Coequational Logic for Finitary Functors

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Motivation

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Deduce properties of (state-based) systems from known properties of these systems.

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deduction of system properties	coequational logic	each state accepts the empty word $\Rightarrow$ each state accepts every word

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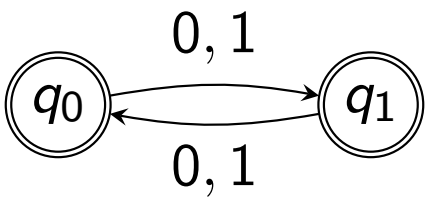

Outline

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Definitions

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Definition

Given a cofree H-coalgebra (Q, α_Q, γ) on C, an H-coalgebra (S, α_S) satisfies $\boxtimes q$ if $c^*(s) \neq q$ for all colourings $c : S \to C$ and for all $s \in S$.

- $c^* : S \to Q$ denotes the unique H-coalgebra homomorphism with $c = \gamma \circ c^*$

Coequations for deterministic automata

Example

Deterministic automata with binary input alphabet $\{0, 1\}$

- coalgebras of the polynomial functor $\{tt, ff\} \times (-)^{\{0,1\}}$
- $Q =$ all coloured ordered binary trees without leaves and with node labels from $\{tt, ff\}$
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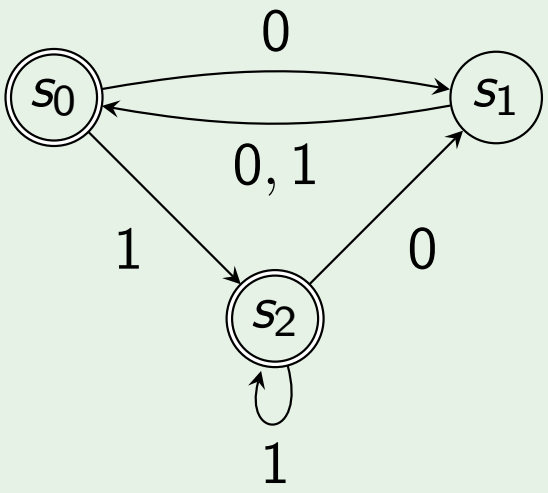

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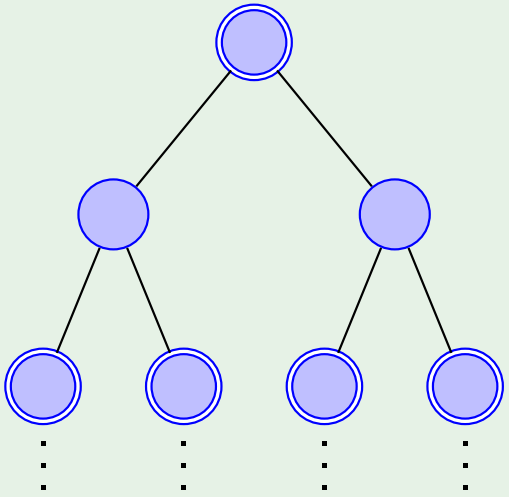

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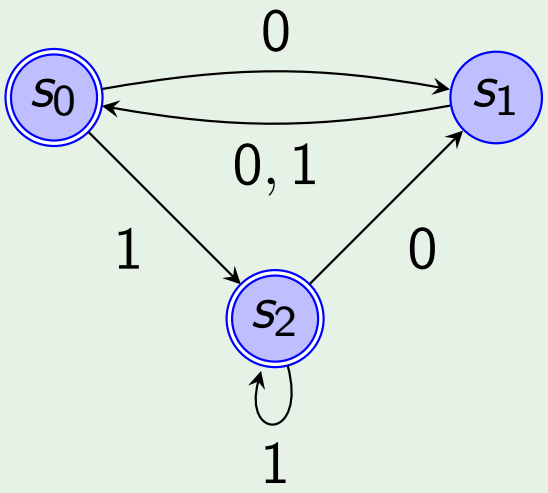

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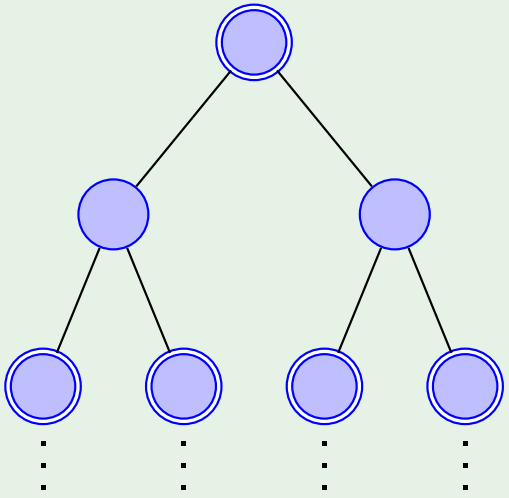

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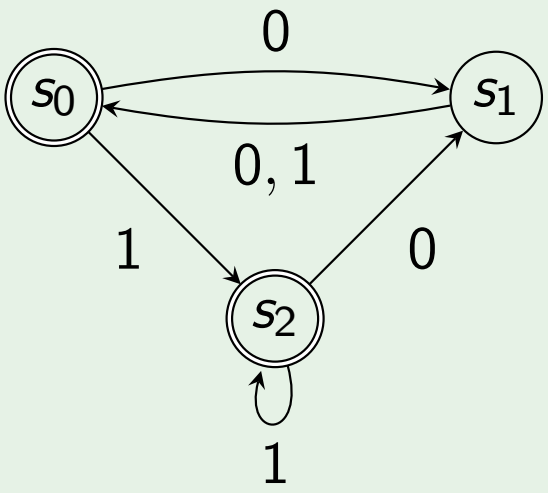

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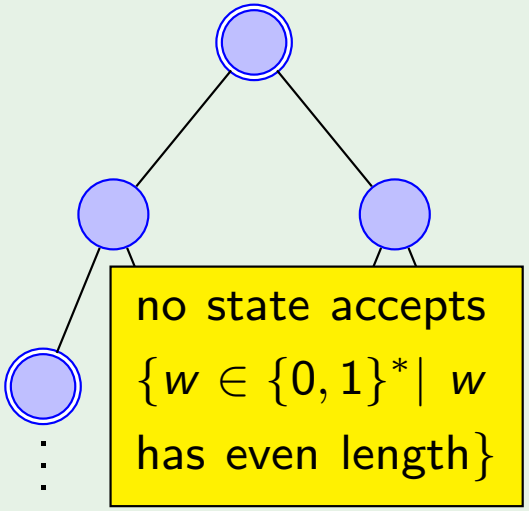

Semantical consequences

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$\boxtimes q$ is a semantical consequence of $\boxtimes q'$ if every coalgebra satisfying $\boxtimes q'$ also satisfies $\boxtimes q$.

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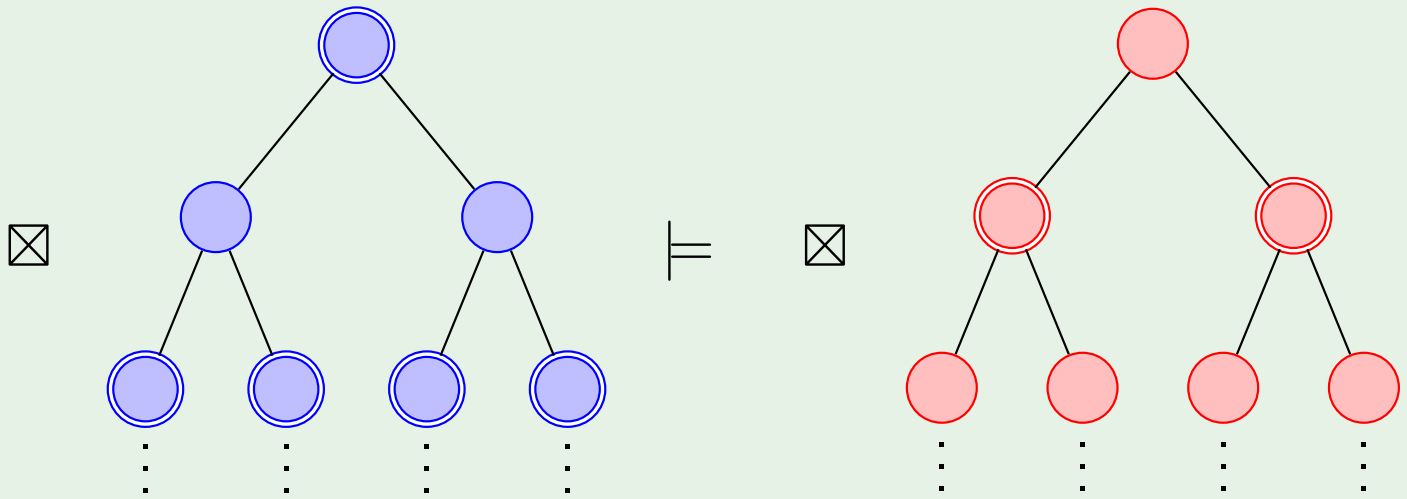

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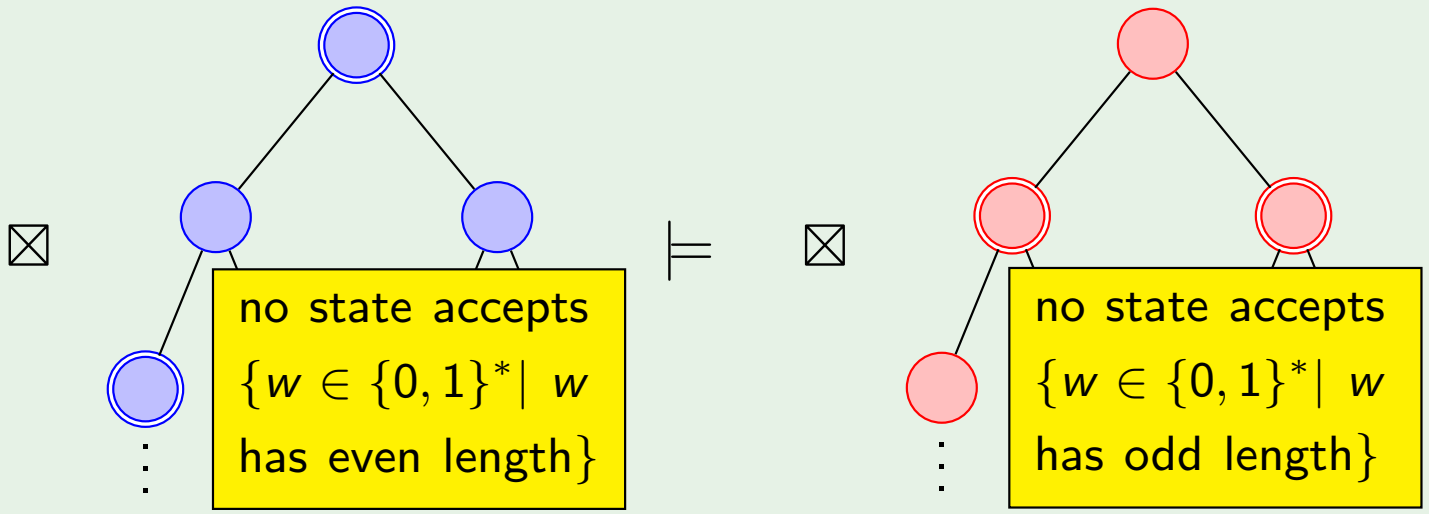

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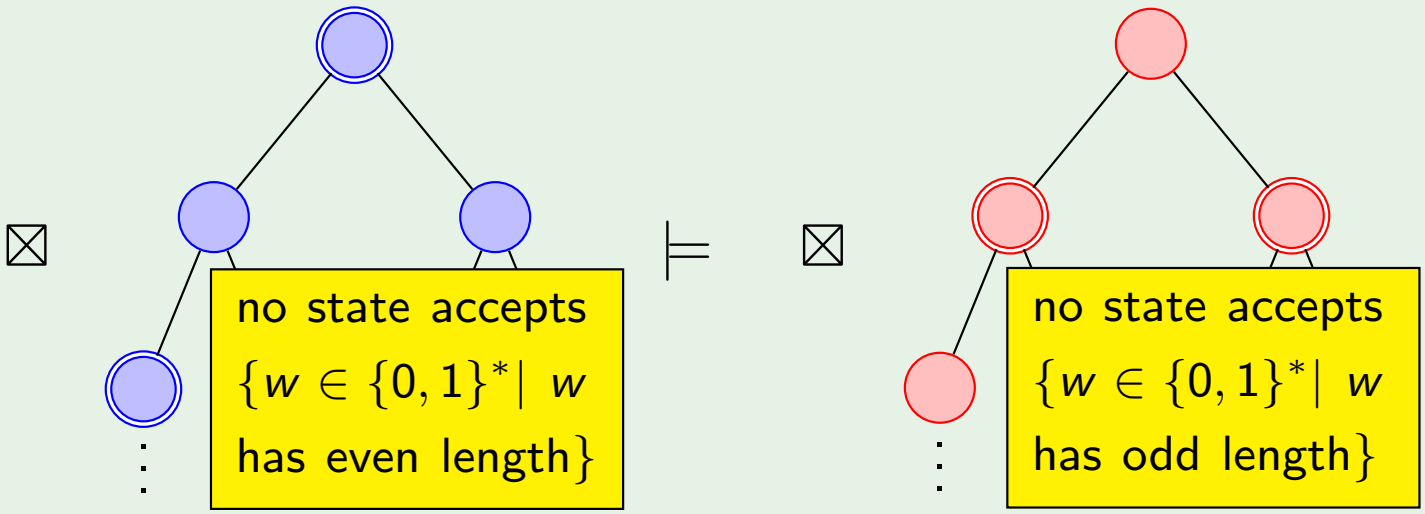

Is this true?

Outline

The relation $\sqsubseteq$

Definition

q' is a recolouring of q if there exists a homomorphism $h : Q \to Q$ with $h(q) = q'$.

- or equivalently: q and q' have the same shape and equivalent subtrees in q are equivalent in q' again
- write $q' \sqsubseteq q$ if q' is a recolouring of a subtree of q

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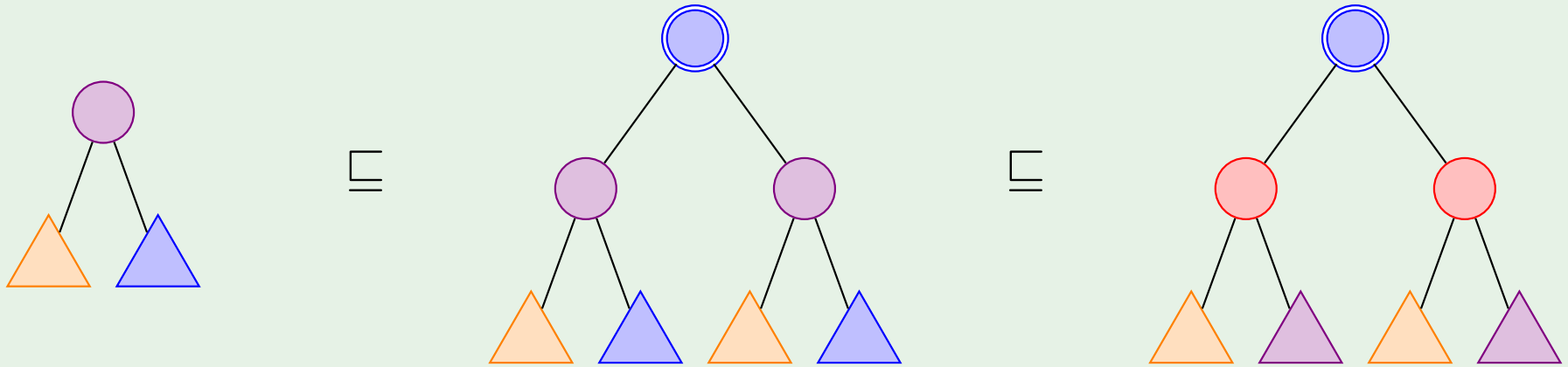

Logic for polynomial functors

Theorem (J. Adámek)

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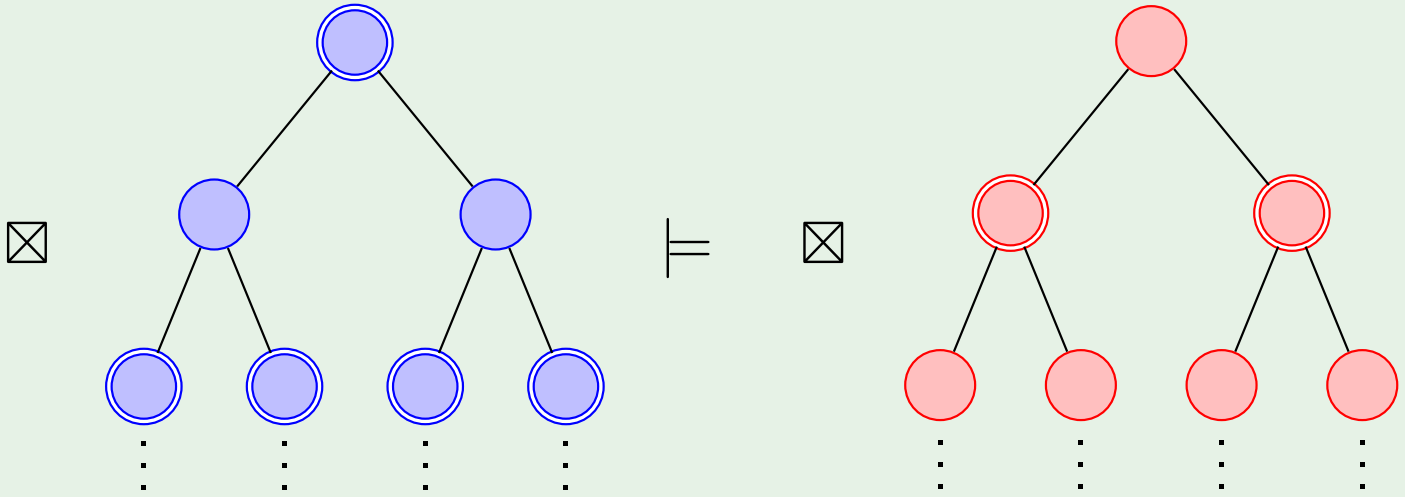

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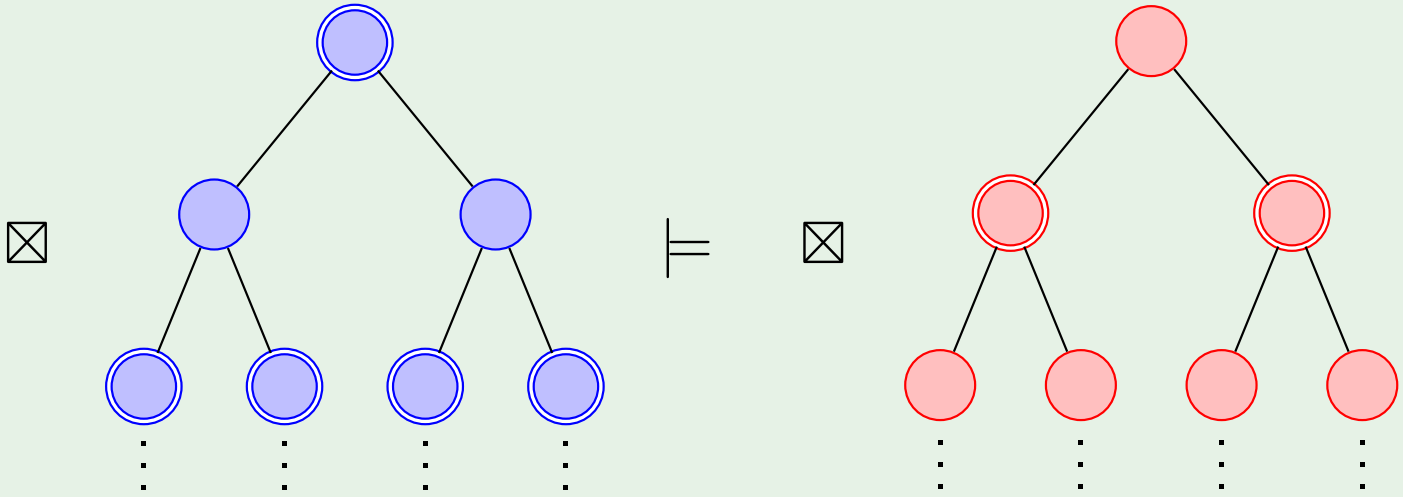

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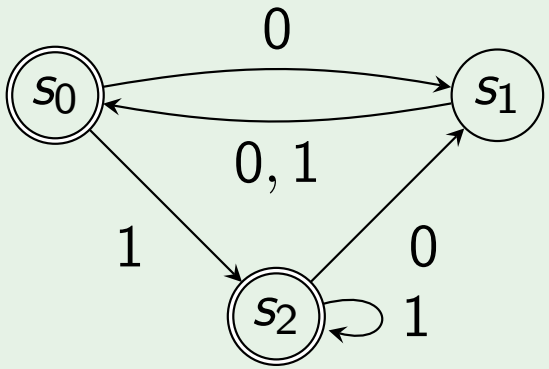

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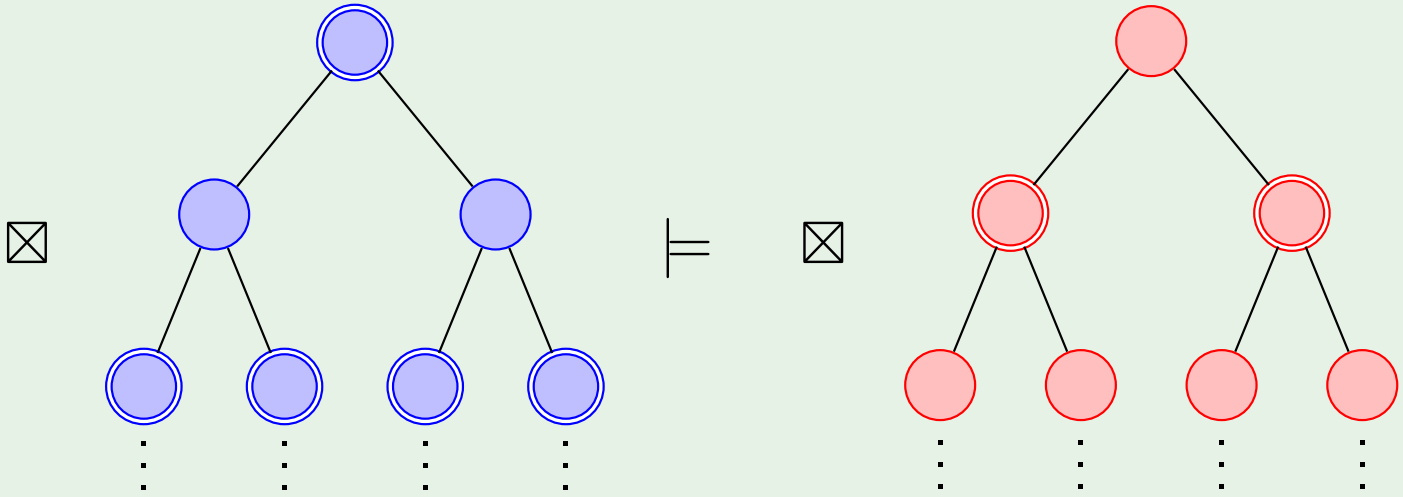

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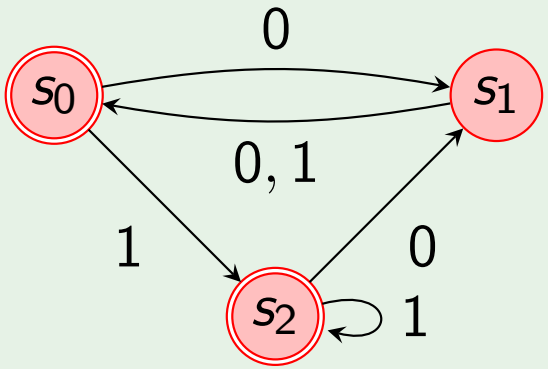

Logic for accessible functors

Accessible functor H

- quotient of some polynomial functor H_Σ
  - via surjective natural transformation $\epsilon_S : H_\Sigma S \twoheadrightarrow HS$
- cofree coalgebra Q quotient of Q_Σ
  - via surjective H-homomorphism $\gamma^*_\Sigma : (Q_\Sigma, \epsilon_{Q_\Sigma} \circ \alpha_{Q_\Sigma}) \twoheadrightarrow (Q, \alpha_Q)$
- notation $[t]$ for the equivalence class of $t \in Q_\Sigma$

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Theorem (J. Adámek)

$$\{\boxtimes q_i | i \in I\} \models \boxtimes q \quad \Longleftrightarrow \quad \forall t.[t] = q \; \exists i \in I \; \exists r.[r] = q_i : r \sqsubseteq t$$

- no simple deduction rules

Outline

Labelled transition systems

Example

Finitely branching LTS with labels from A

- coalgebras of the functor $\mathcal{P}_f(A \times -)$
- logic for accessible functors applies, but:
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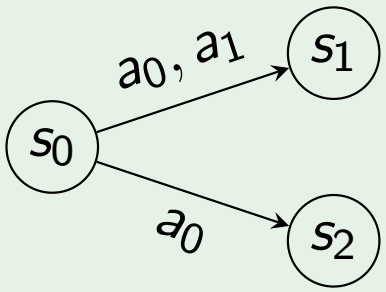

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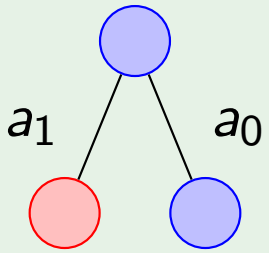

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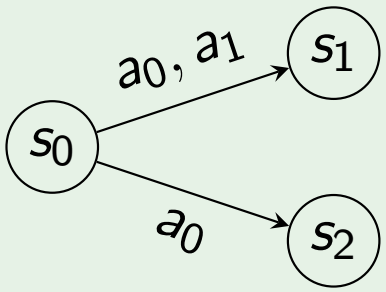

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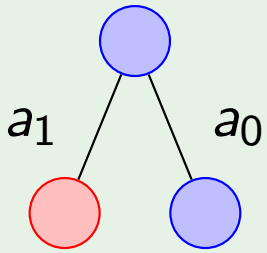

each state with only a_0-/a_1-transitions to halting states has a common a_0-/a_1-successor

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Theorem

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Finitary functors preserving preimages

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- quotient of some polynomial functor H_Σ with finitary signature Σ
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Logic as for LTS

- find a set of representative trees t^q (closed under taking subtrees)
- prove compatibility of equivalence classes with subtrees and recolourings

Equivalence classes and subtrees

Proposition

Let H preserve preimages, then $[t] = [t'] \implies \forall s \exists s' : [s] = [s']$ *(where s, s' are subtrees of t and t' respectively).*

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Proof.

It suffices to show this for child trees s, s' of t, t'.

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- pairs in $ker(\epsilon_{Q_\Sigma} \circ \alpha_{Q_\Sigma})$ and M have same child tree equivalence classes, preserved by $\cup$ and *

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Equivalence classes and recolourings

We try: *"Let H preserve preimages, then* $[t] = [t'] \Longrightarrow \forall r \exists r' : [r] = [r']$ *(where r, r' are recolourings of t and t' respectively)."*

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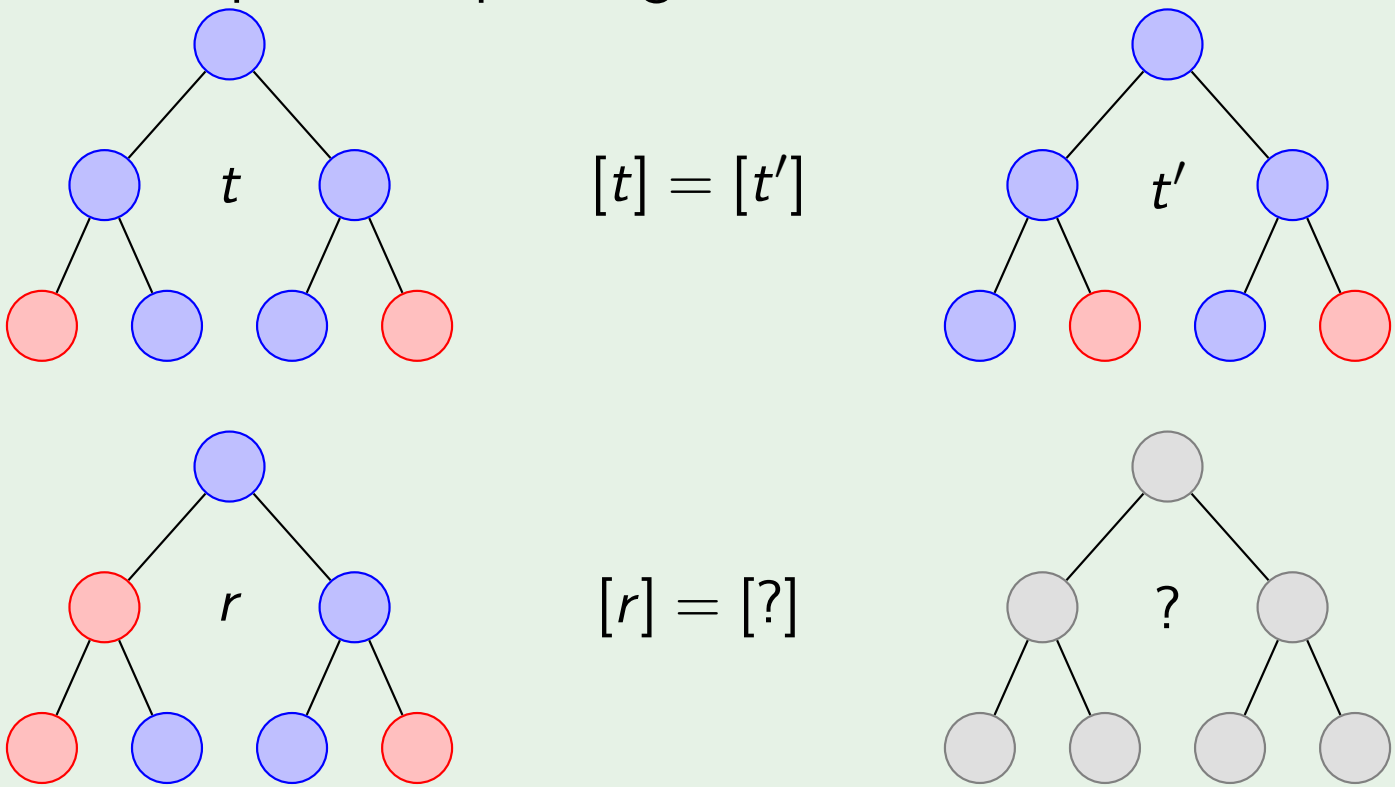
Example

Let H be the quotient of $H_\Sigma = (-)^2 + 1$ modulo the order of the binary operation, then H preserves preimages.



Representative trees

Given H as a quotient of H_Σ (Σ finitary), it holds that

- T_Σ consists of finitely branching Σ-trees
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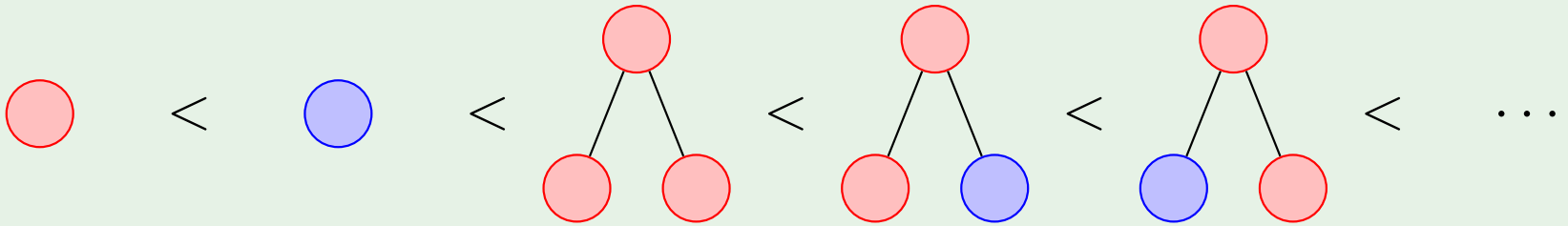
Example

Let again $H_\Sigma = (-)^2 + 1$, i. e. $\Sigma = \{c, \sigma\}$ and $C = \{b, r\}$. Order $\Sigma \times C$ by $(c, r) < (c, b) < (\sigma, r) < (\sigma, b)$. Obtain



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Equivalence classes and recolourings, continued

Proposition

Let H preserve preimages, then $[t^q] = [t'] \implies \forall r \exists r' : [r] = [r']$ *(where r, r' are recolourings of t^q and t' respectively).*

Proof.

- r induces a map $f' : \mathit{subtrees}(t^q) \to C$
- extend f' to $f : Q_\Sigma \to C$ with respect to equivalence classes
- obtain unique homomorphism $f^* : Q_\Sigma \to Q_\Sigma$ with $f = \gamma_\Sigma \circ f^*$
- obviously $r = f^*(t^q)$, choose $r' := f^*(t')$ and prove $[r] = [r']$

□

Logic for preimage preserving finitary functors

Theorem

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$\Leftarrow$ Suppose the formula for accessible functors to be true

- apply it to the representative tree t^q ($[t^q] = q$)

$\square$

Outline

Finitary functors not preserving preimages

Theorem

A finitary functor H preserves preimages if and only if for every linear order of $\Sigma \times C$ we have

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Nevertheless, there are finitary functors not preserving preimages to which the simple logic still applies in case of a special order.

Example

Let H be the quotient of $H_\Sigma = (-)^2 + 1$ ($\Sigma = \{c, \sigma\}$) modulo the equation $\sigma(x, x) = c$. Then H does not preserve preimages, but given an order of $\Sigma \times C$ where $(c, k) < (\sigma, k)$ for all $k \in C$, the simple logic applies.

Summary

- coequations as system properties
- coequational logic for
  - polynomial functors
  - accessible functors
- finitary functors preserving preimages
  - example: LTS
  - how to find a simpler logic

Summary / Literature

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Thank you...

...for your attention!

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Properties presentable by coequations

Definition

A covariety is a class of coalgebras that is closed under sums, subcoalgebras and homomorphic images.

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Covarieties are exactly the classes of coalgebras defined by coequations (where the latter means that the class contains all coalgebras satisfying the coequations).

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Corollary

The system properties coequations stand for are exactly the ones preserved by sums, subcoalgebras and homomorphic images of coalgebras.

A presentation of finitary functors

Definition

Let X be a set of variables. For two operations $\sigma, \tau \in \Sigma$ the expression

$$\sigma(x_1, \ldots, x_n) = \tau(y_1, \ldots, y_m) \qquad x_i, y_j \in X$$

is called an ϵ-equation if $\epsilon_X : H_\Sigma X \twoheadrightarrow HX$ merges the two sides.

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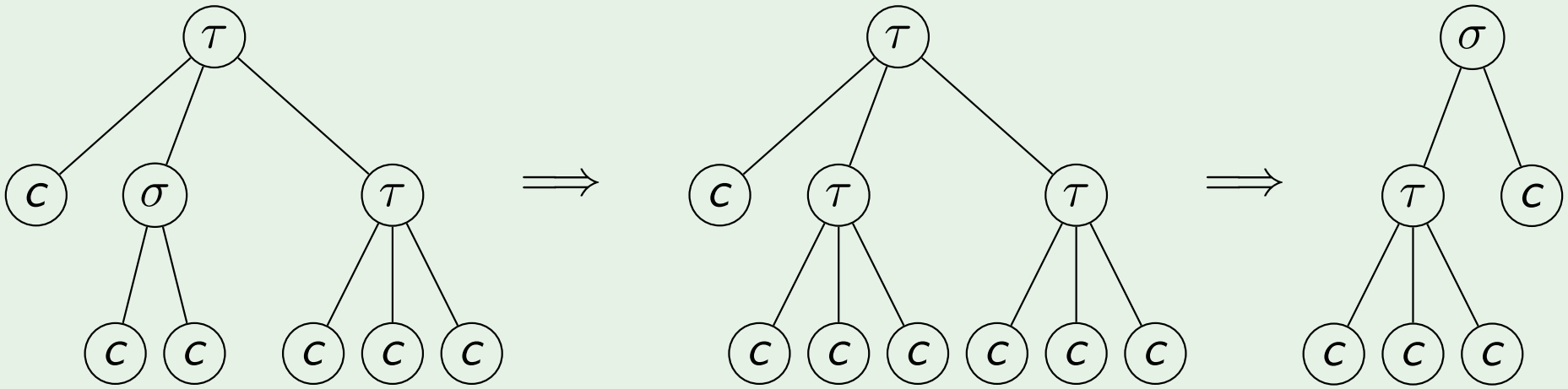
Application of an ϵ-equation (from left to right) to $t \in T_\Sigma$ at node v

- possible if
  - v is labelled by σ and
  - two of its children at positions k, l root the same tree whenever $x_k = x_l$
- it means
  - relabelling v by τ and
  - rearranging the child trees of v according to the rhs of the equation (for new variables arbitrary trees are chosen)

Application of ϵ-equations

Example

Let $\Sigma = \{c, \sigma, \tau\}$ (c constant, σ binary and τ ternary). We apply the ϵ-equation $\sigma(x, y) = \tau(y, x, x)$.



A presentation of finitary functors (continued)

Definition

We say t' can be obtained from t by (possibly infinitely many) applications of ϵ-equations if for every $k \in \mathbb{N}$ the tree $\partial_k t'$ can be obtained from $\partial_k t$ by (finitely many) applications of ϵ-equations.

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Theorem (J. Adámek, S. Milius)

The terminal H-coalgebra T is the quotient of T_Σ modulo the congruence of application of ϵ-equations.

- similar for cofree coalgebras

ϵ-equations and preimages

Definition

An ϵ-equation is called regular if the sets of variables on both sides are the same.

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An ϵ-equation is called regular if the sets of variables on both sides are the same.

Theorem (J. Adámek, D. Lücke, S. Milius)

A finitary functor H preserves preimages if and only if there is a presentation of H as a quotient of some H_Σ via regular ϵ-equations.