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On the Existence of Pure Strategy Equilibria in Games with a Continuum of Players*

M. Ali Khan

Department of Economics, The Johns Hopkins University, Baltimore, Maryland 21218

Kali P. Rath

Department of Economics, University of Notre Dame, Notre Dame, Indiana 46556

and

Yeneng Sun

Department of Mathematics, National University of Singapore, Singapore 0511

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We present results on the existence of pure strategy Nash equilibria in nonatomic games. We also show by means of counterexamples that the stringent conditions on the cardinality of action sets cannot be relaxed, and thus resolve questions which have remained open since Schmeidler's 1973 paper. *Journal of Economic Literature* Classification Number: C72. © 1997 Academic Press

1. INTRODUCTION

In 1973, Schmeidler showed the existence of Nash equilibria in nonatomic games with a finite set of pure strategies or actions.[1] In the concrete setting of a game with two actions, head (H) or tail (T), the set of players' names given by the unit interval endowed with Lebesgue measure λ, and

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[1] In his 1973 paper, Schmeidler uses the terms "pure strategies" and "activities" interchangeably with what we shall also call "actions" here. We avoid his terminology of a *T-strategy*.

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with the payoff functions $u_t(\cdot, \cdot)$ depending on actions as well as on the distribution of the players playing the two actions, Schmeidler showed[2] the existence of a random variable f from $[0, 1]$ to $\{H, T\}$ such that for almost every player t,

$$u_t(f(t), \lambda f^{-1}) \geqslant u_t(a, \lambda f^{-1}), \qquad a = H \text{ or } T,$$

where λf^{-1} is the distribution induced on the set $\{H, T\}$ by the random variable f.

A natural question arises as to the extent to which the assumption of finite action sets can be dispensed with as a hypothesis in this particular formulation of Schmeidler's theorem. This question can be easily answered. First, as shown in [21], the theorem is valid for denumerably many actions without any change in the underlying hypotheses on the game.[3] Second, Example 3 in [36] can be modified in a straightforward way to show that the theorem is false even with the simplest kind of an uncountable set of actions, say an interval in the real line. This example is robust. Thus, there is nothing more to be said in response to the question as posed above.

However, there is another interpretation of Schmeidler's theorem. This concerns the formulation of "societal responses" as averages or integrals rather than as distributions, and follows from the embedding of the actions as unit vectors in a finite-dimensional Euclidean space. An action set consisting of k candidates seeking political office, or of k routes out of a city, can be conceived as the set of extreme points of a simplex in $\mathbb{R}^k$, and the distribution induced on these k actions by a random variable can be viewed simply as the mean of a vector random variable taking values in the simplex. Schmeidler presented his result in this format, and utilized, in particular, Aumann's theory of integration of correspondences [2, 3], to prove his theorem. In this case, even though the employment of the Lebesgue integral utilizes the linear structure of the space, finiteness of actions allows the interpretation of the integral as a distribution—the proportion of the electorate voting for a particular candidate, or the proportion of traffic going through the tunnel. The two formulations are simply matters of interpretation, and their difference carries no substantive significance. However, this is clearly no longer the case when the set of actions has infinite cardinality.

[2] In this introductory statement, we avoid explicit mention of the continuity and measurability hypotheses on payoffs; the reader can see [42] and the theorems presented in the sequel.

[3] See [21, Section 6] and note the additional and obvious requirement that the countably infinite set of actions be compact.

If we stay with an interval as an action set, as in the example mentioned above, and formulate "societal responses" as an average rather than as a distribution, there does exist a Nash equilibrium. In the concrete terms of a game defined on the unit interval endowed with Lebesgue measure and with an interval as an action set A, Rath shows[4] the existence of a random variable f from $[0, 1]$ to A such that for almost every player t,

$$u_t\left(f(t), \int_{t\in T} f(t)\, d\lambda(t)\right) \geqslant u_t\left(a, \int_{t\in T} f(t)\, d\lambda(t)\right), \qquad \text{for all} \quad a \in A,$$

where $\int_{t\in T} f(t)\, d\lambda(t)$ is the Lebesgue integral of f. This is an element of the closed[5] convex hull of A, $\overline{\text{con}}(A)$, and can be looked on as a particular message or statistical measure signalling the actions of all of the players in the game. Indeed, Rath presents his result in the context of an action set which is any compact, but not necessarily convex, subset of a finite-dimensional Euclidean space, $\mathbb{R}^l$. We can now ask why is it that a Nash equilibrium exists when externalities are formulated as averages, but does not exist when they are formulated as distributions, even in the simple and pervasive setting of a compact interval in the real line? This question serves as the departing point for this paper.

The basic observation is that a distribution of a random variable can also be viewed as a particular kind of an integral of a related random variable taking values in a particular infinite-dimensional space. Thus integration and infinite-dimensionality are involved even when they are not explicitly apparent. Once we substantiate the validity of this observation, we are led very naturally beyond finite-dimensional Euclidean space, and into notions of averages that go beyond the Lebesgue integral. Furthermore, the result presented by Rath brings out the fact that once we limit the formalization of "societal responses" to means or averages, cardinality questions on the action set shift in an essential way to the dimension of the space in which that action set is situated. We can then ask whether Schmeidler's theorem is valid for a setting in which the actions sets are the unit vectors in an infinite dimensional linear space, and "societal responses" formalized as the "average" of a vector random variable taking values in such a space? Do there exist pure strategy equilibria when the set of pure strategies is infinite in the specific sense that they cannot be described as

[4] Again, in this introductory statement, we avoid explicit mention of the continuity and measurability hypotheses on payoffs; the reader can see [35, Section 3] and the theorems presented in the sequel. Note also that a variant of this result, one with convex action sets and quasi-concave payoffs, is explicitly stated and proved through a purification argument in [42].

[5] Since the result is set in the context of finite-dimensional Euclidean space, the closure requirement can be omitted. However, we retain it because of the discussion to follow.

linear combinations of a given finite number of vectors? This is to ask, in other words, whether Schmeidler's theorem extends in its original exact form based on integration[6] to action sets in an infinite-dimensional space.

This question is not as easily posed and resolved as was the case under the interpretation of "societal responses" as distributions. The reason is not only that it admits of both an affirmative and a negative answer, and that this mutual intertwining of both makes each difficult, but also because one has to face at the outset the fact that the random variable summarizing "societal responses" now has infinitely many coordinates, and the formulations depend on how the notion of an "average" of such a random variable is conceived and made precise. In rough and ready terms, our basic finding is that in an infinite-dimensional space the results are valid when the actions are denumerable, and generally invalid when they are uncountable, irrespective of how the notion of average is made precise. For the positive results, one cannot appeal directly to the theory of integration in an infinite-dimensional space,[7] but have to exploit the countable structure to construct *ab initio* a sharper theory. Furthermore, in the light of the positive results, the counterexamples have to be constructed with uncountable action sets in an infinite-dimensional space, and whose closed linear span does not have a finite dimension.[8] It is because of this difficulty that we are led to the somewhat elaborate constructions presented here. It is worth stating that the positive results do not overshadow in importance the negative message of the counterexamples, and that the two reinforce each other.

Our emphasis on integration can be motivated on at least two further grounds. The first is in the light of the application of the theory of non-atomic games to existence issues in the theory of perfect competition. Since Arrow-Debreu [13], it is well understood that a competitive (Walrasian) equilibrium can be conceived as a Nash equilibrium if "societal responses" are proxied by prices which in turn depend on aggregate excess demands, the substance of this aggregation simply being a suitable integral of individual actions.[9] It is thus no accident that the two principal notions of integration that we use below have already found application in general equilibrium theory. A second related justification concerns uncertainty and the use of expected utility; this context also invariably relies on integration.

[6] Oblique approximate answers to this question are present in the literature; see [19, 32].

[7] As is laid out in [12, Chapter 2], and extended in [18, 19, 45] to accomodate economic applications. The latter work does not invoke the countability hypothesis, and is therefore limited to approximations.

[8] With a finite-dimensional closed linear span, Theorem 2 in [35] would apply; see columns 3 and 4 of Table 1.

[9] For a general perspective on the relationship of non-cooperative game theory to general equilibrium theory, see the volume [24], and particularly [11] in it.

An overview of the paper is presented in Table 1, which also explicitly relates the results of this paper to earlier work. Note that the emphasis of this entire paper is on the existence of pure strategy Nash equilibria in large non-anonymous games with action sets of infinite cardinality, and with the externalities formulated as averages. In Table 1, our positive results are presented in column 3, the negative ones in column 4, and both are a response to the differing answers in the first two cells of column 4. Note that [36] is exclusively oriented to a setting of large anonymous games, and whereas an example of theirs can be simply recrafted to serve our purposes in Section 2, this is no longer the case regarding the counterexample presented in Section 4. The ancestry of this example can be traced to Lyapunov, and involves an analytical tool-kit which is a lot more sophisticated, and technically not a simple variant of their example. Nevertheless, it is a source of some satisfaction that the basic intuition behind it can be reduced to the argumentation that drives the examples presented in Sections 2 and 3; namely, some sort of an absence of continuity manifested in the impossibility of finding a particular kind of selection from a particular kind of correspondence. Our entire exposition has been tailored to bring out this parallelism. In terms of the positive results, we draw on the technical machinery developed in [21] and, for Section 5, that in [22]. The details of these observations are left for the individual sections in which they arise.

TABLE 1

Existence of Pure Strategy Nash Equilibria in Large Non-Anonymous Games

Externality		Cardinality of the action set: Finite	Countable	Uncountable
Distribution		Yes [42, 25, 35]	Yes [21]	No [36], section 2
Average				
Euclidean space $\mathbb{R}^l$	Lebesgue integral	Yes [42, 25, 35]	Yes [35, 42]	Yes [35, 42]
∞-dimensional space	Gel'fand integral	Yes embed in $\mathbb{R}^l$	Yes sections 5, 6	No sections 2, 3
	Bochner integral	Yes embed in $\mathbb{R}^l$	Yes section 6	No section 4

Note. Sections pertain to this paper. The action set is assumed to be compact.

The outline of the paper is as follows. We begin in Section 2 with the counterexample presented[10] in [36, Section 5]. This allows us to introduce the basic ideas and, in the subsequent Section 3, to offer a reinterpretation of an induced distribution as a suitable integral. This is important in the light of our singling out the mean or average as the basic formulation of "societal responses." In Section 4, we present our counterexample in the setting of a norm compact action set in the space of square-summable sequences, an infinite dimensional space that is the simplest in many senses. With these negative results firmly established, and any expectations of positive results for games with uncountably infinite action sets nullified, all that remains is the question of action sets with denumerably many actions. In Section 5, we present our first and basic positive result in the context of $\mathbb{R}^{\infty}$, the space of all sequences endowed with the topology of coordinatewise convergence.[11] We show that Schmeidler's theorem is valid without any change in the underlying hypotheses on the game if we slot each of the countably infinite number of actions as unit vectors in $\mathbb{R}^{\infty}$. Again, with different formalizations of the notion of an "average," but without any changes in the underlying hypothesis, this theorem can also be set in the variety of contexts available in an infinite-dimensional Banach space.[12] This is done in Section 6. All of these positive results are corollaries of the theory of integration referred to above.[13] In a concluding Section 7, we locate the importance of Schmeidler's theorem both for applications as well as for other formulations of non-cooperative games based on nonatomic measure spaces. The technical proofs and computations are collected in one Appendix, and the basic results on integration stated in another. This is done for the sake of completeness, as well as to focus attention on the more substantive points.

[10] In [36, Example 3], Rath-Sun-Yamashige show that there do not exist symmetric equilibria with uncountably many actions in the setting of large anonymous games of [25]. In particular, they do not use any notion of integration in their paper.

[11] The use of this space to study agent interaction in economic theory goes back at least to [33].

[12] By the variety of contexts, we refer to the norm, weak or weak* topologies. See [40, 41] for definitions; the ideas are basic enough that the reader may choose from several other texts. A Banach space, besides being of interest in its own right, simply offers a unifying framework for the consideration of a variety of infinite-dimensional spaces. Also see footnote 23.

[13] See [22] for a theory of integration for correspondences taking values in a countably infinite subset of a Banach space; we simply state and apply the results here. Such a theory is motivated by the questions posed here, and its basic shape relies essentially on the Bollobás–Varopolous extension of the marriage lemma.

2. A NONATOMIC GAME ON AN INTERVAL

In order to set the stage for the discussion to follow, we begin with an example of a nonatomic game in which "societal responses" are formalized as an induced distribution on the set of actions rather than as an integral.[14]

Consider a game $\mathcal{G}$ in which the set of players T is the unit interval $[0, 1]$ endowed with Lebesgue measure λ, and the action set A is the interval $[-1, 1]$. Let $\mathcal{M}([-1, 1])$ be the set of probability measures on A endowed with the weak topology.[15] Let $\mathcal{U}_A$ be the set of real valued continuous functions on $[-1, 1] \times \mathcal{M}([-1, 1])$. Since $\mathcal{M}([-1, 1])$ is compact in the weak topology,[16] $\mathcal{U}_A$ can be legitimately endowed with the sup norm topology. Since $\mathcal{U}_A$ is endowed with a topology, we can also endow it with the Borel σ-algebra $\mathcal{B}(\mathcal{U}_A)$ generated by this topology,[17] and understand measurability in terms of it. A game $\mathcal{G}$ associates a payoff function with each player, and without going into the details of this association, let the payoff function $\mathcal{G}(t)$ of any player $t \in [0, 1]$ be given by

$$\mathcal{G}(t)(a, v) = h(a, v) - |t - |a||,$$

where $h(\cdot, \cdot)\colon [-1, 1] \times \mathcal{M}([-1, 1]) \to \mathbb{R}$ is a jointly continuous function. Note that with the above specifications, $\mathcal{G}$ is a continuous function from T into $\mathcal{U}_A$, and the family $\{\mathcal{G}(t)\colon t \in T\}$ is equicontinuous.[18]

We have now a complete specification of a nonatomic game: a nonatomic set of players, a common action set, a set of "societal responses," and a payoff function for each player such that the player-payoff association is not only measurable but also continuous.[19] Indeed, more is true. The payoff functions are chosen from an equicontinuous family, furnishing a possible interpretation that there is a "bound on the diversity of payoffs."

[14] As mentioned in the introduction, this is Example 3 in [36]. We simply strip away all the terminology pertaining to symmetric equilibria of a large anonymous game.

[15] The reference to this topology as the weak topology on the set of probability measures is now established usage; see [8] or [30]. However, from functional-analytic point of view, this is really the weak* topology. We conform to convention and the reader should bear this in mind when in a subsequent section, we shall have occasion to use both topologies. Also see footnote 12.

[16] See [41, Theorem 1.10] for the construction of a Borel σ-algebra.

[17] See [8] or [30, Theorem 6.4, p. 45].

[18] Fix any a and v. Then $\mathcal{G}(t)(a, v) - \mathcal{G}(t')(a, v) = |t' - |a|| - |t - |a||$. Since $|(|t' - |a|| - |t - |a||)| \leqslant |(t' - |a|) - (t - |a|)| = |t' - t|$, we obtain $|\mathcal{G}(t)(a, v) - \mathcal{G}(t')(a, v)| \leqslant |t' - t|$. This implies that $\|\mathcal{G}(t) - \mathcal{G}(t')\| \leqslant |t' - t|$ and completes the proof of the first assertion. The second assertion follows from the elementary facts that the continuous image of a compact set is compact, and that compact subsets of the space of continuous functions over a compact metric space are equicontinuous.

[19] This simply takes advantage of the topological structure already present on the set $[0, 1]$ of players' names.

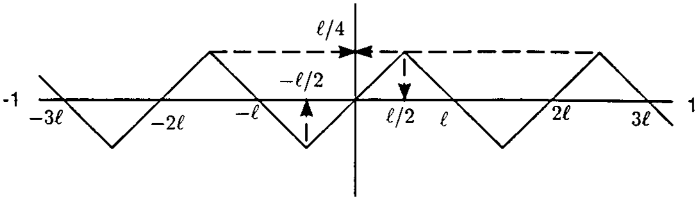

FIGURE 1

If we now assume in addition that for any $a \in [-1, 1]$, $h(a, \lambda^*) = 0$, λ^* the uniform distribution on $[-1, 1]$, we already have enough information to conclude that the uniform distribution cannot be induced by a Nash equilibrium. This is to say that the uniform distribution cannot be a summary of "societal responses" in equilibrium. If it was, then the payoff function $\mathscr{G}(t)$ of player t is given simply by $-|t - |\cdot||$, and accordingly her best response is the set $\{-t, t\}$, which leads to the best response correspondence exhibited in Figure 2. It is well known[20] that there does not exist any measurable selection from this correspondence and which induces λ^*.

The point, however, is that a further specification of $h(\cdot, \cdot)$ allows us to show that $\mathscr{G}$ does not have *any* Nash equilibrium. Towards this end, consider an uncountable family of periodic functions on $[-1, 1]$, with period $2l$, $l \in (0, 1]$, and defined as

$$g(a, l) = \begin{cases} a/2 & \text{for} \quad 0 \leqslant a \leqslant (l/2) \\ (l-a)/2 & \text{for} \quad (l/2) \leqslant a \leqslant l \\ -g(a-l, l) & \text{for} \quad l \leqslant a \leqslant 2l. \end{cases}$$

A typical representative with subsequent periodic extensions in both directions is presented in Figure 1. Note that $g(\cdot, l)$ is also an odd function, i.e., $g(a, l) = -g(-a, l)$, $a < 0$. When $l = 0$, we simply let $g(a, l) \equiv 0$. We leave it to the reader to check that $g(\cdot, \cdot)$ is jointly continuous on $[-1, 1] \times [0, 1]$. Now define

$$h: [-1, 1] \times \mathscr{M}([-1, 1]) \to \mathbb{R}, \qquad \text{where} \quad h(a, v) = g(a, \beta d(\lambda^*, v)),$$

where β is a number between 0 and 1, and $d(\lambda^*, v)$ is the Prohorov distance[21] between λ^* and v.

[20] Let f be such a measurable selection, and let $E = \{t \in [0, 1]: f(t) \in [0, 1]\}$. Then $\lambda^*(E) = \lambda(f^{-1}(E)) = \lambda(E)$. Since $\lambda^* = (1/2)\lambda$, $\lambda(E) = 0$, and hence $\lambda^*([-1, 0]) = \lambda f^{-1}([-1, 0]) = \lambda(\{t \notin E\}) = 1$, a contradiction.

[21] Recall from [8, pp. 237–238] that this metric on the space of probability measures is defined as $d(\rho, v) = \inf\{\varepsilon > 0: \rho(E) \leqslant v(B_\varepsilon(E)) + \varepsilon$ and $v(E) \leqslant \rho(B_\varepsilon(E)) + \varepsilon\}$, for all Lebesgue measurable sets E in $[-1, 1]$, and where, for any $\varepsilon > 0$, $B_\varepsilon(E) = \{x \in [-1, 1]: |x - y| < \varepsilon$, $y \in [-1, 1]\}$. Typically, it suffices to establish only one of these inequalities; see the details in Dudley referenced in [36, p. 345].

There are two elementary facts associated with a positive periodicity parameter l in the definition of g. For the first, divide the unit interval $[0, 1]$ into intervals of length l, so that we obtain $[0, 1] \cap (\bigcup_{n \in \mathbb{N}} [nl, (n+1)l])$, where $\mathbb{N}$ is the set of all non-negative integers. Consider the probability measure v on $[-1, 1]$, induced by the function which is the identity on the even intervals and the negative identity on the odd intervals. Then the support S of v oscillates between intervals of length l, moving outwards from the origin in both directions, and on the support, v is the same as the Lebesgue measure. It is now easy to check[22] that the Prohorov distance $d(\lambda^*, v)$ is bounded from above by l.

The second fact represents the characterization of the best response correspondence and is even easier to verify. Suppose v is a possible summary of "societal responses." Let $l = \beta d(\lambda^*, v)$. Then the payoff $\mathcal{G}(t)$ function of player t satisfies

$$\mathcal{G}(t)(a, v) = h(a, v) - |t - |a|| = g(a, l) - |t - |a||.$$

It can be checked that the best action in the set $[-1, 1]$ for any player t, $0 \leqslant t \leqslant l$, is given by t. First note that $g(t, l) > 0$ for such a t. For all a in the interval $[0, 1]$, we appeal to the fact that $g(\cdot, l)$ is Lipschitz continuous of modulus $(1/2)$ and conclude that

$$\mathcal{G}(t)(a, v) - \mathcal{G}(t)(t, v) = g(a, l) - g(t, l) - |t - |a|| \leqslant 0,$$

with equality only when $a = t$. For $a \in [-1, 0)$, we cannot apply the same proof since $|t - |a|| < |t - a|$. However, as noted in [36, Remark 4], one can use the oddness of g to obtain that $g(a, l) - g(t, l) = -g(|a|, l) - g(t, l)$, which is certainly less than $|t - |a||$ if $g(|a|, l) \geqslant 0$. Otherwise, $g(|a|, l) < 0$, and in this case $|a| > l$; hence, by the fact that $g(l, l) = 0$, we have

$$\begin{aligned} -g(|a|, l) - g(t, l) &= g(l, l) - g(|a|, l) - g(t, l) \\ &\leqslant (|a| - l)/2 + (l - t)/2 < -|t - |a||. \end{aligned}$$

[22] In the light of the preceeding footnote, we have only to show that for any E in $[-1, 1]$, $v(E) \leqslant \lambda^*(B_\varepsilon(E)) + \varepsilon$ for any $\varepsilon > l$. Without loss of generality, assume E to be a subset of S and which does not contain any endpoints of the subintervals. List the intervals in S as $S_1, S_2, ..., S_m$ in an increasing order, with S_1 or S_m possibly of length less than l, and let $E_i = E \cap S_i$. Then $E_i + l$ is a subset of the open subinterval with length l on right of S_i for $1 \leqslant i \leqslant m-1$ (note that S_m may not be followed by a subinterval of length l). It is clear that all the E_i, $E_i + l$ for $1 \leqslant i \leqslant m-1$ are disjoint and also their union is a subset of $B_\varepsilon(E)$. Since $\lambda(E_m) \leqslant l$ and also $\lambda^*(E_i) = \lambda^*(E_i + l) = \lambda(E_i)/2$, we obtain $v(E) = \sum_{i=1}^{m} v(E_i) = \sum_{i=1}^{m-1} \lambda(E_i) + \lambda(E_m) \leqslant \sum_{i=1}^{m-1} (\lambda^*(E_i) + \lambda^*(E_i + l)) + l$, which implies the assertion.

This argument then extends to a t in any other odd interval. As regards t in any even interval, a similar argument shows that the best action is given by $-t$.

But now the non-existence argument can be easily completed. Suppose there is an equilibrium for the game $\mathcal{G}$. Let v_0 be the distribution induced in this equilibrium. We have already seen that v_0 cannot be the uniform distribution λ^* on $[-1, 1]$. Then $d(\lambda^*, v_0)$ must be positive. By the definition of the Prohorov metric, certainly $d(\lambda^*, v_0) \leqslant 1$. However, the function $h(\cdot, \cdot)$ in the specification of payoffs is defined with respect to $\beta d(\lambda^*, v_0) = l_0$, and thus the unit interval is divided into intervals of length l_0. Now, from the second fact discussed above, the distribution induced by the best singleton responses for each player is concentrated on alternate intervals of length l_0, and this distribution must be the same as v_0. From the first fact discussed above, this implies that the Prohorov distance of v_0 from λ^* is less than or equal to $l_0 = \beta d(\lambda^*, v_0)$, a contradiction since $\beta < 1$. To summarize the argument another way, β ensures that any element v of $\mathcal{M}([-1, 1])$ leads to a best response that induces a measure whose distance from λ^* is less than the distance of v from λ^*, setting up a contraction map tending towards λ^*, but this has already been shown not to be an equilibrium. Thus the non-existence argument revolves in some way on the absence of closure.

3. A NONATOMIC GAME BASED ON ANOTHER NOTION OF AN AVERAGE

At one level, the example in the previous section shows that the reason for the non-existence of Nash equilibrium in pure strategies lies solely in the way "societal responses" are formalized. If they had been formalized as an average rather than as an induced distribution, there would be no difficulty in showing existence. In order to see this, define another game $\mathcal{G}_1$, identical to the game $\mathcal{G}$ of Section 2, but now defined in terms of the function $g(\cdot, \cdot)$ rather than $h(\cdot, \cdot)$, and with the Lebesgue integral playing the determining role. This is to say that for any random variable f: $[0, 1] \to [-1, 1]$, and for any t in $[0, 1]$,

$$\mathcal{G}_1(t)\left(a, \int_{-1}^{1} f(t)\, d\lambda(t)\right) = g\left(a, \beta \int_{-1}^{1} f(t)\, d\lambda(t)\right) - |t - |a||$$

and specifies the particular representative chosen from the uncountable family $g(\cdot, \cdot)$. It is now easy to see that there exists a Nash equilibrium of the game $\mathcal{G}_1$; it is simply any measurable selection with zero integral chosen from the correspondence $t \to \{-t, t\}$ pictured in Figure 2. Indeed, this is

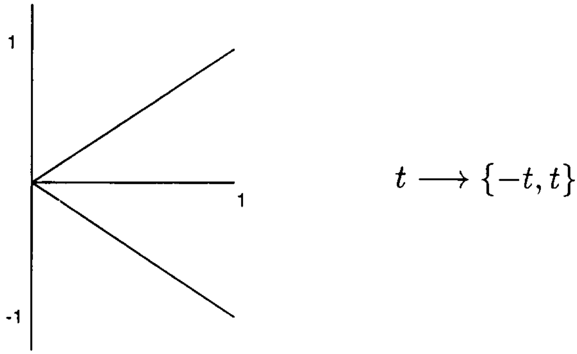

FIGURE 2

precisely what we would expect since we have a theorem on the existence of Nash equilibria in nonatomic games with *any* nonempty compact subset of $\mathbb{R}^k$ as the common action set; see [35, Theorem 2].

However, at another level, this way of looking at the example puts too much emphasis on the "average versus distribution" difference, and not enough on the fact that the induced distribution of a random variable can *also* be seen as an integral. Naturally, this is not the Lebesgue integral, but the so-called Gel'fand integral of an induced random variable into the space of probability measures. It is this aspect of our example that grounds it to the integration point of view being developed here.

The basic idea is to focus on the set of mixed strategies and to conceive of a pure strategy simply as an extreme point of this set, which is to say, as a Dirac or point-measure in the space of probability measures. Under this conception, the random variable listing players' responses is brought "upstairs" into the space of probabilities, and it is the "average" of this function, suitably conceived, that enters a particular player's payoff function. More formally, through this embedding of the action set into the space of probability measures on the set, one acquires a linear structure even in a setting where it is not present to begin with. If the number of actions is a finite integer k, as in [42], the Dirac or point-measure on each of the actions represent the set of extreme points of the set of probability measures on a finite set, and hence again reduce to the extreme points of a simplex in $\mathbb{R}^k$. The notion of an "average" is thus clear in this case. However, if the set of actions consists of an uncountable number, and we have embedded them, as we must, in an infinite-dimensional space, we are dealing with a uncountable compact set of probability measures, and the notion of an "average response" has to formalized as a suitable integral of a function taking probabilities as its values.

For concreteness, consider again the action set $[-1, 1]$ and $([0, 1], \lambda)$ as the set of players' names. The set of probability measures $\mathscr{M}([-1, 1])$ on

$[-1, 1]$ is a set in the space of signed measures on $[-1, 1]$, a space which is "mathematically identical" to the space of real-valued bounded linear functionals on the space of real-valued continuous functions $C([-1, 1])$ on $[-1, 1]$ endowed with the sup norm.[23] But now the notions of measurability and integrability of a function from $[0, 1]$ to $\mathscr{M}([-1, 1])$ have to be spelt out.

Before we do this, however, it is important to point out why traditional notions of Bochner measurability and integrability[24] are no longer adequate for our purpose. First, as is clear from the discussion of our counterexample in Section 2, we are working with the weak topology[25] on the space of probability measures, and this topology is not generated by a norm unless the set of actions is finite and the topology reduces to one generated by the Euclidean norm. Second, even if we worked with the variation norm, the space of probabilities is not separable, unless the actions are countable, and we cannot therefore approximate a measurable function by a simple function. Thus we have look elsewhere for the notion of a "societal average."

The notion of Gel'fand measurability and integrability exploits the fact already mentioned that any probability measure μ over $[-1, 1]$ represents a continuous linear functional, and its value on the domain $C([-1, 1])$ is given by

$$(\mu, h) = \int_{-1}^{1} h(a)\, d\mu(a), \qquad h \in C([-1, 1]).$$

A function f from $[0, 1]$ to $\mathscr{M}([-1, 1])$ is said to be *Gel'fand measurable* if for each $h \in C([-1, 1])$, $(f(t), h) = \int_{-1}^{1} h(a)\, f(t)(da)$ is measurable with respect to the Borel σ-algebra in $[0, 1]$. f is said to be *Gel'fand integrable* if there exists a unique element x^* in $\mathscr{M}([-1, 1])$ with the property that for each $h \in C([-1, 1])$, $(f(t), h)$ is integrable over $[0, 1]$ and such that

$$\begin{aligned}(x^*, h) &= \int_0^1 (f(t), h)\, d\lambda(t)\\ &= \int_0^1 \left(\int_{-1}^{1} h(a)\, f(t)(da) \right) d\lambda(t) \qquad \text{for all} \quad h \in C([-1, 1]).\end{aligned}$$

[23] In technical terms, we are simply referring to the Reisz representation theorem whereby the space of signed measures on a compact Hausdorff space is the norm dual of the Banach space of continuous functions on that space; see [41, Chapter 6]. This chapter also has a good reference to *signed measures* and to the *variation norm*, a concept we refer to below. Note also how the concept of a Banach space has naturally been brought into the discussion simply by introducing randomization.

[24] As exposited in [12, Chapter 2].

[25] Footnote 15 on the terminological confusion between weak and weak* topologies is also relevant here.

x^* is said to be the Gel'fand integral[26] of the function f and also denoted by $\int_0^1 f(t)\, d\lambda(t)$. It bears emphasis that, by definition, $f(t)$ is a probability measure for all players $t \in [0, 1]$, and hence the notation $f(t)(da)$ could alternatively be expressed as $df(t)(a)$ in analogy with $d\lambda(t)$.

This technical discussion can be simply summarized. Randomization on an action set A involves a function from the set of players' names to the set of probabilities on A, and such a function can be reduced to a real-valued function by integrating a particular continuous function on A with respect to its values. For different continuous functions we obtain different real-valued functions, but we say that the original function is Gel'fand measurable if *all* of these real-valued functions are measurable in the conventional sense. Furthermore, we say that the original function is Gel'fand integrable if there exists a probability measure x^* on A such that the integral of a continuous function h on A with respect to x^* is identical to the integral of the real-valued function that is obtained by reduction with respect to h, and that this identity holds irrespective of the choice of the function h. The question to ask, of course, is the relevance of these definitions to the game-theoretic phenomena being modelled. The answer, not altogether unsurprising, is that the induced distribution of a random variable and the Gel'fand integral of the corresponding probability-valued function are identical! The following lemma makes this precise.[27]

LEMMA 1. *Let $(T, \mathscr{T}, \lambda)$ be a probability space, and for any measurable function $f: T \to [-1, 1]$, let $f^m: T \to \mathscr{M}([-1, 1])$ be defined by $f^m(t) = \delta_{f(t)}$ for all $t \in T$, where $\delta_{f(t)}$ denotes the Dirac measure at $f(t)$. Then*

$$\lambda f^{-1} = \int_T f^m(t)\, d\lambda(t),$$

which is to say that the induced distribution of f and the Gel'fand integral of f^m are identical.

An alternative, and possibly more conventional, perspective on the Gel'fand integral is also worth stating. A function from the set of players' names to the set of probabilities on A can be reduced to a real-valued function by evaluating each probability on a particular Borel subset B of A, and the Lebesgue integral of this function can be used to associate a real

[26] See [12, p. 53] for details. The use of Gel'fand integration in economic theory goes back at least to Bewley's 1973 formulation of the identity of the set of core and competitive allocations in [7]; also see [18, 19, 32] for application to the question of the existence of approximate Nash equilibria in nonatomic games, and [29] and their references for application to the existence of competitive equilibrium.

[27] The proof of Lemma 1 is deferred to Appendix 1 below.

number with B. For each B in the Borel σ-algebra on A, this association furnishes a probability measure, and more importantly, the Gel'fand integral of the original function.[28] As such, this chracterization is a further complement to that spelt out by Lemma 1.

We now obtain an alternative perspective on the counterexample of Section 2: the lack of existence of Nash equilibria in pure strategies has to do with the fact that "societal responses" are not being formalized as a Lebesgue integral but rather as a Gel'fand integral, taking us from an uncountable action set in a finite-dimensional space to one in an infinite-dimensional one. This observation can be precisely pinned down by exhibiting how the game $\mathscr{G}$ can be parlayed into another one in which the dependence of players' payoffs on induced distributions is replaced by their dependence on Gel'fand integrals, and by asking whether there are any Nash equilibria in such a redefined game. Towards this end, let $A^m = \{\delta_a \in \mathscr{M}([-1, 1]) : a \in A \equiv [-1, 1]\}$, and $\mathscr{U}_A^m$ the set of real valued continuous functions $A^m \times \mathscr{M}([-1, 1])$, and endowed with the sup norm topology. Note that $\mathscr{M}([-1, 1]) = \overline{\mathrm{con}}(A^m)$ which in turn equals $\{\int_0^1 f(t)\, d\lambda(t) : f$ a Gel'fand integrable mapping from $[0, 1]$ to $A^m\}$, the integral to be interpreted as the Gel'fand integral,[29] and where the second equality follows from the simple fact that every probability measure on $[-1, 1]$ is induced by some measurable function from $[0, 1]$ to $[-1, 1]$. Let $\psi : A \to A^m$ such that for any $a \in A$, $\psi(a) = \delta_a$. It is easily checked that ψ is a bijective continuous mapping from A to A^m, and since both of these sets are compact Hausdorff, that it is a homeomorphism. Next, define a function H from $\mathscr{U}_A$ to $\mathscr{U}_A^m$ such that for any $u \in \mathscr{U}_A$,

$$H(u)(x, y) = u(\psi^{-1}(x), y) \qquad \text{for all} \quad x \in A^m, \text{ and for all } y \in \mathscr{M}([-1, 1]).$$

It is easy to show that H is a continuous map.

We are now in a position to specify our game $\mathscr{G}^m$ to be $H \circ \mathscr{G}$, where $\mathscr{G}$ is the game presented in Section 2. Since H is one-to-one, $\mathscr{G}^m$ is still a non-atomic game, and any function $f : [0, 1] \to A^m$ is its Nash equilibrium if the Lebesgue measure of the set

$$\left\{ t \in [0, 1] : \mathscr{G}^m(t) \left(f(t), \int_0^1 f(t)\, d\lambda(t) \right) \right.$$
$$\left. \geqslant \mathscr{G}^m(t) \left(x, \int_0^1 f(t)\, d\lambda(t) \right) \text{for all } x \in A^m \right\}$$

is one. We can now assert that the game $\mathscr{G}^m$ has no Nash equilibrium.

[28] The validity of this procedure and the assertion to which it leads is established in [44, especially Appendix A2].

[29] All integral signs will refer to the Gel'fand integral till the end of this section.

Suppose f is a Nash equilibrium of $\mathscr{G}^m$. Then for almost all $t \in T$,

$$H(\mathscr{G}(t))\left(f(t), \int_0^1 f(t)\, d\lambda(t)\right)$$

$$\geqslant H(\mathscr{G}(t))\left(x, \int_0^1 f(t)\, d\lambda(t)\right) \qquad \text{for all} \quad x \in A^m.$$

From the definition of H, it follows that for almost all $t \in T$,

$$\mathscr{G}(t)\left(\psi^{-1}(f(t)), \int_0^1 f(t)\, d\lambda(t)\right)$$

$$\geqslant \mathscr{G}(t)\left(\psi^{-1}(x), \int_0^1 f(t)\, d\lambda(t)\right) \qquad \text{for all} \quad x \in A^m.$$

We now appeal to Lemma 1, and to the fact that ψ is a homeomorphism, to obtain for almost all $t \in T$,

$$\mathscr{G}(t)(z(t), \lambda z^{-1}) \geqslant \mathscr{G}(t)(a, \lambda z^{-1}) \qquad \text{for all} \quad a \in A,$$

where $z = \psi^{-1} \circ f$ and $\lambda z^{-1} = \int_0^1 \psi \circ z(t)\, d\lambda(t) = \int_0^1 f(t)\, d\lambda(t)$. This contradicts the nonexistence of a Nash equilibrium in $\mathscr{G}$, and completes the argument.

Next, we present two additional results[30] on the robustness of our basic counterexample: they show how the nonexistence of Nash equilibria in specific nonatomic games implies their lack of existence in more general games. Let $(T, \mathscr{T}, \lambda)$ denote a measure space of players' names but one no longer assumed to be atomless. Let A be a compact metric space, and $\mathscr{M}(A)$, A^m, $\mathscr{U}_A$ and $\mathscr{U}_A^m$ be as defined above. Let Meas$(T; X)$ denote the set of $\mathscr{T} - \mathscr{B}(X)$ measurable functions from T to a topological space X. Then once the space X is specified, a game is simply an element of Meas$(T; X)$. Note, however, that a game in Meas$(T; \mathscr{U}_A)$ is defined in terms of induced distributions, whereas that in Meas$(T; \mathscr{U}_A^m)$ is defined in terms of Gel'fand integrals. We can now present

Proposition 1. *If there exists a game in* Meas$(T; \mathscr{U}_A)$ *whose set of Nash equilibria are empty, then there exists a game in* Meas$(T; \mathscr{U}_A^m)$ *whose set of Nash equilibria are also empty.*

Proof. We leave it to the reader to check that Lemma 1 and the mapping H above can be modified to cover the general case asserted in the proposition. ∎

[30] The basic formulation and proof of the second of these is due to [36] and it is presented here for the sake of completeness.

Our next result allows a further generalization of the claim of non-existence of Nash equilibria in games based on the interval $[-1, 1]$ to games based on any uncountable compact metric space A when the space $(T, \mathscr{T}, \lambda)$ of players' names is the unit Lebesgue interval.

PROPOSITION 2. *If there exists a game $\mathscr{G}$ in* $\mathrm{Meas}(T; \mathscr{U}_{[-1,1]})$ *whose set of Nash equilibria are empty, then there exists a game $\mathscr{G}'$ in* $\mathrm{Meas}(T; \mathscr{U}_A)$ *whose set of Nash equilibria are also empty.*

Proof. In [36, Section 6], Rath–Sun–Yamashige show that there exists a continuous surjective mapping F from A to $[-1, 1]$, and that the mapping G_1 from $\mathscr{U}_{[-1,1]}$ to $\mathscr{U}_A$ defined by $G_1(u)(x, y) = u(F(x), y \circ F^{-1})$ is injective. We can now define a new game in $\mathrm{Meas}(T; \mathscr{U}_A)$ by $\mathscr{G}' = G_1 \circ \mathscr{G}$. Suppose f is an equilibrium for the new game $\mathscr{G}'$. Then it can be checked that $F \circ f$ is an equilibrium for the original game $\mathscr{G}$. Hence the set of Nash equilibria of $\mathscr{G}'$ is empty. ▮

We conclude this section with the observation that with a compact metric space of actions A, the existence of a mixed strategy equilibrium can be translated into a pure strategy equilibrium of a modified game based on $\mathscr{M}(A)$ as the action set and with payoff functions which are linear in these actions. However, Proposition 2 shows the non-existence of Nash equilibrium if the linearity assumption is relaxed to continuity, barring the trivial case of A a singleton set.

4. A NONATOMIC GAME IN HILBERT SPACE l_2

In all of the examples that we have presented so far, "societal responses" are formalized, in one way or another, as Gel'fand integrals, and hence revolve around Gel'fand measurability. A natural question then arises as to the extent to which these examples draw essentially on the fact that the underlying random variable cannot be conceived as the pointwise limit of simple functions in the norm topology. To put the matter another way, can one construct counterexamples when the set of actions are uncountably infinite, but "societal responses" are measurable in the stronger sense that their average is a Bochner integral. We present such an example here; it takes off from an example due to Lyapunov for showing that the range of a vector measure can fail to be convex,[31] but the reader will also find affinity with the ideas behind the example in Section 2.

Our example is set in the space l_2 of real square-summable sequences. This is a Hilbert space with the norm $\|a\|$ of any element $a = (a_1, ..., a_n, ...)$

[31] See [12, Example 2, p. 262].

given by $\sum_{n=1}^{\infty} a_n^2$ and generated by an inner product in the standard way. The action set of the game that we construct is a norm compact set in l_2 such that it contains the range of a function ψ from $[0, 1]$ to l_2. It is this function that will chart out the uncountably many actions of the game, and its description is the first step in the specification of our game. Towards this end, let $\{W_n(t)\}_{n=1}^{\infty}$ be the Walsh system of real valued functions on l_2, which is to say

$$W_1(t) \equiv -1, \qquad W_n(t) = (-1)^{[2^{n-1}t]} \qquad \text{for } n \geqslant 2, \ [x] \text{ the integer part of } x.$$

It is well known[32] that $\{W_n\}_{n=1}^{\infty}$ is a complete orthogonal basis of $L_2([0, 1], \lambda)$. Now for each $t \in [0, 1]$, and for each $n \geqslant 1$, let

$$\psi_n(t) = \frac{1 - W_n(t)}{2^n} \qquad \text{and} \qquad \psi(t) = (\psi_1(t), ..., \psi_n(t), ...).$$

In order to get intuition into the mapping ψ, focus on the sequence $i = \{1/2^n\}_{n \in \mathbb{N}}$. It is square-summable and hence an element of l_2. Now consider tracking down any player $t \in [0, 1)$ in terms of a binary sequence, where the nth digit is zero or unity depending on whether t is in the left or right interval obtained in the nth subdivision of $t \in [0, 1)$, each interval constructed to be open from the right. Thus the player 0 is tracked by a sequence of zeros, the player (1/2) by 1 followed by a sequence of zeroes, and so on. Corresponding to any player t, consider a sequence $i(t)$ obtained by modifying the sequence i so that there are zero elements "sprinkled" in all of those places where there is a zero digit in the binary expansion of t described above. $\psi(t)$ is simply the sequence $\{1, i(t)\}$; it is clear that it is also square-summable. We thus have our mapping ψ.

It is easily checked that

$$\int_0^1 \psi_n(t)\, d\lambda(t) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 1/2^n & \text{if} \quad n \geqslant 2 \end{cases}$$

and that ψ is a Bochner integrable[33] function in l_2. Also note that the range of ψ is contained in the norm compact and convex set $\{(a_1, ..., a_n, ...): |a_n| \leqslant 1/2^{n-1} \text{ for } n \geqslant 1\}$. Finally, let

$$e = \int_0^1 \psi(t)\, d\lambda = (1, \tfrac{1}{4}, \tfrac{1}{8}, ...).$$

[32] This fact is not elementary, however; see, for example, [37, pp. 129–131], and also [40] and [41, Chapters 3 and 4].

[33] Hence it is also Gel'fand integrable. Indeed, it is this property that allows a proof of the fact that the Bochner integral is simply the integral taken coordinatewise; see [12, Chapter 2] for details. For the use of Bochner integration in economic theory, see [18, 19, 23, 39].

Now consider the correspondence $t \to \{0, \psi(t)\}$, with the integral of this correspondence $\int_0^1 \{0, \psi(t)\}\, d\lambda(t)$ defined in the standard way.[34] We can assert[35]

LEMMA 2. $(e/2) \notin \int_0^1 \{0, \psi(t)\}\, d\lambda(t)$.

We are now ready to specify our nonatomic game $\mathcal{G}_{\mathcal{B}}$. Let the set of players be the unit interval endowed with Lebesgue measure. Let A be any norm compact set in l_2 which contains the origin and the range of ψ. Since the latter is contained in a norm compact set, such a set can certainly be found. A will be the common action set of all the players in our game. Let $M = \max\{\|x\|: x \in A\}$, and note that since $\psi_1(t) \equiv 1$, $\|\psi(t)\| \geqslant 1$, and that this implies that $M \geqslant 1$. Note also that $\overline{\mathrm{con}}(A)$ is also norm compact;[36] it serves as the space of "societal responses." Finally, note that for any $b \in \overline{\mathrm{con}}(A)$, $\|b\| \leqslant M$ and that $\|e\| \leqslant M$. These observations imply $\|b - (e/2)\| \leqslant 2M$. Denote the square of the distance, in l_2-norm, between any two elements of l_2 by $d(\cdot, \cdot)$. Hence,

$$\beta d(b, (e/2)) \equiv \frac{1}{4M^2} \|b - (e/2)\|^2 \leqslant 1, \qquad \text{where} \quad \beta \equiv \frac{1}{4M^2}.$$

Certainly, for any $b \in \overline{\mathrm{con}}(A)$, $0 \leqslant \beta d(b, e/2) \leqslant 1$. Note that since $M \geqslant 1$, we have $\beta = (1/4M^2) \leqslant \frac{1}{4}$.

Next, we turn to individual payoffs. Let $C(A \times \overline{\mathrm{con}}(A))$ be the space of continuous functions on $A \times \overline{\mathrm{con}}(A)$, and endowed with the sup norm topology and its corresponding Borel σ-algebra. A game $\mathcal{G}_B$ is a measurable mapping from the set of players' names to the set of payoff functions such that for any player $t \in [0, 1]$,

$$\mathcal{G}_B(t)(a, b) = -h(t, a, \psi(t), \beta d(b, e/2)) - \|a\| \cdot \|a - \psi(t)\| \equiv u_t(a, b),$$

where $a \in A$, $b \in \overline{\mathrm{con}}(A)$, and h: $[0, 1] \times l_2 \times l_2 \times \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$h(t, x, y, \alpha) = \begin{cases} \alpha \left| \sin \dfrac{t}{\alpha} \pi \right| \cdot (\|x\| + 1 - (-1)^{[t/\alpha]}) \cdot (\|x - y\| + 1 + (-1)^{[t/\alpha]}) \\ \qquad \text{if} \quad \alpha > 0 \\ 0 \qquad \text{if} \quad \alpha = 0. \end{cases}$$

[34] As in [2]; also as in the proof of Theorem 2 in [42]. Of course, here the integral has infinitely-many coordinates.

[35] The proof of Lemma 2 is relegated to Appendix 1.

[36] This is Mazur's theorem; see [12, Theorem 12, p. 51].

All that remains to be shown is that $\mathcal{G}_{\mathcal{B}}$ is a game. This is to say that $\mathcal{G}_{\mathcal{B}}$ is measurable, and this is a consequence of the result asserted below.[37]

LEMMA 3. *ψ is a measurable function and h is a continuous function.*

The measurability of ψ and continuity of h guaranteed by the above lemma, ensures that $\mathcal{G}_{\mathcal{B}}$ is a measurable mapping from $([0,1], \lambda)$ to $C(A \times \overline{\text{con}}(A))$, where the latter is equipped with its Borel σ-algebra. Note, however, the rationale for the inclusion of the term $\alpha\,|\sin(t/\alpha)\pi|$ in the correction term h. Note that without this term,

$$(\|x\| + 1 - (-1)^{[t/\alpha]}) \cdot (\|x - y\| + 1 + (-1)^{[t/\alpha]})$$

is not continuous in α and t because of the truncation of t/α to its integer part $[t/\alpha]$. The term $\alpha\,|\sin(t/\alpha)\pi|$ is added to ensure that the function is continuous for all $\alpha \geqslant 0$. For the case that t/α is not an integer, $\alpha\,|\sin(t/\alpha)\pi| > 0$, and the maximizing values of the original function are also preserved.

In comparison with the game $\mathcal{G}$ presented in Section 2, the game $\mathcal{G}_{\mathcal{B}}$ is technically more intricate; in particular, unlike $\mathcal{G}$ the payoff functions of the game $\mathcal{G}_{\mathcal{B}}$ explicitly depend on the parameter β pertaining to the action set A. Nevertheless, it is easiest to understand the structure of $\mathcal{G}_{\mathcal{B}}$ in terms of $\mathcal{G}$. The parameters β, d, h, as well as l_0 to follow, play analogous roles. In $\mathcal{G}$, the uniform distribution served as an important benchmark in the argument, and its place is taken by the point $(e/2)$ here. If equilibrium "societal responses" are equal to $(e/2)$, the resulting best responses constitute a correspondence that has no selection whose Bochner integral is equal to it. On the other hand, if equilibrium "societal responses" do not equal $(e/2)$, the resulting best responses constitute a function whose Bochner integral is different from the value on the basis of which they were generated. It is this argument that is made precise below.

Suppose there is a Nash equilibrium f of $\mathcal{G}_{\mathcal{B}}$. If $\int_0^1 f(t)\, d\lambda(t) = e/2$, the payoff function reduces to

$$u_t(a, (e/2)) = -\|a\| \cdot \|a - \psi(t)\|,$$

and each player's optimal response is the doubleton set $\{0, \psi(t)\}$, and hence f is a measurable selection of the correspondence $t \to \{0, \psi(t)\}$. But Lemma 2 ensures that there is no measurable selection from this correspondence whose Bochner integral is $(e/2)$.

[37] The proof of Lemma 3 is relegated to Appendix 1.

We are thus led to the interesting case when the integral $\int_0^1 f(t)\,d\lambda(t)$ of the equilibrium "societal response" is no longer given by $(e/2)$, and hence the fourth argument α in the correction term h is no longer zero. Let $l_0 = \beta d(\int_0^1 f(t)\,d\lambda(t), e/2)$. Certainly, $0 < l_0 \leqslant 1$. Now, as in the argument in Section 2, there are two facts associated with the number l_0, and each depends on the division of the unit interval into intervals of length l_0, so that we again obtain $[0, 1] \cap (\bigcup_{n \in \mathbb{N}} [nl_0, (n+1) l_0])$.

The first fact at issue represents the characterization of the best response correspondence. Focus on any player $t \in (nl_0, (n+1) l_0)$. If n is even, then $[t/l_0]$ is even, and the correction term reduces to

$$h(t, x, \psi(t), l_0) = l_0 \left| \sin \frac{t}{l_0} \pi \right| \cdot (\|x\|) \cdot (\|x - \psi(t)\| + 2).$$

This is zero only when $x = 0$, which is to say that for $x \neq 0$, $h(t, x, \psi(t), l_0) > 0 = h(t, 0, \psi(t), l_0)$. Similarly, for n odd, $[t/l_0]$ is odd, and the correction term reduces to

$$h(t, x, \psi(t), l_0) = l_0 \left| \sin \frac{t}{l_0} \pi \right| \cdot (\|x\| + 2) \cdot (\|x - \psi(t)\|).$$

Again, this is zero only when $x = \psi(t)$, which is to say that for $x \neq \psi(t)$, $h(t, x, \psi(t), l_0) > 0 = h(t, \psi(t), \psi(t), l_0)$. With the values of the correction term specified, we can turn to u_t and observe that for almost all $t \in [0, 1]$, the optimal responses are singletons, and that they oscillate between 0 and $\psi(t)$ on odd and even intervals respectively. They can thus be written in the form

$$f(t) = \frac{1 - (-1)^{[t/l_0]}}{2} \psi(t).$$

The second fact concerns the distance, in l_2 norm, between the integral of f as characterized above and $e/2$. This is no longer an elementary computation and hinges on the following inequality.[38]

Lemma 4. *For any α such that $0 < \alpha \leqslant 1$,*

$$\sum_{n=1}^{\infty} \left(\tfrac{1}{2} \int_0^1 (-1)^{[t/\alpha]} \psi_n(t)\, d\lambda(t) \right)^2 < \alpha.$$

[38] The proof of Lemma 4 is relegated to Appendix 1.

Now by virtue of the inequality in Lemma 4, we obtain

$$\begin{aligned} d\left(\int_0^1 f(t)\, d\lambda(t), \frac{e}{2}\right) &= \left\| \int_0^1 f(t)\, d\lambda(t) - \frac{e}{2} \right\|^2 \\ &= \left\| \int_0^1 f(t)\, d\lambda(t) - \frac{1}{2}\int_0^1 \psi(t)\, d\lambda(t) \right\|^2 \\ &= \left\| \frac{1}{2}\int_0^1 (-1)^{[t/l_0]}\, \psi(t)\, d\lambda(t) \right\|^2 \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\int_0^1 (-1)^{[t/l_0]}\, \psi_n(t)\, d\lambda(t) \right)^2 < l_0. \end{aligned}$$

Hence $d(\int_0^1 f(t)\, d\lambda(t), e/2) < \beta d(\int_0^1 f(t)\, d\lambda(t), e/2)$ by the definition of l_0. This implies $\beta > 1$, a contradiction since we have shown $\beta \leqslant 1/4$. Therefore f cannot be an equilibrium for the game $\mathcal{G}_{\mathcal{B}}$. To summarize the argument another way, β ensures that any element g from l_2 formalizing a possible "societal responses" leads to a best response function whose Bochner integral has a distance from $(e/2)$ that is less than the distance of the integral of f from $(e/2)$, setting up a contraction map tending towards $(e/2)$, but this has already been shown not to be an equilibrium. Thus, here again, the non-existence argument revolves in some way on the absence of closure.

We conclude with an observation on the robustness of our counterexample. Given the isomorphism between l_2 and any separable infinite dimensional L_2, our counterexample extends to the latter space. This is useful in the light of the use of L_2 in models with information and uncertainty. It is also not difficult to see that the same counterexample can be modified to cover the case when the target space is l_p for some $p \geqslant 1$. Furthermore, by appealing to [12, Corollary 6, p. 265], one should hopefully be able to set a version of the counterexample in any arbitrary infinite dimensional Banach space.

5. NONATOMIC GAMES IN $\mathbb{R}^\infty$

In the light of the counterexamples presented above, the obvious question is raised as to whether there are any positive results on the existence of pure strategy Nash equilibria in nonatomic games beyond the case of finite action sets in infinite-dimensional spaces. We turn to this question.

Our point of departure in this section is to follow Schmeidler as closely as we can, and consider a setting which is identical to his except for the fact that the set of pure strategies is assumed to be countably infinite. If we follow Schmeidler's procedure of conceiving each pure strategy as a unit vector in a suitable space, we are naturally led to the space of infinite sequences. However, this requires our having to make explicit the topology in which the action set is assumed to be compact.[39] We work in $\mathbb{R}^\infty$, the space of real sequences equipped with the product topology, also referred to as the topology of coordinate-wise convergence.[40] Let $\{e_n\}_{n\geqslant 1}$ be the standard basis vectors in $\mathbb{R}^\infty$, $e_0=0$, $A=\{e_n\}_{n\in\mathbb{N}}$, and $\overline{\mathrm{con}}(\{e_n\}_{n\in\mathbb{N}})=\overline{\mathrm{con}}(A)$ the closed convex hull of the set A; the set of all infinite sequence of real numbers chosen from the closed unit interval. As above, we fix an atomless probability measure space $(T, \mathscr{T}, \lambda)$ as the space of players.

In order to define a nonatomic game on $\mathbb{R}^\infty$, we have to make precise the notion of "societal responses," and the attendant notion of an integral. Towards this end, for any function $f\colon T\to\mathbb{R}^\infty$; $f(t)\in\{e_n\}_{n\in\mathbb{N}}$, we shall say that f is measurable if

$$f^{-1}(\{e_n\})\in\mathscr{T} \qquad \text{for any} \quad n\in\mathbb{N}.$$

The integral of any measurable function f over T is then given by

$$\int_T f(t)\, d\lambda(t) = \sum_{n\in\mathbb{N}} e_n \lambda(f^{-1}(\{e_n\})).$$

We can define the integral of a correspondence F from T into A by

$$\int_T F(t)\, d\lambda(t) = \left\{\int_T f(t)\, d\lambda(t)\colon f \text{ is an integrable selection of } F\right\}.$$

Note that our definition of measurability and of integrability directly exploits the fact that we are working with probability measures and with functions whose range is denumerable. Thus, unlike the more general classical case, we do not resort to any approximations by simple functions, or to the reduction to real-valued functions.[41] We preserve the feature of the finite action theory whereby we can go back and forth between the interpretation of the integrals as an average or as an induced distribution on the action set. This informal observation has a technical counterpart.

[39] It is a trivial observation that a continuous function on a countably infinite space need not necessarily have a maximal element. This observation was already invoked in footnote 3.

[40] An early application of $\mathbb{R}^\infty$ as the space of commodities is available in [33].

[41] As respectively in the case of Bochner and Gelf'and integration; see [12, Chapter 2] and [41, Chapter 2]. [40, Chapter 3] also presents a theory of vector-valued integration for functions taking values in a Fréchet space, but unlike the treatment therein, we do not require that T be compact and that the function be continuous.

On appealing to the standard "change of variables" formula, we observe that the integral of the correspondence F as defined here is identical to

$$\left\{\int_{\mathbb{R}^\infty} \mathrm{id}_{\mathbb{R}^\infty}\, d(\lambda f^{-1})\colon f \text{ an integrable selection of } F\right\}.$$

Since this is simply the induced distribution of the correspondence F, we have access to the mathematical results now available for this object; see [21, Section 3].

We have the mathematical preliminaries that we need to define a nonatomic game and to develop our existence result. Let $C(A \times \overline{\mathrm{con}}(A)) \equiv \mathscr{C}_A$ be the space of continuous functions on $A \times \overline{\mathrm{con}}(A)$. Since $A \times \overline{\mathrm{con}}(A)$ is compact,[42] we can endow this space with the sup-norm topology and its attendant Borel σ-algebra. We can now present

THEOREM 1. *If $\mathscr{G}$ is a measurable map from T to $\mathscr{C}_A$, there exists a measurable function $f\colon T \to A$ such that for all $t \in T$,*

$$u_t\left(f(t), \int_T f\, d\lambda\right) \geqslant u_t\left(a, \int_T f\, d\lambda\right) \qquad \textit{for all} \quad a \in A,$$

where $u_t \equiv \mathscr{G}(t) \in \mathscr{C}_A$.

The function f represents a Nash equilibrium of the nonatomic game $\mathscr{G}$, its integral specifies the proportion of the players playing a particular pure strategy, and its measurability is deduced as a consequence of the measurability hypothesis on the game. We turn to the simple proof of the theorem.

Proof of Theorem 1. The proof of Schmeidler's theorem as presented in [35] revolves around the best-response correspondence

$$F\colon T \times \overline{\mathrm{con}}(A) \to A, \qquad \text{where} \quad F(t, x) = \mathrm{Argmax}_{a \in A}\, u_t(a, x),$$

and the integral correspondence

$$\phi\colon \overline{\mathrm{con}}(A) \to \overline{\mathrm{con}}(A), \qquad \text{where} \quad \phi(x) = \int_{t \in T} F(t, x)\, d\lambda.$$

For his finite dimensional setting, Rath applies the basic ingredients of general equilibrium theory to the map ϕ: namely, Berge's maximum theorem to guarantee the upper semicontinuity of $F(t, x)$ in terms of x; and then Aumann's result on the preservation of upper semicontinuity through

[42] This simply follows from Tychonoff theorem; see, for example, [40].

integration to obtain the upper semicontinuity of the integral $\phi(x)$ in terms of x; Aumann's measurable selection theorem to obtain its non-emptiness; the Lyapunov–Richter theorem to obtain its convexity; and finally, the Kakutani theorem to obtain the fixed point. All of these steps transpose to our in finite dimensional setting. Once we have the convexity and upper semicontinuity results stated as Theorems A and B in Appendix 2, we can appeal to the selection theorem in Castaing–Valadier [9, Theorem III.39], to the full power of Berge's theorem [5, Section VI.3], and to the Fan–Glicksberg fixed point theorem, [13, 14], to complete the argument. ∎

In conclusion, we leave it to the reader to satisfy herself that Theorem 1 extends to any countably infinite compact set A, and furthermore, by a suitable modification of the game $\mathcal{G}$ of Section 4, that there are nonatomic games in $\mathbb{R}^\infty$ with uncountable action sets and without any Nash equilibria.[43]

6. EXISTENCE RESULTS FOR GENERAL NONATOMIC GAMES

Next, we turn to a situation when the action set is already equipped with a linear structure, as well as with a metric structure induced by a norm. Indeed, it is the absence of a norm in $\mathbb{R}^\infty$ that prevents us from relying on conventional notions of integration in infinite dimensional spaces, and leads us to pursue an alternative formalization. The question therefore remains as to whether there exist Nash equilibria in pure strategies when "societal responses" are formalized as integrals in the conventional Bochner or Gel'fand sense. Furthermore, nonatomic games with action sets lying in normed linear spaces are objects that sit naturally between the Euclidean context of [35] and the Hilbert space setting of the counterexample in Section 4.

We shall work with a Banach space, a normed linear space which is also complete,[44] and here we have a choice of at least two topologies, the norm and the weak topology. The latter, in the context of a Hilbert space, allows us to consider the important case of the action set A being an arbitrary closed subset of the unit sphere $\{x \in X: \|x\| \leqslant 1\}$, a set which is not compact in the norm topology. Note that because of the counterexample of Section 4, we have to confine ourselves to the case of a countably infinite set of actions.

[43] For the second observation, simply note that the restriction to the set A, defined in Section 4, of the topology of coordinate-wise convergence coincides with the l_2 norm topology, and thus all of the arguments carry through.

[44] We need the completeness hypothesis to define all of the classical spaces of integrable functions; see [40, 41]. These texts may also be consulted for elementary introductions to the functional-analytic notions used here.

For our first theorem, let A be a countably infinite weakly compact set in a Banach space $(X, \|\cdot\|)$. Since we only work with correspondences with a countable range, we can assume X to be separable without loss of generality. Note that $\overline{\text{con}}(A)$, the closed convex hull of A is also weakly compact.[45] Let $\mathscr{U}^w_A$ be the space of weakly continous real-valued functions on $A \times \overline{\text{con}}(A)$ endowed with the sup-norm topology and the corresponding Borel σ-algebra.

THEOREM 2. *Theorem* 1 *is valid with* A *a nonempty countable weakly compact subset of a Banach space,* $\mathscr{U}^w_A$ *is substituted for* $\mathscr{C}_A$, *and the integrals interpreted as Bochner integrals.*

Proof of Theorem 2. The proof of Theorem 1 applies directly other than the fact that we have to find the relevant analogues of the convexity and upper semicontinuity theorems. Such analogues are reported as Theorems C and D in Appendix 2 below. ▮

It is interesting that Theorem 2 already covers the case of action sets which are norm compact, and payoff functions which are norm continuous. If A is a norm compact subset of X, $\overline{\text{con}}(A)$ is also norm compact.[46] If $\mathscr{U}^n_A$ denotes the space of norm continuous real-valued functions on $A \times \overline{\text{con}}(A)$ endowed with the sup-norm topology and with the corresponding Borel σ-algebra generated by this topology, we can present the following corollary to Theorem 2 in light of the fact that norm continuous functions on norm compact sets are weakly continuous.

COROLLARY 1. *Theorem* 2 *is valid with the space* $\mathscr{U}^n_A$ *substituted for* $\mathscr{U}^w_A$ *and with* A *a norm compact set.*

We remark here that one can also give a direct proof of the corollary with the usual steps; we simply need an analogue in the norm topology for the upper semicontinuity result presented as Theorem D in Appendix 2. This is reported there as Theorem E. Note also that the counterexample in Section 4 is phrased in terms of the norm topology.

We now revert to our discussion in Section 3, and place it in its proper technical context. The space of probability measures on a compact set is a subset of the dual of a Banach space. As such it can be equipped with the weak* topology, in addition to the norm and weak topologies.[47] It is precisely this weak* topology that motivated the notions of Gel'fand measurability and integrability, and one that is referred to by probabilists

[45] This is a consequence of the Krein-Smulian theorem; see [12, Theorem 11, p. 51].

[46] We have already mentioned that this is a consequence of Mazur's theorem; see [12, Theorem 12, p. 51].

[47] See [40, p. 66] for a description and properties of the weak* topology.

as the weak topology on the space of probability measures. It is then natural to ask whether Theorem 2 is valid when set in this context. We conclude this section with a positive answer to this question.

Let X^* be the dual of a separable Banach space X, and A be a countable weak* compact subset of X^*. We shall continue to use the notation $\overline{\text{con}}(A)$ to denote the weak* closed convex hull of A. It is also clear that $\overline{\text{con}}(A)$ is weak* compact. Let $\mathcal{U}_A^{w^*}$ be the space of weak* continous real-valued functions on $A \times \overline{\text{con}}(A)$ endowed with the sup-norm topology and with the corresponding Borel σ-algebra. We can now present

THEOREM 3. *Theorem 2 is valid with the space $\mathcal{U}_A^{w^*}$ substituted for $\mathcal{U}_A^{w}$, with A a weak* compact countable subset of the dual of a separable Banach space, and with the integrals all interpreted as Gel'fand integrals.*

Proof. All one needs are Theorems F and G reported in Appendix 2. ∎

There are two final points to be noted. First, Corollary 1 above already covers the case of the norm topology in X^* provided we revert to the Bochner integral. Second, Theorem 3 yields a result for the setting of an action set A^m consisting of probability measures on a compact countably infinite set A, and with $\mathcal{U}_A^m$ defined as in Section 3. Such a result is the best that can be done in light of the counterexample of Section 3.

COROLLARY 2. *Theorem 3 is valid with $\mathcal{U}_A^m$ substituted for $\mathcal{U}_A^{w^*}$, and with the integrals interpreted as Gel'fand integrals.*

7. CONCLUDING REMARKS

In terms of a summary position, the existence of pure strategy Nash equilibria in games with an atomless set of players and with compact and denumerable action sets is a robust finding, but there seems to be no hope for any theory beyond this case. The question that arises is whether the effort towards obtaining this conclusion, and the detailed treatment of pure strategy equilibria in Schmeidler's formulation of nonatomic games, is justified. In the introduction, we have already touched on the relevance of nonatomic games for general competitive analysis; we now conclude the paper with our answer to this question in the context of non-cooperative game theory more generally.

The importance of pure strategies is by now well-appreciated, with some authors arguing that "one of the reasons that game-theoretic ideas have not found more widespread application is that randomized strategies play a significant role in the theory of games but have limited appeal in many practical situations."[48] Indeed, the whole point of Schmeidler's paper was to show how the assumption of a continuum of players yields a worthwhile existence theorem concerning pure strategy equilibria.[49] In any case, even if an analyst's concern is solely with mixed strategy equilibria, it is reasonable to suppose that she would be interested in distinguishing between those that can be purified from those that cannot. We have seen how such a concern leads to an infinite-dimensional setting, even in the simplest context of an interval.

Leaving aside the question of mixed versus pure strategies, it is worth noting that Schmeidler's model occupies a central position in the theory of games based on nonatomic measure spaces. When agent interdependence is formulated in terms of induced distributions on the action set, the analytical argumentation applies almost *verbatim* to the proof of the existence of pure strategy equilibria in finite player games with diffused and independent information. One simply works with the induced distribution of a correspondence rather than its integral.[50] There is also a direct implication of Schmeidler's theorem for Mas-Colell's formulation of a large anonymous game. The existence of a pure strategy equilibrium in Schmeidler's model *automatically* implies through the identity mapping the existence of a symmetric equilibrium in an anonymous game.[51]

Moving beyond questions of analysis, the applications of Schmeidler's theorem have been many and diverse. It has been applied, for example, to financial markets, as in [10]; monopolistic competition and the economics of firm entry, as in [38, 31]; restaurant pricing and the economics of "social influences," as in [16]; and to the economics of fashion, as in [17]. Furthermore, once the consequences of the model for large anonymous games is understood, much of recent work undertaken from this point of view also becomes directly relevant. This work has involved extensions to time, as in [26, 27]; as well as to uncertainty, as in [15, 6] and their references. Indeed, [15, Section 3] sketch out an entire research program on the possible application of large games to a variety of micro- and macroeconomic contexts. It seems to us somewhat myopic to argue that all these applications, and presumably those to follow in the future, will be

[48] See the first sentence in [34] and in [4].

[49] Thus Schmeidler was explicit that the main result of his work was his Theorem 2 (see his Remark 4 in [42]), and added that "the importance of Theorem 2 lies in the fact that in many real, gamelike situations, a mixed strategy has no meaning." Indeed, a theorem on the existence of a mixed strategy equilibrium for a *finite* set of players was already available in [28].

[50] See [21, Sections 4 and 5]. For the recognition of the importance of Schmeidler's work in the original formulations of such games, see [34, especially Footnote 3].

[51] Simply put, player names are reinterpreted as player characteristics. This observation constitutes one direction of the synthesis between the two formulations presented in [20] for a finite-action setting.

adequately described in terms of a finite number of actions. As such, a detailed study of the model, as is conducted here, may well have payoffs that spill over to this substantial amount of work.

8. APPENDIX 1

Proof of Lemma 1. Pick any real-valued continuous function ϕ on $[-1, 1]$. Then the function $f^m(\cdot)(\phi) = \phi \circ f(\cdot)$ is measurable as a simple consequence of the measurability of f.

In order to prove the assertion, we again pick an arbitrary real-valued continuous function ϕ on $[-1, 1]$. By a simple application of the Riesz representation theorem, this follows from the fact that $(\lambda f^{-1})(\phi)$ equals

$$\begin{aligned}\int_{-1}^{1} \phi(x)\, d(\lambda f^{-1}) &= \int_T \phi(f(t))\, d\lambda(t)\\ &= \int_T (\delta_{f(t)}(\phi)\, d\lambda(t)\\ &= \left(\int_T \delta_{f(t)}\, d\lambda(t)\right)(\phi)\\ &= \int_T f^m(t)\, d\lambda(t)(\phi). \quad \blacksquare\end{aligned}$$

Proof of Lemma 2. If $(e/2) \in \int_0^1 \{0, \psi(t)\}\, d\lambda(t)$, then there is a measurable subset E of $[0, 1]$ such that

$$(e/2) \in \int_E \psi(t)\, d\lambda(t) = \left(\int_E \psi_1(t)\, d\lambda(t), ..., \int_E \psi_n(t)\, d\lambda(t), ...\right),$$

which implies that the function

$$r(t) = \begin{cases} 1 & \text{if} \quad t \in E \\ -1 & \text{if} \quad t \notin E \end{cases}$$

is orthogonal to all the ψ_n. This contradicts[52] the completeness of the system $\{\psi_n\}$. $\blacksquare$

[52] For a more detailed argument, see [12, Example 2, p. 262], [43, Example 1] and [45, Example 6.1].

Proof of Lemma 3. The measurability of $\psi(\cdot)$ is straightforward. We shall show that h is continuous at an arbitrary point $(t_0, x_0, y_0, \alpha_0)$.

Case 1. $\alpha_0 = 0$. In this case, $0 \leqslant h(t, x, y, \alpha) \leqslant \alpha(\|x\| + 2)(\|x - y\| + 2)$, which converges to zero as $(t, x, y, \alpha) \to (t_0, x_0, y_0, \alpha_0)$.

Case 2. $\alpha_0 > 0$, (t_0/α_0) is an integer. Here $0 \leqslant h(t, x, y, \alpha) \leqslant \alpha\,|\sin(t/\alpha)\pi| \cdot (\|x\| + 2)(\|x - y\| + 2)$, which again converges to zero as desired.

Case 3. $\alpha_0 > 0$, (t_0/α_0) is not an integer. Then $[t/\alpha]$ is continuous in a neighbourhood of (t_0, α_0), which implies the desired continuity.
Thus we have shown that h is continuous on its entire domain. ∎

Before we offer a proof of Lemma 4, we shall need two additional inequalities.

LEMMA 5. *For* $\alpha > 0$ *and any real number* b, $|\int_0^1 (-1)^{[(t+b)/\alpha]}\, d\lambda(t)| \leqslant \alpha$.

Proof. Routine!

LEMMA 6. *For* $\alpha > 0$, $n \geqslant 1$, $|\int_0^1 (-1)^{[t/\alpha]}\,(1 - (-1)^{[2^n t]})\, d\lambda(t)| \leqslant \min\{2, 2^n\alpha\}$.

Proof.

$$\left|\int_0^1 (-1)^{[t/\alpha]}\,(1 - (-1)^{[2^n t]})\, d\lambda(t)\right|$$

$$= \left|\sum_{k=0}^{2^{n-1}-1} \int_{(2k+1)/2^n}^{(2k+2)/2n} (-1)^{[t/\alpha]} \cdot 2d\lambda(t)|\right|$$

$$= \left|\sum_{k=0}^{2^{n-1}-1} \frac{1}{2n-1} \int_0^1 (-1)^{[(u+2k+1)/(2^n \cdot \alpha)]}\, du\right|$$

$$\leqslant \sum_{k=0}^{2^{n-1}-1} \frac{1}{2^{n-1}} \left|\int_0^1 (-1)^{[(u+2k+1)/(2n \cdot \alpha)]}\, du\right|$$

$$\leqslant \sum_{k=0}^{2^{n-1}-1} \frac{1}{2^{n-1}} \cdot 2^n\alpha = 2^n\alpha.$$

It is also clear that the integral is bounded by 2, and hence by $\min\{2, 2^n\alpha\}$. ∎

Proof of Lemma 4.

$$\Lambda = \left(\frac{1}{2}\int_0^1 (-1)^{[t/\alpha]}\, d\lambda(t)\right)^2 + \sum_{n=2}^{\infty}\left(\frac{1}{2}\int_0^1 (-1)^{[t/\alpha]} \cdot \frac{1-(-1)^{[2^{n-1}t]}}{2^n}\, d\lambda(t)\right)^2$$

$$\leqslant \frac{\alpha^2}{4} + \frac{1}{4}\sum_{n=1}^{\infty}\left(\int_0^1 (-1)^{[t/\alpha]}\,\frac{1-(-1)^{[2^n t]}}{2^{n+1}}\, d\lambda(t)\right)^2$$

$$\leqslant \frac{\alpha^2}{4} + \frac{1}{4}\cdot\sum_{n=1}^{\infty}\frac{1}{4^{n+1}}\cdot(\min\{2, 2^n\alpha\})^2$$

$$\leqslant \frac{\alpha^2}{4} + \sum_{n=1}^{\infty}\frac{1}{4^{n+2}}\min\{4, 4^n\alpha^2\}.$$

Let $m = [\ln(2/\alpha)/\ln 2]$. Then

$$\frac{\ln\frac{2}{\alpha}}{\ln 2} - 1 < m \leqslant \frac{\ln\frac{2}{\alpha}}{\ln 2},$$

which implies

$$2^m\alpha \leqslant 2 \qquad \text{and} \qquad 2^{m+1}\alpha > 2, \qquad \alpha > \frac{1}{2^m}.$$

Thus

$$\Lambda \leqslant \frac{\alpha^2}{4} + \sum_{n=1}^{m}\frac{1}{4^{n+2}}\cdot 4^n\alpha^2 + \sum_{n=m+1}^{\infty}\frac{1}{4^{n+2}}\cdot 4$$

$$= \frac{\alpha^2}{4} + \frac{m}{4^2}\alpha^2 + \frac{\frac{1}{4^{m+2}}}{1-\frac{1}{4}}$$

$$= \left(\frac{1}{4} + \frac{m}{16}\right)\alpha^2 + \frac{4}{3}\cdot\frac{1}{4^2}\cdot\left(\frac{1}{2^m}\right)^2$$

$$< \left(\frac{1}{4} + \frac{m}{16}\right)\alpha^2 + \frac{1}{12}\alpha^2$$

$$= \left(\frac{1}{3} + \frac{m}{16}\right)\alpha^2$$

$$\leqslant \left(\frac{1}{3} + \frac{1}{16}\,\frac{\ln(2/\alpha)}{\ln 2}\right)\alpha^2.$$

Now consider the function $a(x) = x - \frac{1}{3} - \frac{1}{16}(\ln 2x/\ln 2)$ on $\mathbb{R}$. For $x \geqslant 1$,

$$a'(x) = 1 - \frac{1}{x}\frac{1}{\ln 2^{16}} \geqslant 1 - \frac{1}{\ln 2^{16}} > 0$$

$$a(1) = 1 - \frac{1}{3} - \frac{1}{16} > 0$$

Hence

$$a(x) > 0 \Rightarrow x > \frac{1}{3} + \frac{1}{16}\frac{\ln 2x}{\ln 2}$$

Therefore, for $0 < \alpha \leqslant 1$,

$$\frac{1}{3} + \frac{1}{16}\frac{\ln \frac{2}{\alpha}}{\ln 2} < \frac{1}{\alpha},$$

which implies, for $0 < \alpha \leqslant 1$, $\Lambda < \alpha$. ∎

9. APPENDIX 2

In this Appendix, we collect recent results on the integration of correspondences with a countable range. In all of the results below, $(T, \mathscr{T}, \lambda)$ is an atomless probability space.

For our first two results, the integral is defined as in Section 5.

THEOREM A. *If F is a correspondence from T into $\mathbb{R}^\infty$ with a countable range, then the integral $\int_T F\, d\lambda(t)$ of F is convex.*

Proof. Follows from [21, Theorem 7] once we observe that the integral of the correspondence F as defined here is identical to

$$\left\{ \int_{\mathbb{R}^\infty} \mathrm{id}_{\mathbb{R}^\infty}\, d(\lambda f^{-1}) : f \text{ an integrable selection of } F \right\}. \quad ∎$$

THEOREM B. *Let Y be a metric space and F a correspondence from $T \times Y$ to a countable compact set in $\mathbb{R}^\infty$ such that for each fixed $y \in Y$, $F(\cdot, y)$ is a measurable compact valued correspondence from T to $\mathbb{R}^\infty$. If $F(t, y)$ is upper semicontinuous on Y for each fixed t, then $\int_T F(t, y)\, d\lambda(t)$ is upper semicontinuous on Y.*

Proof. Follows from [21, Theorem 9] once we make the same observation as in the proof of Theorem A. ∎

The next five results are all taken from [22]; the first three are based on the Bochner integral, and the remaining two on the Gel'fand integral.

THEOREM C. *If F is a correspondence from T to a countable set in a Banach space X, $\int_T F\, d\lambda(t)$ of F is convex.*

THEOREM D. *Let Y be a metric space and F a correspondence from $T \times Y$ to a countable weakly compact set in a Banach space X such that for each fixed $y \in Y$, $F(\cdot, y)$ is a measurable weakly compact valued correspondence from T to X. If $F(t, y)$ is norm upper semicontinuous on Y for each fixed t, then $\int_T F(t, y)\, d\lambda(t)$ is norm upper semi-continuous on Y.*

THEOREM E. *Let Y be a metric space and F a correspondence from $T \times Y$ to a countable norm compact set in a Banach space X such that for each fixed $y \in X$, $F(\cdot, y)$ is a measurable norm compact valued correspondence from T to X. If $F(t, y)$ is norm upper semicontinuous on Y for each fixed t, then $\int_T F(t, y)\, d\lambda(t)$ is norm upper semicontinuous on Y.*

THEOREM F. *If F is a correspondence from T into the dual Banach space X^*, with a countable range, $\int_T F\, d\lambda(t)$ of the correspondence F is convex.*

THEOREM G. *Let Y be a metric space and F a correspondence from $T \times Y$ to a countable weak* compact set in the dual Banach space X^* such that for each fixed $y \in Y$, $F(\cdot, y)$ is a measurable weak* compact correspondence from T to X^*. If $F(t, y)$ is weak* upper semicontinuous on Y for each fixed t, then $\int_T F(t, y)\, d\lambda(t)$ is weak* upper semicontinuous on Y.*

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