). From divisibility conditions for \( p + q \) and \( p + q^2 \) we get \( p \in \{1, 3, 6, 9, 15, 21\} \). The case \( p = 1 \) is ruled out by Jurišić and Koolen [8, Theorem 6.4]. For \( p = 3 \) and 9 we get strongly regular graphs with parameters \( (126, 45, 12, 18) \) and \( (378, 117, 36, 36) \), respectively. The known antipodal distance-regular covers with diameter 4 of these graphs are \( 3.\Omega\kappa_3 \) and \( 3.\Omega\tau_3 \), respectively. For \( p = 6, 15 \) and 21 we get the remaining three possible strongly regular graphs.

\[ \square \]

Lemma 3.2. Let \( H \) be an antipodal distance-regular graph with diameter 4 and \( G \) its antipodal quotient. If \( (n, k, \lambda, \mu) \) are parameters of \( G \) and if \( H \) is not tight, then \( \lambda < k/4 \).

Proof. Let us suppose that \( \lambda \geq k/4 \). Then we have \( \theta_1 + \theta_2 = \lambda \geq k/4 \), so \( \theta_1 \geq k/4 - \theta_2 > k/4 \). On the other hand, we have \( k = \theta_1(\theta_2) > k/4(\theta_2) \), so \( \theta_1 \in \{-1, -2, -3\} \). The first choice, \( \theta_2 = -1 \), is not good because of (2). Therefore, by (2) and the fact that \( H \) is not tight, we need to consider only the following cases separately: \( \theta_2 = -2, \theta_2 = -3 \) and \( \theta_2 = -3, \theta_4 \in \{-4, \ldots, -8\} \). Note, that \( \theta_1 = \lambda - \theta_2 \) and \( \theta_2 = (\theta_1 + 1)(\theta_3 + 1)/(\theta_4 + 1) \). If \( \ell = 1 \) we get the complement of triangular graph \( T(6) \) with \( \lambda < k/4 \). Cases \( \ell = 2 \) and 4 are not possible by integrality of \( m_1 \). The case \( \ell = 10 \) is ruled out by integrality of \( m_2 \).

(i) \( \theta_2 = -2, \theta_4 = -3 \). We have \( \theta_1 = \lambda + 2 \) and \( \theta_2 = (\lambda + 3)/2 - 1 \), thus \( \lambda = 2\ell - 1 \) for some positive integer \( \ell \). Furthermore, we have \( k = -\theta_1\theta_2 = 2(2\ell + 1), \mu = \lambda + 2\theta_4 = \ell + 2 \) and \( k(\lambda + 1)/\mu = 4(2\ell + 1)(\ell + 1)/\ell + 2 = 8\ell - 4 + 12/(\ell + 2) \). Thus \( \ell \in \{1, 2, 4, 10\} \). If \( \ell = 1 \) we get the complement of triangular graph \( T(6) \) with \( \lambda < k/4 \). Cases \( \ell = 2 \) and 4 are not possible by integrality of \( m_2 \). Thus \( \ell = 10 \) is the multiplicity \( m_2 \) an integer, but in these cases \( m_1 \notin \mathbb{N} \).

(ii) \( \theta_2 = -3, \theta_4 = -4 \). We obtain \( \theta_1 = \lambda + 3 \) and \( \theta_2 = (\lambda + 4)/3 - 1 \), thus \( \lambda = 3\ell - 1 \) for some positive integer \( \ell \). Furthermore, \( k = 9\ell + 6, \mu = \ell + 2 \) and \( k(\lambda + 1)/\mu = 18(3\ell - 1)/\ell + 2 = 54\ell - 18 + 72/(\ell + 2) \). So \( \ell \in \{1, 2, 4, 6, 7, 10, 16, 22, 34, 70\} \). Only for \( \ell \in \{2, 10\} \) is the multiplicity \( m_2 \) an integer, but in these cases \( m_1 \notin \mathbb{N} \).

(iii) \( \theta_2 = -3, \theta_4 = -5 \). Similarly as above we obtain \( k = 6\ell + 9, \lambda = 2\ell, \mu = \ell + 4 \) and \( k(\lambda + 1)/\mu = 12(\ell + 3)/\ell + 2 = 24\ell - 12 + 120/(\ell + 4) \) for some integer \( \ell \geq 0 \). Thus \( \ell \in \{0, 1, 2, 4, 6, 8, 11, 16, 20, 26, 36, 56, 116\} \). Only for \( \ell \in \{0, 6, 16\} \) is the multiplicity \( m_2 \) an integer. The case \( \ell = 0 \) is ruled out by the Krein condition (3). Cases \( \ell = 6 \) and 16 are ruled out by integrality of \( m_1 \) and condition (C).

(iv) \( \theta_2 = -3, \theta_4 = -6 \). We obtain \( k = 15\ell - 3, \lambda = 5\ell - 4, \mu = 3\ell + 3 \) and \( k(\lambda + 1)/\mu = 10(5\ell - 1)/\ell + 1 = 50\ell - 60 + 60/(\ell + 1) \) for some positive integer \( \ell \). So \( \ell \in \{1, 2, 3, 4, 5, 9, 11, 14, 19, 29, 59\} \). Only for \( \ell \in \{2, 5, 9, 11\} \) is the multiplicity \( m_2 \) an integer. For \( \ell = 2 \) we get a possible strongly regular graph with parameters \( (88, 27, 6, 9) \) with \( \lambda < k/4 \). Cases \( \ell = 5, 9 \) and 11 are ruled out by integrality of \( m_1 \).

(v) \( \theta_2 = -3, \theta_4 = -7 \). We obtain \( k = 9\ell + 6, \lambda = 3\ell - 1, \mu = 2\ell + 6 \) and \( k(\lambda + 1)/\mu = 9(3\ell + 2)/\ell + 3 = 27\ell - 36 + 126/(\ell + 3) \). So \( \ell \in \{3, 4, 6, 11, 15, 18, 39, 60, 123\} \). Only for \( \ell \in \{3, 11, 18\} \) is the multiplicity \( m_2 \) an integer. These cases are ruled out by integrality of \( m_1 \) and condition (C).

(vi) \( \theta_2 = -3, \theta_4 = -8 \). We obtain \( k = 3(7\ell + 6), \lambda = 7\ell + 3, \mu = 5(\ell + 2) \) and \( k(\lambda + 1)/\mu = 42(7\ell + 6)/5(\ell + 2) \). For
some integer $\ell$, $\ell \geq 0$. Thus $\ell \in \{2, 4, 12, 14, 19, 22, 54, 82, 334\}$. Only for $\ell = 4$ is the multiplicity $m_2$ an integer. But for $\ell = 4$ the multiplicity $m_1$ is not an integer.

**Lemma 3.3.** Let $H$ be an antipodal distance-regular graph with diameter 4 and $G$ its antipodal quotient. If $(n, k, \lambda, \mu)$ are parameters of $G$ and $\mu < k$, then $\mu < k/2$, unless $G$ is the complement of triangular graph $T(6)$, the complement of triangular graph $T(7)$ or the folded Johnson graph $J(8, 4)$.

**Proof.** Suppose $\mu \geq k/2$. If $\lambda \geq k/4$, then we get, by Lemma 3.1, the complement of triangular graph $T(7)$ and the folded $J(8, 4)$. Now assume $\lambda < k/4$. Because $-\theta_2\theta_4 = k - \mu$, we have $-\theta_2\theta_4 \leq k/2$. From $\theta_2 + \theta_4 = \lambda - \mu$, we conclude $2\theta_2 + 2\theta_4 < -k/2$. By summing these two inequalities we get

$$\theta_2 < -\frac{2\theta_4}{2 - \theta_4} = 2 - \frac{4}{2 - \theta_4}.$$ 

But the right-hand side of this inequality is always less than 2, so the eigenvalue $\theta_2$ must be 1. The complement $G$ of the graph $G$ is again a strongly regular graph, and has eigenvalues $n - k - 1, -\theta_4 - 1$ and $-\theta_2 - 1 = -2$, see [2, Theorem 1.3.1]. By Seidel [10], [2, Theorem 3.12.4], $G$ is one of the following graphs: a triangular graph $T(n)(n \geq 5)$, a lattice graph $L_2(n)(n \geq 3)$, a complete multipartite graph $K_{n \times 2}(n \geq 2)$, or one of the graphs of Petersen, Clebsch, Schlafli, Shrikhande or Chang. So $G$ must be the complement of one of these graphs. We know that the complement $G$ has parameters $k = n - k - 1$, $\lambda = n - 2k + \mu - 2$ and $\mu = n - 2k + \lambda$. We first rule out the graphs of Petersen, Schlafli, Shrikhande and Chang, because the eigenvalues $\theta_1$ and $\theta_3$ for their complements are not integral. The complement of the Clebsch graph is a folded 5-cube with $\mu < k/2$. Furthermore, the complete multipartite graphs $K_{n \times 2}$ are ruled out because their complements are not connected.

If $G$ is isomorphic to the complement of the triangular graph $T(n)$, then $G$ has no distance-regular antipodal covers for $n \geq 7$ according to Van Bon and Brouwer [1, Proposition 4.2]. If $n = 6$ or 7, then $G$ has a unique distance-regular antipodal cover. If $n = 5$, then $G$ is a Petersen graph. By Van Bon and Brouwer [1, Proposition 4.1], a Petersen graph has no antipodal distance-regular cover with diameter 4.

The case $G = L_2(n)$ has already been ruled out by Van Bon and Brouwer [1, p 148].

**Proof of Theorem 1.1.** If $\lambda \geq k/4$ or $\mu \geq k/2$, then we have nine possibilities for $G$: $T(6)$, $T(7)$, the folded Johnson graph $J(8, 4)$, the folded halved 8-cube, the strongly regular graph with parameters $(126, 45, 36, 36)$, or $(378, 117, 36, 36)$, or the possible strongly regular graph with parameters $(250, 81, 24, 27)$, $(638, 189, 60, 54)$ or $(900, 261, 84, 72)$. It is a straightforward calculation to check that, only for $T(7)$, the folded Johnson graph $J(8, 4)$ and folded halved 8-cube inequality $k \leq 2(n + 1)/5$ is not valid. Now if $\lambda < k/4$ and $\mu < k/2$, we have

$$n = 1 + k + \frac{k(k - \lambda - 1)}{\mu} > 1 + k + \frac{3k}{2} - 2 = \frac{5k}{2} - 1.$$ 

Therefore, $k < 2(n + 1)/5$. By integrality of $k$, we obtain the desired inequality.

It is easy to see that, for $T(7)$, the folded Johnson graph $J(8, 4)$ and the folded halved 8-cube the inequality $k \leq (n - 1)/2$ holds. Also, if $2(n + 1)/5 \geq (n - 1)/2$, then $n \leq 9$. But there is no antipodal quotient of antipodal distance-regular graph with diameter 4, for which $n \leq 9$ and $\mu < k$, see [2, p 421]. Thus, it is a trivial consequence of Theorem 1.1 that inequality $k \leq (n - 1)/2$ holds for all antipodal quotients of antipodal distance-regular graphs of diameter 4, for which $\mu < k$. 


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