Two Spanning Disjoint Paths with Required length in Generalized Hypercubes

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Abstract

This work investigates 2RP-property of a generalized hypercube G. Given any four distinct vertices u, v, x and y in G, let l₁ and l₂ be two integers such that l₁ (l₂) is not less than the distance between u and v (x and y), and l₁+l₂ is equal to the number of vertices in G minus two. Then, there exist two vertex-disjoint paths P₁ and P₂ such that (1) P₁ is a path joining u and v with length of l₁; (2) P₂ is a path joining x and y with length of l₂, and (3) P₁ ⊙ P₂ spans G except some special conditions. This work shows that a G(m₁, m₂, ..., m_r) satisfies 2RP-property, where m_r ≥ 4 for all 1 ≤ r ≤ s.

1 Introduction

The topological structure of a multiprocessor system can be modeled by an interconnection network (or a graph), the interconnection network plays an important role in some issues such as communication performance [3], hardware cost [1], embedding and fault tolerant capabilities [16], and those are all driven by a mathematical model. Paths are suitable for designing simple algorithms with low communication costs. The path embedding problem is a very important issue for a network and is widely discussed in many researches [2], [5], [6], [12], [13], [14]. Generally, an interconnection network can be modeled by a graph, and edges (vertices) in the graph represent links (vertices) in the interconnection network.

An interconnection network is usually represented by an undirected simple graph G = (V, E). V(G) (E(G)) denotes the vertex (edge) set of G, where V(G) (E(G)) is a finite set. An edge between two vertices u and v in G is denoted by (u, v). For two distinct vertices u, v ∈ V(G), u and v are adjacent if (u, v) ∈ E(G). A path in G is a sequence of edges that connect adjacent vertices. A Hamiltonian path is a path that contains every vertex of G exactly once. A graph G is Hamiltonian-connected if every two distinct vertices of G are connected by a Hamiltonian path. A graph G is k-vertex fault-tolerant Hamiltonian-connected (k-Hamiltonian-connected for short) if it remains Hamiltonian-connected after removing no more than k vertices from G.

The interconnection network considered in this work is the generalized hypercube which has excellent topological properties, such as logarithmic diameter [9], vertex-symmetry [9], edge-symmetry [9], efficient communication [7], [9], [17] and high degree of fault tolerance [16]. Some basic properties of a generalized hypercube, such as diameter, wide diameter, and fault diameter have been determined in [4].

In [10], Lee et al. introduced an interesting property, called 2RP-property, as described below. Let l(P) denote the length of a path P, i.e. the number of edges which connect adjacent vertices in P. The distance between vertices u and v in G, denoted by d_G(u, v), is the minimum l(P) for every P joining u and v. Given any four distinct vertices u, v, x, and y in a graph G, let l₁ and l₂ be two integers such that l₁ ≥ d_G(u, v), l₂ ≥ d_G(x, y), and l₁+l₂ = |V(G)|−2. Then, there exist two vertex-disjoint (disjoint for short) paths P₁ and P₂ such that (1) P₁ is a path joining u and v with l(P₁) = l₁; (2) P₂ is a path joining x and y with l(P₂) = l₂, and (3) P₁ ⊙ P₂ spans G. Two paths are vertex-disjoint if they do not share any common vertex. The 2RP-property have been studied on various graphs, such as Hypercubes [10], augmented cubes [11], arrangement graphs [15]. Since a generalized hypercube is a generalization
of a hypercube, this work further shows that a
generalized hypercube satisfies the 2RP-property.

The rest of this paper is organized as follows.
Section 2 formally introduces the definition and some
properties of a generalized hypercube.
Section 3 defines the 2RP-property for the
generalized hypercube and proves that a
generalized hypercube satisfies the 2RP-property.
Conclusions are finally drawn in Section 4.

2 Background and Notations

This section discusses the structure and some
properties of a generalized hypercube. Some
notations frequently used in this work are also
presented.

2.1 Generalized Hypercube graphs

Generalized hypercubes are the generalization of
hypercubes [1] which were proposed for
building massively parallel computer systems. Let
\( G(m_0, m_1, \ldots, m_r) \) denote a \( r \)-dimensional
generalized hypercube of order \( \prod_{i=0}^{r} m_i \), where
\( r \geq 1 \) and \( m_i \geq 2 \) for all \( 1 \leq i \leq r \). In other words, there
are \( m_0 \times m_1 \times \cdots \times m_r \) vertices in a
\( G(m_0, m_1, \ldots, m_r) \) and each vertex is assigned a \( r \)-digit
identifier \( x_0x_1\cdots x_r \), where \( x_i \in \{0, 1\} \) for all \( 1 \leq i \leq r \). In a
\( G(m_0, m_1, \ldots, m_r) \), two vertices are adjacent if and
only if their identifiers differ at exactly one
digit position. Two adjacent vertices whose
identifiers differ at \( i \) position are connected by an
\( i \)-edge and they are \( i \)-neighbors, where \( 1 \leq i \leq r \).
Each vertex in a \( G(m_0, m_1, \ldots, m_r) \) has degree
\( 2^{m_i} \).

There exist \( m_i \) supernodes \( G_i[j] \) for \( 0 \leq j \leq m_i - 1 \)
with the vertex sets \( V(G_i[j]) = \{x_i \ldots x_1 \mid 0 \leq x_i \leq m_i - 1 \} \) for
\( 1 \leq i \leq r \) that form a partition of \( V(G(m_0, m_1, \ldots, m_r)) \). A
\( G(m_0, m_1, \ldots, m_r) \) is a complete graph of\( m_i \) vertices in \( G_i[j] \).

Let \( V(G_i[j]) = N_i \) for all \( 0 \leq i \leq m_i - 1 \). Let \( x = x_0x_1\cdots x_r \) be a vertex in a
\( G(m_0, m_1, \ldots, m_r) \). \( \sigma(x) = x_i \), for \( 1 \leq i \leq r \).
Also, let \( x' \) in \( G_i[j] \) be the \( i \)-neighbor of \( x \) in \( G_i[x_i] \) for
\( 0 \leq j \leq m_i - 1 \) and \( i \neq x_r \). Notably, the order of
dimensions of a \( G(m_0, m_1, \ldots, m_r) \) can be exchanged without changing its structure. For instance, a \( G(4, 3, 2) \) is isomorphic to a \( G(2, 3, 4) \).

Figure 1 shows the structure of a \( G(4, 3, 2) \). Let \( N_u(u) \) denote the set of vertices adjacent to vertex
\( u \) in \( G \). For convenience, let \( \langle u, u_0, u_1, \ldots, u_m, v \rangle \) denote a path joining \( u \) and \( v \), where all the vertices \( u, u_0, u_1, \ldots, u_m \) and \( v \) are distinct.
Additionally, \( \langle u, u_0, u_1, \ldots, u_m, v \rangle = \langle u, P, v \rangle \) if \( P = \langle u, u_0, u_1, \ldots, u_m, v \rangle \).

2.2 Some properties of Generalized Hypercube graphs

This section addresses some properties of a
generalized hypercube, and proves that it is a
1-Hamiltonian-connected graph.

Lemma 1 [8]. A \( G(m_0, m_1, \ldots, m_r) \) is
Hamiltonian-connected, except \( m_i = 2 \) for all \( 1 \leq i \leq r \).

Lemma 2. A \( G(m_0, m_1, \ldots, m_r) \) is
1-Hamiltonian-connected, where \( m_i \geq 3 \) for all \( 1 \leq i \leq r \).

Proof. This lemma is proved by induction on
dimension \( r \). For \( r = 1 \), \( G(m_1) \) is a complete graph of
\( m_1 \) vertices and the lemma clearly holds.
Assume the lemma holds for any \( G(m_0, m_1, \ldots, m_r) \) with \( r < k \).

Let \( f \) be the faulty vertex in a \( G(m_0, m_1, \ldots, m_r) \),
and \( u, v \in V(G(m_0, m_1, \ldots, m_r)) \) be two
distinct vertices. Since \( uv \notin \sigma(v) \), there exists a digit
\( l \) such that \( \sigma^l(u) \neq \sigma^l(v) \), where \( 1 \leq l \leq k - 1 \). Without loss of
generality, assume that \( l = k + 1 \). Since \( \sigma^l(u) \neq \sigma^l(v) \), we may assume that \( u \notin \sigma^l(V(G_{k+2}[1])) \) and
\( v \in \sigma^l(V(G_{k+2}[1])) \) without loss of generality. There are
two scenarios to be considered in the following.

Case 1. \( \sigma^k(u) \neq \sigma^k(v) \) and \( \sigma^l(u) \neq \sigma^l(v) \) : Without loss of
generality, assume that \( \sigma^k(u) = 2 \), and thus \( f \) is
included in \( G_{k+2}[2] \).

Case 1.1. \( N^l[1] \geq 4 \): Since \( |V(G_{k+2}[1])| \geq 4 \) for all \( 1 \leq l \leq m_{k+1} \), there exists \( p^2 \) in \( G_{k+2}[2] \), such that
\( p^2 \neq v \) and \( q^2 \) in \( G_{k+2}[2] \), such that
\( q^2 \neq v \). By the induction hypothesis, there
exists a Hamiltonian path \( H_1 \) in \( G_{k+2}[2] \), joining
\( p^2 \) and \( q^2 \). By Lemma 1, there exists two
Hamiltonian paths \( H_2 \) in \( G_{k+2}[0] \) and \( H_3 \) in
\( G(m_{k+1}, m_{k+2}, \ldots, m_{k+2}) \), such that
\( H_2 \) joins \( p^2 \) and \( q^2 \), and \( H_3 \) joins \( q^2 \) and \( v \).

Obviously, \( \langle u, H_2, p^2, p^2, q^2, q^2, v \rangle \) is
a Hamiltonian path in \( G(m_{k+1}, \ldots, m_{k+2}) \). The
Hamiltonian path construction of Case
1.1 is illustrated in Figure 2.
Figure 2. The Hamiltonian path constructed by Case 1.1.

Case 1.2. ($N^3=3$): There exists a vertex $p^2_1$ in $G^{+1}[2] \setminus \{f\}$ such that $p^0p^2_u$ and $q^2$ is included in $G^{+1}[2] \setminus \{f, p^2\}$. The discussions depend on the locations of $q^1$ and $q^2$. When $q^2uv$, the proof is the same as that in Case 1.1. When $q^1=uv$ and $q^0u$, by Lemma 1, there exists a Hamiltonian path $H$ in $(m_{k+1}, m_k, \ldots, m_1)G^{+1}[0]G^{+1}[2]$ joining $p^1$ and $v$. Hence, $(u, p^1, q^2, q^1, \cdots, m_1, v)$ is the Hamiltonian path in $G(m_{k+1}, m_k, \ldots, m_1)G^{+1}[0]G^{+1}[2]$ as shown in Figure 3(a). When $q^1=uv$ and $q^0u$, by Lemma 1, there exists a Hamiltonian path $H$ in $(m_{k+1}, m_k, \ldots, m_1)G^{+1}[0]G^{+1}[2]$ joining $p^1$ and $v$. Hence, $(u, q^2, p^1, q^0, q^1, \cdots, m_1)G^{+1}[0]G^{+1}[2]$ as depicted in Figure 3(b).

Figure 3. Hamiltonian paths constructed by Case 1.2. (a) $q^1=uv$ and $q^0u$. (b) $q^1=uv$ and $q^0u$.

Case 2. $(f^+ \setminus \{f\}) \subseteq (f^+ \setminus \{u\})$; Without loss generality, assume that $f^+ \setminus \{f\} = f^+ \setminus \{u\}$, and then assume that both $f$ and $u$ are included in $G^{+1}[0]$. There exists a vertex $p^0$ in $G^{+1}[0] \setminus \{f, u\}$. By Lemma 1, there exists a Hamiltonian path $H_1$ in $G(m_{k+1}, m_k, \ldots, m_1)G^{+1}[0]G^{+1}[2]$ joining $p^2$ and $v$. By the induction hypothesis, there exist a Hamiltonian path $H_2$ in $G^{+1}[0] \setminus \{f\}$ joining $u$ and $p^2$. Obviously, $(u, H_2, p^2, p^3, H_1, v)$ is the Hamiltonian path in $G(m_{k+1}, m_k, \ldots, m_1)G^{+1}[0]G^{+1}[2]$.

Lemma 3. Let $P$ be a path in a $G(m_r, m_{r-1}, \ldots, m_1)$. If $|P| \geq m_r$, then $P$ contains an $m_r$-edge where $m_r \neq r$.

Proof. Let $P=(u^0, u^1, \ldots, u^4)$ be a path in a $G(m_r, m_{r-1}, \ldots, m_1)$, where $k \geq m_r$, and any two vertices of $P$ is connected by $r$-edges. In other words, $\sigma(u^i) \neq \sigma(u^j)$ where $0 \leq i, j \leq k$ and $i \neq j$. Thus, there should be $k+1\geq m_r$ distinct symbols. It is a contradiction because there are only $m_r$ distinct symbols in $r$ dimension. Therefore, this lemma follows.

Lemma 4. Let $P$ be a path from $u$ to $v$, and $w$ and $x$ be any two distinct vertices in a $G(m_r, m_{r-1}, \ldots, m_1)$, where $r \geq 2$ and $m_r=4$ for all $1 \leq s < r$. If $|P| \geq 4\cdot s$, then there exist two adjacent vertices $y$ and $z$ in $P$, such that $N_G(y, m_r, \ldots, m_1)(w) \cap N_G(m_r, \ldots, m)(x) \neq \emptyset$ and $\{w, x\} \cap \{y, z\} = \emptyset$.

Proof. Since $|N_G(y, m_r, \ldots, m_1)(w)| = 3r$, $|V(P)\setminus N_G(y, m_r, \ldots, m_1)(w)| = |\{u, v, w, x\}| \geq (4\cdot s-3) \cdot 4 \geq 1$. Thus, there exist a vertex $y$ in $V(P)\setminus N_G(y, m_r, \ldots, m_1)(w)$ and $w \notin N_G(y, m_r, \ldots, m_1)(w)$, there exists a vertex $z \in N_P(y)$ in such that $z \notin \{w, x\}$. Therefore, there exist two adjacent vertices $y$ and $z$ in $P$ such that $N_G(z, m_r, \ldots, m_1)(w) \cap N_G(m_r, m_{r-1}, \ldots, m_1)(x) \neq \emptyset$ and $\{w, x\} \cap \{y, z\} = \emptyset$.

3 2RP-property

This section demonstrates that a $G(m_r, m_{r-1}, \ldots, m_1)$ satisfies the 2RP-property, when $m_r \geq 4$ for all $1 \leq s < r$. Let $u, v, x$ and $y$ be any four distinct vertices of a $G(m_r, m_{r-1}, \ldots, m_1)$, and $l_1$ and $l_2$ be two integers with $l_1 \geq d_G(u, v) \geq d_G(x, y)$ and $l_1+l_2 = |V(G(m_r, m_{r-1}, \ldots, m_1))|$. Then, there exist two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $u$ and $v$ with $l(P_1) = l_1$; (2) $P_2$ is a path joining $x$ and $y$ with $l(P_2) = l_2$; and (3) $P_1 \cup P_2$ spans $G(m_r, m_{r-1}, \ldots, m_1)$, except the following two cases: (a) $l_1 = 2$ or $l_2 = 2$ and $|V(G(m_r, m_{r-1}, \ldots, m_1))| \neq 4$ with $d_G(u, v) \geq 1$ such that $N_G(m_r, m_{r-1}, \ldots, m_1)(u, v) = \{x, y\}$; (b) $l_1 = 2$ or $l_2 = 2$ and $|V(G(m_r, m_{r-1}, \ldots, m_1))| \neq 4$ with $d_G(u, v) \geq 2$ such that $N_G(m_r, m_{r-1}, \ldots, m_1)(u) \cap N_G(m_r, m_{r-1}, \ldots, m_1)(v) = \{x, y\}$.
Lemma 5. A $G(m_1, m_2, \ldots, m_l)$ satisfies
2RP-property, where $r \geq 2$ and $m_l=4$ for all $2 \leq s \leq r$.

Proof. This lemma is proved by induction on $r$.

With the aid of a computer program, the lemma holds for $r=2$. Suppose the lemma holds for any $G(m_1, m_2, \ldots, m_l)$ with $r=k+1$. To prove the lemma holds, let $r=k+1$ and $G$ denote $G(m_1, m_2, \ldots, m_l)$ for simplicity. Without loss of generality, assume that $l_1 \leq s_2$, thus $l_1 \leq (4^{s_1-2})/2 = 2(4^r-1)$. Let $u, v$, $x$ and $y$ be $4$ arbitrary distinct vertices in $G$. Since $xy$, there exists a dig $l$ such that $d(x)v \notin d(y)$, where $1 \leq l \leq 4k+1$. Without loss of generality, assume $l = k+1$. Since $d(x)v \notin d(y)$, we may assume $x \in V(G^{k+1}[0])$ and $y \in V(G^{k+1}[1])$ without loss of
generality. Five cases should be considered as follows.

Case 1. $(d^{k+1}(u) \neq d^{k+1}(v))$ and $(d^{k+1}(u), d^{k+1}(v)) \cap$
$(d^{k+1}(x), d^{k+1}(y)) = \emptyset$. Without loss of generality, assume
that $d^{k+1}(u)=2$, $d^{k+1}(v)=3$, such that $u \in G^{k+1}[2]$, $v \in G^{k+1}[3]$. Three cases
are necessary to be discussed depending on the
values of $d(u,v)$ and $l_i$.

Case 1.1. $(d^{k+1}(u) \leq l_i \leq 4^r-1$ with $d(u,v) = 1$):

For $l_i=1$, we have $P_1 = (u,v)$. There is a vertex $a^1 \in G^{k+1}[0] \setminus \{v\}$ such that $a^1 \neq x$. By Lemma 1, there are two Hamiltonian paths $H$
in $G^{k+1}[0]$ and $Q$ in $G^{k+1}[1]$ such that $H$ joins $x$
and $a^1$, and $Q$ joins $a^1$ and $y$. Since $l(Q) = 4^r-1$, $Q
$can be written as $(a^1, Q_1, b^1, c^1, Q_2, y)$ for
some vertices $b^1$ and $c^1$ such that $\{b^1, c^1\} \cap \{u\} = \emptyset$. According to Lemma 2, there is a
Hamiltonian path $R$ in $G^{k+1}[2] \setminus \{u\}$ joining $b^2$
and $c^2$, and $R$ can be written as $(b^2, R_1, d^2, c^2,$
$R_2, c^2)$ for some vertices $d^2$ and $e^2$. By Lemma
2, there is a Hamiltonian path $S$ in $G^{k+1}[3] \setminus \{v\}$
joining $d^3$ and $e^3$. Hence, $P_2 = (x, H, a^1, Q_1,$
$b^1, b^2, R_1, d^2, d^3, S, e^2, R_2, e^2, c^2, Q_2, y)$,
and $P_1$ and $P_2$ are the required paths as shown in
Figure 4(a).

For $l_i=2$, only $N_d(u) \cap N_d(v) = \{x, y\}$ needs to be discussed. Since $N_d(u) \cap N_d(v) = \{u', u''\}$,
we have $\{u', u''\} \neq \{x, y\}$. Without loss of
generality, assume that $u' \neq x$. Hence, $P_1 = (u, u'', v)$. By Lemma 2, there is a Hamiltonian path $H$
in $G^{k+1}[0] \setminus \{u''\}$ joining $x$ and $a^2$. Hence, $P_2 = (x,$
$H', a^2, a^1, Q_1, b^1, b^2, R_1, d^2, d^3, S, e^2, R_2, c^2,$
$c^1, Q_2, y)$, and $P_1$ and $P_2$ are the required paths
as shown in Figure 4(b).

For $3 \leq l_i \leq 4^r-1$, there are two vertices $a^2 \in
N_G^{k+1}[u]$ and $b^3 \in G^{k+1}[0] \setminus \{x\}$ such that $b^3 \neq x$.
By Lemma 1, there are two Hamiltonian paths $H$
in $G^{k+1}[0]$ and $Q$ in $G^{k+1}[1]$ such that $H$ joins
$x$ and $b^3$, and $Q$ joins $b^3$ and $y$. Since $l(Q) = 4^r-1$, $Q$
can be written as $(b^1, Q_2, c^1, d^2, Q_2, y)$ for
some vertices $c^1$ and $d^2$ such that $\{c^1, d^2\} \cap \{u, a^2\} = \emptyset$. By the induction hypothesis,
there exist two disjoint paths $R_1$ and $R_2$ such that
(1) $R_1$ is a path joining $u$ and $a^2$ with
\[l(R_1) = 1; (2) R_2$ is a path joining $c^2$ and $d^3$ with
\[l(R_2) = 4^r-3; (3) R_3 \cap R_2$ spans $G^{k+1}[2]$. By

Lemma 4, $R_2$ can be written as $(c^2, R_2, c^1, d^2,
R_2, d^3)$. Hence, $P_1 = (u, R_1, a^1, v, R_2, a^2,$
$c^2, d^3)$, $P_2 = (x, H, b^1, b^2, R_1, d^2, d^3,$
$S, e^2, R_2, e^2, c^2, Q_2, y)$, and $P_1$ and $P_2$ are the required paths as shown in
Figure 4(c).

Figure 4. Paths $P_1$ and $P_2$ constructed by Case 1.1.
(a) $l_i=1.$ (b) $l_i=2.$ (c) $3 \leq l_i \leq 4^r-1.$

Case 1.2. $(d^{k+1}(u) \leq l_i \leq 4^r-1$ with $d^{k+1}(u,v) = 2$):

For $d^{k+1}(u,v) = l_i = 4^r-2$, there is a vertex
$a^6$ in $G^{k+1}[0]\{x\}$ such that $a^6\neq y$ and $a^6\notin \{u^3, v\}$. By Lemma 1, there are two Hamiltonian paths $H$ in $G^{k+1}[0]$ and $Q$ in $G^{k+1}[1]$ such that $H$ joins $x$ and $a^6$, and $Q$ joins $a^6$ and $v$. Since $l(Q)=4^k-1$, $Q$ can be written as $(b_1', Q_1, c_1', d_1', Q_2, y)$ for some vertices $c_1'$ and $d_1'$ such that $(c_1', d_1')\cap \{u, a^6\} = \emptyset$. By the induction hypothesis, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $u$ and $a^6$ with $l(R_1)=l-1$; (2) $R_2$ is a path joining $c_1'$ and $d_1'$ with $l(R_2)=4^k-3$, and (3) $R_1\cup R_2$ spans $G^{k+2}$ [2]. By Lemma 2, there exists a Hamiltonian path $S$ in $G^{k+2}[1]$ joining $a^6$ and $v$. Hence, $P_1=(u, R_1, a^6, S_1, v)$ and $P_2=(x, H, b_1', b_1, R_1, a^6, S_1, v)$ are the required paths as shown in Figure 6(a).

For $l=4^{k+1}$, there is a Hamiltonian path $S$ of $G^{k+1}[3]\{a^6\}$ joining $u^3$ and $v$ by Lemma 2. Hence, $P_1=(u, u', S, v)$ and $P_2=(x, H, a^6, a^6, a^6, Q_1, b_1, b_1, R, c^6, c^6, Q_2, y)$ are the required paths as shown in Figure 5(b).

**Case 1.3.** $(4^k \leq l \leq 2(4^k)-1)$

For $l=4^k$, there are two vertices $a^6 \in N_{G^{k+1}[1]}(u)$ and $b^6$ in $G^{k+1}[0]\{x\}$ such that $a^6\neq y$, $b^6\neq y$, and $b^6\notin \{v, a^6\}$. By Lemma 1, there are two Hamiltonian paths $H$ in $G^{k+1}[0]$ and $Q$ in $G^{k+1}[1]$ such that $H$ joins $x$ and $b^6$, and $Q$ joins $b^6$ and $v$. Since $l(Q)=4^k-1$, $Q$ can be written as $(b_1', Q_1, c_1', d_1', Q_2, y)$ for some vertices $c_1'$ and $d_1'$ such that $(c_1', d_1')\cap \{u, a^6\} = \emptyset$. By the induction hypothesis, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $u$ and $a^6$ with $l(R_1)=l-1$; (2) $R_2$ is a path joining $c_1'$ and $d_1'$ with $l(R_2)=4^k-3$, and (3) $R_1\cup R_2$ spans $G^{k+2}$ [2]. By Lemma 2, there exists a Hamiltonian path $S$ in $G^{k+2}[1]$ joining $a^6$ and $v$. Hence, $P_1=(u, R_1, a^6, S_1, v)$ and $P_2=(x, H, b_1', b_1, Q_1, c_1', c_1, R_2, d_1', d_1', Q_2, y)$ are the required paths as shown in Figure 6(b).

For $l=2(4^k)-2$, there is a Hamiltonian path $R$ in $G^{k+2}[2]$ joining $u$ and $a^6$ by Lemma 1. Hence, $P_1=(u, R, a^6, S, v)$ and $P_2=(x, H, b^6, b_1', Q_1, c^6, c_1', R_2, d^6, d_1', Q_2, y)$ are the required paths as shown in Figure 6(c).

For $l=2(4^k)-1$, $P_1=(u, R, a^6, S, v)$ and $P_2=(x, H, b^6, b_1', Q_1, c^6, c_1', R_2, d^6, d_1', Q_2, y)$ are the required paths as shown in Figure 6(d).

**Case 2.** $(\sigma^{d+1}(u)\cap \sigma^{d+1}(v)) \cap (\sigma^{d+1}(u), \sigma^{d+1}(v)) = \emptyset$: Without loss of generality, assume that $\sigma^{d+1}(u)\cap \sigma^{d+1}(v)=\emptyset$. $\sigma^{d+1}(v)=\emptyset$ such that $u$ and $v$ are included in $G^{k+1}[1]$ and $G^{k+2}[2]$, respectively. Three cases are necessary to be discussed depending on the values of $d(u, v)$ and $l$.

**Case 2.1.** $(d(u, v) \leq l \leq 4^k-1)$

For $l=1$, we have $P_1=(u, v)$. There is a vertex $a^6$ in $G^{k+1}[0]\{x\}$ such that $a^6\neq y$, $u$. By Lemma 1, there is a Hamiltonian path $H$ in $G^{k+1}[0]$ joining $x$ and $a^6$. By Lemma 2, there is a Hamiltonian path $Q$ in $G^{k+1}[1]\{u\}$ joining $a^6$ and $y$. Besides, $Q$ can be written as $(a^6, Q_1, b^6, c^6, Q_2, y)$ for some vertices $b^6$ and $c^6$. By Lemma 2, there exist a Hamiltonian path $R$ in $G^{k+2}[2]\{v\}$ joining $b^6$ and $c^6$. Besides, $R$ can be written as $(b^6, R_1, d_2, e_2, R_2, c^6)$ for some vertices $d_2$ and $e_2$.
$G^k[3]$ joining $d^i$ and $e^i$. Hence, $P_2 = (x, H, d^i, a^i, Q_1, b^i, d^i, S, e^i, c^i, R_1, c^i, Q_2, y)$, and $P_1$ and $P_2$ are the required paths.

For $3 \leq l_1 \leq 4^k-1$, there are two vertices $a^i \in N_{G^k[1]}(u)$ and $b^i$ in $G^k[0]\{x\}$ such that $a^i \not\in y$ and $b^i \not\in \{y, u, a^i\}$. By Lemma 1, there is a Hamiltonian path $H$ in $G^k[0]$ joining $x$ and $b^i$. By the induction hypothesis, there exist two disjoint paths $Q_1$ and $Q_2$ such that (1) $Q_1$ is a path joining $u$ and $a^i$ with $l(Q_1)=1$; (2) $Q_2$ is a path joining $b^i$ and $y$ with $l(Q_2)=4^{k-1}-3$, and (3) $Q_1 \cap Q_2$ spans $G^k[1]$. By Lemma 4, $Q_2$ can be written as $(b^i, Q_{21}, c^i, d^i, Q_{22}, y)$ for some vertices $c^i$ and $d^i$ such that $N_{G^k[1]}(b^i) \cap N_{G^k[1]}(c^i) \not\in \{c^i, d^i\}$ and $\{v, a^i\} \cap \{c^i, d^i\} = \emptyset$. By the induction hypothesis, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $a^i$ and $v$ with $l(R_1)=l_1-2$; (2) $R_2$ is a path joining $c^i$ and $d^i$ with $l(R_2)=4^k-2-l(R_1)$, and (3) $R_1 \cup R_2$ spans $G^k[2]$. $R_2$ can be written as $(c^i, R_{21}, e^i, f^i, R_{22}, d^i)$ for some vertices $e^i$ and $f^i$. If $l(R_2)=1$, then $e^i = c^i$ and $f^i = d^i$. By Lemma 1, there is a Hamiltonian path $S$ in $G^k[3]$ joining $e^i$ and $f^i$. Hence, $P_1 = (u, d^i, a^i, R_1, v)$ and $P_2 = (x, H, b^i, b^i, Q_{21}, e^i, c^i, R_{21}, e^i, S, f^i, f^i, R_{22}, d^i, d^i, Q_{22}, y)$ are the required paths.

**Case 2.2.** $(d(u, v) \leq l_1 \leq 4^k-1)$ with $d(u, v) \geq 2$.

For $d(u, v) \leq l_1 \leq 4^k-2$, there is a vertex $a^i$ in $G^k[0]\{x\}$ such that $a^i \not\in \{y, u\}$. By Lemma 1, there is a Hamiltonian path $H$ in $G^k[0]$ joining $x$ and $a^i$. By Lemma 2, there is a Hamiltonian path $Q$ of $G^k[1]\{u\}$ joining $a^i$ and $y$. By Lemma 4, $Q$ can be written as $(a^i, Q_{11}, b^i, c^i, Q_{22}, y)$ for some vertices $b^i$ and $c^i$ such that $N_{G^k[1]}(a^i) \cap N_{G^k[1]}(b^i) \not\in \{b^i, c^i\}$ and $\{v, a^i\} \cap \{b^i, c^i\} = \emptyset$. By the induction hypothesis, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $a^i$ and $v$ with $l(R_1)=l_1-1$; (2) $R_2$ is a path joining $b^i$ and $c^i$ with $l(R_2)=4^k-2-l(R_1)$, and (3) $R_1 \cup R_2$ spans $G^k[2]$. Besides, $R_2$ can be written as $(b^i, R_{21}, e^i, f^i, R_{22}, d^i, d^i, d^i, Q_{22}, y)$ for some vertices $e^i$ and $f^i$. If $l(R_2)=1$, then $e^i = b^i$ and $f^i = c^i$. By Lemma 1, there is a Hamiltonian path $S$ in $G^k[3]$ joining $d^i$ and $e^i$. Hence, $P_1 = (u, u^i, R_1, v)$ and $P_2 = (x, H, a^i, a^i, Q_{11}, b^i, b^i, Q_{21}, d^i, d^i, S, e^i, e^i, c^i, Q_{22}, y)$ are the required paths.

For $l_1 = 4^k-1$, there is a Hamiltonian path $R$ in $G^k[2]\{c^i\}$ joining $u^i$ and $v$ by Lemma 2. By Lemma 1, there is a Hamiltonian path $S$ in $G^k[3]$ joining $b^i$ and $c^i$. Hence, $P_1 = (u, u^i, R, v)$ and $P_2 = (x, H, a^i, a^i, Q_{11}, b^i, b^i, b^i,$
Case 3. For $|u| = 1$, we have $P_1 = \langle u, v \rangle$. There is a vertex $a^1 \in G^{k[0]}[u]$, such that $a^1 \neq x$. By Lemma 2, there are two Hamiltonian paths $H$ in $G^{k[0]}[u]$ and $Q$ in $G^{k[1]}[v]$ such that $H$ joins $x$ and $a^1$, and $Q$ joins $a^1$ and $y$. $Q$ can be written as $\langle a^1, Q_1, b^1, c^1, Q_2, y \rangle$ for some vertices $b^1$ and $c^1$. By Lemma 1, there is a Hamiltonian path $R$ in $G^{k[2]}$ joining $b^1$ and $c^1$, and $R$ can be written as $\langle b^1, R_1, d^1, e^1, R_2, e^2, c^2, Q_2, y \rangle$, and $P_1$ and $P_2$ are the required paths.

For $l = 2$, by Lemma 1, there is a Hamiltonian path $R$ in $G^{k[2]}[v]$ joining $b^2$ and $c^2$, and $R$ can be written as $\langle b^2, R_1, R_2, d^2, e^2, e^2, c^2, c^2, Q_2, y \rangle$ for some vertices $d$ and $e$. Hence, $P_1 = (u, u', v)$ and $P_2 = \langle x, H, a^1, a', Q_1, b^1, b', e^1, c^1, e^2, e^2, Q_2, y \rangle$ are the required paths.

For $3 \leq |u| \leq 4^k - 1$, there is a vertex $v' \in N_{G^{k[0]}[u]}(x)$ on $G^{k[0]}[u]$. Within $G^{k[0]}[u]$, we have $|N_{G^{k[0]}[u]}(x) \cap N_{G^{k[0]}[u]}(y)| \geq 2k - 2 \geq 4$. Thus, there is a vertex $v' \in N_{G^{k[0]}[u]}(x) \cap N_{G^{k[0]}[u]}(y)$ in $G^{k[0]}[u]$. Again, by the induction hypothesis, there exist two disjoint paths $Q_1$ and $Q_2$ such that (1) $H_1$ is a path joining $u$ and $v'$ with $l(H_1) = l_1 - 2$; (2) $H_2$ is a path joining $x$ and $v'$ with $l(H_2) = 4^k - 2 - l(H_1)$, and (3) $H_1 \cup H_2$ spans $G^{k[0]}[u]$. Again, by the induction hypothesis, there exist two disjoint paths $Q_1$ and $Q_2$ such that (1) $Q_1$ is a path joining $f^1$ and $v$ with $l(Q_1) = 1$; (2) $Q_2$ is a path joining $g^1$ and $v$ with $l(Q_2) = 4^k - 3$, and (3) $Q_1 \cup Q_2$ spans $G^{k[1]}[v]$, besides $Q_1$. $Q_1$ can be written as $\langle a', Q_1, b^1, c^1, Q_2, y \rangle$ for some vertices $b$ and $c$. Hence, $P_1 = (u, u', v)$ and $P_2 = \langle x, H, a^1, a', Q_1, b^1, b', c^1, e^1, c^2, e^2, e^2, Q_2, y \rangle$ are the required paths.

Case 3. \(d_3(u, v) \leq l_1 \leq 4^k - 1\) with $d_3(u, v) = 1$:

For $l_1 = l_2$, we have $P_1 = \langle u, v \rangle$, and $P_2 = \langle x, H, x \rangle$. Since $N_{G^{k[0]}[u]}(v) = \{u, v\}$, we have $\{u, v^0\} = \{x, y\}$. Without loss of generality, assume $u \neq y$. There are two vertices $a', a'' \in N_{G^{k[0]}[u]}(x) \cap N_{G^{k[0]}[v]}(y)$ in $G^{k[0]}[x, y, x']$, and $b^1$ in $G^{k[2]}[a']$. By the induction hypothesis, there exist two disjoint paths $Q_1$ and $Q_2$ such that (1) $Q_1$ is a path joining $u$ and $v$ with $l(Q_1) = l_1 - 1$; (2) $Q_2$ is a path joining $x$ and $y$ with $l(H_2) = 4^k - 2 - l(Q_1)$, and (3) $Q_1 \cup Q_2$ spans $G^{k[1]}[v]$. By Lemma 2, there is a Hamiltonian path $H$ in $G^{k[0]}[u]$.
joining and $a^0$. By Lemma 1, there are two Hamiltonian paths $R$ in $G^{k+1}[2]$ and $S$ in $G^{k+1}[3]$ such that $R$ joins $a^0$ and $b^0$, and $S$ joins $a^0$ and $b^0$. Hence, $P_1 = \{u, u^1, Q_1, v\}$ and $P_2 = \{x, H, a^0, b^0, S, b^0, R, a^0, Q_2, y\}$ are the required paths.

For $I = d_0(u, v) = d_0(u, v) + 3$. If $u^1 \not\in y$ or $v^1 \not\in x$, then the proof is the same as the situation that $I = d_0(u, v)$ with $d_0(u, v) = 2$. On the other hand, when $u^1 \not\in y$ and $v^1 \not\in x$, there are four vertices $a^0$, $b^0$, $c^0$, and $d^0$ such that $a^0 \not\in N_{G^{k+1}[0]}(u)$ and $d^0(u, v) = d_0(u, v) + 1$, $b^0 \not\in N_{G^{k+1}[0]}(u) \cap N_{G^{k+1}[0]}(v)$ is included in $G^{k+1}[0]$, where $a^0$, $d^0$, $v$, $c^0 \not\in N_{G^{k+1}[0]}(a^0) \cap N_{G^{k+1}[0]}(v)$ is included in $G^{k+1}[1]$, and $d^0$ is included in $G^{k+1}[2]$.

By the induction hypothesis, there exist two disjoint paths $H_1$ and $H_2$ such that (1) $H_1$ is a path joining $u$ and $d^0$ with $l(H_1) = 1$; (2) $H_2$ is a path joining $a^0$ and $d^0$, and (3) $H_1 \cup H_2$ spans $G^{k+1}[0]$. Again, by the induction hypothesis, there exist two disjoint paths $Q_1$ and $Q_2$ such that (1) $Q_1$ is a path joining $a^0$ and $c^0$ with $l(Q_1, v) = 1$; (2) $Q_2$ is a path joining $c^0$ and $l(Q_2) = 1$; and (3) $Q_1 \cup Q_2$ spans $G^{k+1}[1]$. By Lemma 1, there are two Hamiltonian paths $R$ in $G^{k+1}[2]$ and $S$ in $G^{k+1}[3]$ such that $R$ joins $c^0$ and $d^0$, and $S$ joins $b^0$ and $d^0$. Hence, $P_1 = \{u, H, a^0, b^0, Q_1, v\}$ and $P_2 = \{x, H, a^0, b^0, S, b^0, Q_1, c^0, R, a^0, d^0, Q_2, y\}$ are the required paths.

For $4^1 \leq 4^1 \leq 2^4$, there exists a Hamiltonian path $Q'$ in $G^{k+1}[1] \{v\}$ joining $b^0$ and $v$ by Lemma 2. By Lemma 4, $Q'$ can be written as $(b^0, Q', b^0, c^0, Q', y)$, and $y$ are the required paths. By the induction hypothesis, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $u$ and $d^0$ with $l(R_1) = 1$; (2) $R_2$ is a path joining $c^0$ and $d^0$, and (3) $R_1 \cup R_2$ spans $G^{k+1}[2]$. Hence, $P_1 = \{u, H, a^0, b^0, Q_1, v\}$ and $P_2 = \{x, H, a^0, b^0, Q_1, c^0, R, a^0, d^0, Q_2, y\}$ are the required paths.
$Q_2, y)$ are required paths.

For $l=4^2 - 2$, there is a Hamiltonian path $R$ in $G^{4^2}[2]\{\alpha_1, a\}$ joining $u$ and $v$ by Lemma 2. Hence, $P_1 = (u, R, v)$ and $P_2 = (x, H_1, b, b', b, S, c, c', H_2, a, d, a', a', Q, y)$ are the required paths.

For $l=4^2 - 1$, there is a Hamiltonian path $R'$ in $G^{4^2}[2]$ joining $u$ and $v$ by Lemma 1. Hence, $P_1 = (u, R, v)$ and $P_2 = (x, H_1, b, b', b, S, c, c', H_2, a, d, a', a', Q, y)$ are the required paths.

For $4^2 ≤ l ≤ 2(4^2) - 4$, obviously, path $R$ can be written as $\langle u, R, f', g', R, v \rangle$ for some vertices $f'$ and $g'$. By Lemma 4, $H$ can be written as $\langle u, H_1, b, b', b', S, a', c, c, H_2, a, d, a', a', Q, y \rangle$ for some vertices $b'$ and $c'$ such that $N_{G^{k+1}}[1][f'] ∩ N_{G^{k+1}}[1][g'] ≠ \{b', c'\}$ and $\{f', g'\} ∩ \{b', c'\} = \emptyset$. By the induction hypothesis, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $f'$ to $g'$ with $l(S_1) - l_1 = 4^2 - 1$, (2) $S_2$ is a path joining $b'$ to $c'$ with $l(S_2) = 4^2 - 2 - l(S_1)$, and (3) $S_1 ∪ S_2$ spans $G^{k+1}[3]$. Hence, $P_1 = (u, R, f', f', S_1, a, c, g', R_2, v)$ and $P_2 = (x, H_1, b, b', b', S, b', c, c', H_2, a, d, a', a', Q, y)$ are the required paths.

For $l=2(4^2) - 3$, there is a Hamiltonian path $S'$ in $G^{k+1}[3]\{\alpha_1, \alpha_2\}$ joining $f'$ and $g'$ by Lemma 2. Hence, $P_1 = (u, R, f', f', S, g', g', R_2, v)$ and $P_2 = (x, H, a, d, a, a', Q, y)$ are the required paths.

For $l=2(4^2) - 2$, there exist a Hamiltonian path $S''$ in $G^{k+1}[3]$ joining $f'$ and $g'$ by Lemma 1. Hence, $P_1 = (u, R, f', f', S, g', g', R_2, v)$ and $P_2 = (x, H, a, d, a, a', Q, y)$ are the required paths.

For $l=2(4^2) - 1$, obviously, path $R$ can be written as $\langle u, R, f', g', R_2, v \rangle$ for some vertices $f'$ and $g'$. Hence, $P_1 = (u, R, f', f', S, g', g', R_2, v)$ and $P_2 = (x, H, a, d, a', a', Q, y)$ are the required paths.

Case 5: $(\sigma^{k+1}(u) - \sigma^{k+1}(v)) ∩ (\sigma^{k+1}(x), \sigma^{k+1}(y)) = \emptyset$. Without loss of generality, assume that $\sigma^{k+1}(u) - \sigma^{k+1}(v) ∩ \{u, v\}$ are both included in $G^{k+1}[0]$.

For $d_\delta(u, v) ≤ l ≤ 4^2 - 3$. Since $|N_{G^{k+1}[0]}(x) - N_{G^{k+1}[1]}(u) ∩ N_{G^{k+1}[1]}(v)| ≥ 4$, there is a vertex $a^1 \in N_{G^{k+1}[1]}(x) - N_{G^{k+1}[1]}(u) ∩ N_{G^{k+1}[1]}(v)$ in $G^{k+1}[0](u, v)$ such that $a^1 ∉ N_{G^{k+1}[1]}(u)$ and $N_{G^{k+1}[1]}(u) ∩ N_{G^{k+1}[1]}(v) = \{x, a^1\}$. By the induction hypothesis, there exist two disjoint paths $H_1$ and $H_2$ such that (1) $H_1$ is a path joining $u$ to $v$ with $l(H_1) = l_1$; (2) $H_2$ is a path joining $x$ to $a^1$ with $l(H_2) = 4^2 - 2 - l(H_1)$, and (3) $H_1 ∪ H_2$ spans $G^{k+1}[1]$. By Lemma 1, there is a Hamiltonian path $Q$ in $G^{k+1}[1]$ joining $a'$ and $y$, and $Q$ can be written as $\langle a', Q, b, b', Q, y \rangle$ for some vertices $b'$ and $c'$. By Lemma 1, there is a Hamiltonian path $R$ in $G^{k+1}[2]$ joining $b'$ and $c'$, and $R$ can be written as $\langle b', R_1, d', e', c', d, R_2, v \rangle$ for some vertices $d'$ and $c'$. Again, by Lemma 1, there is a Hamiltonian path $S$ in $G^{k+1}[3]$ joining $d'$ and $e'$. Hence, $P_1 = (u, H_1, v)$ and $P_2 = (x, H_2, a, d', Q, b, b', R_1, d', e', c', R_2, c', e', Q_2, y)$ are the required paths.

For $l=4^2 - 2$, there is a Hamiltonian path $H$ in $G^{k+1}[0]\{x\}$ joining $u$ to $v$ by Lemma 2. By Lemma 1, there exists a Hamiltonian path $S'$ in $G^{k+1}[3]$ joining $x'$ and $a'$. Hence, $P_1 = (u, H, v)$ and $P_2 = (x, x', S', a', a', Q_1, b, b', R, c, c', Q_2, y)$ are the required paths.

For $4^2 - 1 ≤ l ≤ 2(4^2) - 5$, $(H_1)$ and $(H_2)$ can be reset as $4^2 - 3$ and 1, and $H_1$ can be written as $\langle u, H_1, f, f', g', H_2, d \rangle$ for some vertices $f'$ and $g'$. By Lemma 4, $R$ can be written as $\langle b', R_1, d', e', c', R_2, c' \rangle$ for some vertices $d'$ and $e'$ such that $N_{G^{k+1}[1]}(f') ∩ N_{G^{k+1}[1]}(g') ≠ \{d', e'\}$ and $\{f', g'\} ∩ \{d', e'\} = \emptyset$. By the induction hypothesis, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $f'$ and $g'$ with $l(S_1) = l_1 - 4^2 + 2$, (2) $S_2$ is a path joining $d'$ and $e'$ with $l(S_2) = 4^2 - 2 - l(S_1)$, and (3) $S_1 ∪ S_2$ spans $G^{k+1}[3]$. Hence, $P_1 = (u, H_1, f, f', S_1, g', g', H_2, v)$ and $P_2 = (x, H_2, d', a', a', Q_1, b, b', R, c, c', Q_2, y)$ are the required paths.

For $l=2(4^2) - 3$, there is a Hamiltonian path $S''$ in $G^{k+1}[3]$ joining $f'$ to $g'$ by Lemma 1. Hence, $P_1 = (u, H_1, f', f', S', g', g', H_2, v)$ and $P_2 = (x, H_2, a', d', a', Q_1, b, b', R, c, c', Q_2, y)$ are the required paths.

For $l=2(4^2) - 2$, Since $(H) = 4^2 - 2$, path $H$ can be written as $\langle u, H, f, f', g', H_2, v \rangle$ for some vertices $f'$ and $g'$ such that $g' ∉ N_{G^k}(x')$. By Lemma 1, there is a Hamiltonian path $R'$ in $G^{k+1}[2]$ joining $x'$ and $a'$. Hence, $P_1 = (u, H_1, f, f', S', g', g', H_2, v)$ and $P_2 = (x, x', R', a', a', Q, y)$ are the required paths.

For $l=2(4^2) - 1$, there is a Hamiltonian path $R''$ in $G^{k+1}[2]\{g'\}$ joining $x'$ and $a'$ by Lemma 1. Hence, $P_1 = (u, H_1, f, f', S', g', g', H_2, v)$ and $P_2 = (x, x', R', a', a', Q, y)$ are the required paths.

Lemma 6. If $G(4, m_1, ..., m_l)$ satisfies the 2RP property, then $G(m_1, m_2, ..., m_l)$ satisfies the 2RP property, where $r ≥ 2$ and $m_2 ≥ 4$ for all $1 ≤ r ≤ l$.

Proof. This lemma is proved by induction on $m_r$. Obviously, the lemma holds for $m_r = 4$. Suppose that $G(m_1, m_2, ..., m_l)$ satisfies the 2RP property for all $4 ≤ m_r ≤ 5k$. In the following, we prove that $G(k + 1, m_1, ..., m_l)$ satisfies the 2RP property. Let $G$ denote $G(k + 1, m_1, ..., m_l)$ for brevity. Without loss of generality, assume that $l ≤ k$ and thus $l_1 ≤ (k + 1)N^k - 2)/2$. Moreover, $l_1 ≤ ...
(k×N^2−2)−max{k, d_δ(x, y)}, as explained below. If k ≥ d_δ(x, y), then l_1 ≤ (k×N^2−2)−k because ((k+1)×N^2−2)/2 ≤ (k×N^2−2)−k can be assured by k≥4, N′≥4, and r≥2. On the other hand (k < d_δ(x, y)), we have l_1 ≤ (k×N^2−2)−d_δ(x, y) because ((k+1)×N^2−2)/2 ≤ (k×N^2−2)−d_δ(x, y) can be assured by k≥4, N′≥4, d_δ(x, y)≤r, and r≥2.

Let u, v, x, and y be four distinct vertices in G. Since G'[j], G'[k] are vertex-disjoint to each other, there exists a G'[j] such that none of u, v, x, and y is included in G'[j], where 0≤j≤k. Without loss of generality, assume that j=k.

Since d_δ(u, v) ≤ l_1, d_δ(u, v) ≤ l_1 ≤ (k×N^2−2)−max{k, d_δ(x, y)}. The following discussions first exclude the situation that l_1=2 with N_{G'[k]}(u)∩N_{G'[k]}(v) = {x, y} and N_{G'[k]}(u)∩N_{G'[k]}(v) ≠ {x, y}. Since G′[k] \cong G(k, m_1,…, m_4), by the induction hypothesis, there exist two disjoint paths H_1 and H_2 such that (1) H_1 is a path joining u and v with l(H_1) = l_1; (2) H_2 is a path joining x and y with l(H_2) = (k×N^2−2)−l_1, (3) H_1 ∪ H_2 spans G′[k]. Since l(H_2) ≥ 2k, by Lemma 3, there exists an m-edge in H_2 where m≠r. Thus, H_2 can be written as (x, H_2, a, b, H_2, y) for some vertices a, b such that a and b is connected by m-edge and a≠b. By Lemma 1, there is a Hamiltonian path Q in G′[k] joining a and b. Hence P_1 = (u, H_1, v) and P_2 = (x, H_2, a, d, Q, b, H_2, y) are the required paths as shown in Figure 7(a).

The rest of this proof considers the situation that l_1=2 with N_{G'[k]}(u)∩N_{G'[k]}(v) = {x, y} and N_{G'[k]}(u)∩N_{G'[k]}(v) ≠ {x, y}. Since N_{G'[k]}(u)∩N_{G'[k]}(v) \subset N_{G'[k]}(u)∩N_{G'[k]}(v) and v is not in G′[k], there is a vertex r ∈ N_{G'[k]}(u)∩N_{G'[k]}(v) in G′[k], such that u(v) ∈ N_{G'[k]}(r); u and v are r-neighbors, and u and v are r-neighbors. Therefore, u^2 = v^2 = r and d_δ(u, v)=1. Since N_{G'[k]}(u)∩N_{G'[k]}(v) = {x, y} and u and v are r-neighbors, only four vertices u, v, x, and y are connected by r-edge in G′[k] such that d_{r}(u)=d_{r}(v)=d_{r}(x)=d_{r}(y) and k≠4. Without loss of generality, assume that d_{r}(x)=0, d_{r}(y)=1, d_{r}(u)=2, and d_{r}(v)=3, such that u, v, x, and y are included in G′[0], G′[1], G′[2], and G′[3], respectively. There is a vertex d in G′[0]\{x\} such that d≠x.

By Lemma 1, there are two Hamiltonian paths H in G′[0] and Q in G′[1] such that H joins x and d^2, and Q joins a and y. Since l(Q) ≥ 4^{k−1}, Q can be written as (a, Q_1, b, c, v_2, y) for some vertices b and c such that \{b, c\} \subset Q_1 = \varnothing. By Lemma 2, there is a Hamiltonian path R in G′[2]\{u\} joining b and c, and R can be written as (b, R_1, d, e, R_2, c^2) for some vertices d and e. By Lemma 2, there is a Hamiltonian path S in G′[3]\{v\} joining d and e^2, and S can be written as (d_5, S_1, f, g, S_2, e^3) for some vertices f and g. Hence, P_1 = (u, t, v) and P_2 = (x, H, a^2, a^3, Q_1, b, R_1, d, S_1, e^3, f, T, g, S_2, e^3, R_2, Q_2, y) are the required paths as shown in Figure 7(b).

**Theorem 1.** A G(m_1,…, m_4) satisfies 2RP-property, where m_2≥4 for all 1≤s≤r.

**Proof.** If r=1, then G(m_1) is a complete graph of m_1 vertices and thus this theorem holds. Let 4(a) represent 4, 4,…, 4 of length x. On the other hand (r≥2), by Lemma 5, G(4^q) satisfies 2RP-property. According to Lemma 6, G(m_1, 4^{r−1}) satisfies 2RP-property. Since G(m_1, 4^{r−1}) \cong G(4^{r−1}, m_1), G(4^{r−1}, m_1) = G(4^{r−1}, m_1) \cong G(m_1, 4^{r−1}) satisfies 2RP-property, where m_2=m_4=4. Again, by Lemma 6, G(m_1, 4^{r−2}) satisfies 2RP-property, where m_2=m_4≥4. By repeating the deduction above r times, we have that G(m_1, m_2,…, m_r) satisfies 2RP-property, where m_2=m_4≥4 for all 1≤s≤r.

**4 Conclusion**

The 2RP-property of an interconnection network indicates the path embedding capability of the network. This work first demonstrates that a G(m_1, m_2,…, m_r) is 1-Hamiltonian-connected, where m_2≥3 for all 1≤s≤r. Then, by the aids of this
property, our study shows that a $G(m_r, m_{r-1}, \ldots, m_1)$ satisfies 2RP-property, where $m_i \geq 4$ for all $1 \leq i \leq r$.

Acknowledgments

The authors would like to thank the National Science Council of the Republic of China, Taiwan for financially supporting this research under Contract No. NSC-99-2221-E-260-010-.

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