A Procedure for Experimental Evaluation of Cartesian Stiffness Matrix of Multibody Robotic Systems

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Abstract—In this paper a general procedure is proposed for experimental evaluation of Cartesian Stiffness matrix. The proposed procedure is based on a trilateration technique that is applied for a new tracking system named as Milli-CaTraSys (Milli Cassino Tracking System). A case of study for experimental evaluation of the stiffness matrix is presented in order to show the soundness of the proposed procedure.

Keywords- Robotics, Stiffness Matrix, Experimental Mechanics

I. INTRODUCTION

The manipulating performance of a robot is strictly related to stiffness properties of the mechanical design. In fact, if the stiffness of links and joints are inadequate, external forces and moments may cause large deflections in the links, which are undesirable from the viewpoint of both accuracy and payload performance, [1-5]. Thus, numerical and experimental evaluation of stiffness performance should be part of a general procedure for evaluating positioning errors and guaranteeing the effectiveness of a robotic system for given tasks, as proposed for example in [6].

Several stiffness analyses and numerical evaluations of stiffness performance have been carried out for different robotic architectures such as in [1-4, 7-12]. Preliminary comparisons of numerical results with experimental evidence have been proposed for example in [10-12]. Nevertheless, the basic experimental approach consists in measuring one component of the linear compliant displacements for a step external force applied in static conditions. This approach requires one trial for each perturbation direction. Thus, it can be tedious and can be affected by trial-to-trial repeatability errors. Moreover, the measurement of small angular compliant displacements can be very complex due to the lack of sensors having suitably high accuracy.

In this paper a general procedure is proposed in order to evaluate all the linear and angular components of compliant displacements in the overall workspace with only one set-up. The proposed procedure is based on the use of a new tracking system that has been developed at LARM Laboratory of Robotics and Mechatronics in Cassino. It is named Milli-CATRASYS (Milli-Cassino Tracking System) and it is based on the design of a wire tracking system capable of measuring displacements and orientation variations of complex movable multi-body systems. This tracking system has been named as CATRASYS (Cassino Tracking System), since it has been designed and built at LARM, [13] in the form shown in Fig.1.

By using the Milli-CaTraSys connected to the end-effector of a robotic system it is possible to evaluate the linear and angular compliant displacements occurring to a reference frame attached to the end-effector of a robot by measuring only linear distances. Moreover, the Milli-CaTraSys can be used to apply a known wrench on the end-effector of the robot.

The main advantage of using Milli-CaTraSys consists in the possibility of applying a wrench and measuring its effect on a multibody robotic system in terms of compliant displacements at the same time. Moreover, this can be achieved at any pose that is assumed by the robot with the same experimental set-up. Thus, one can compute the 6x6 Cartesian stiffness matrix at any pose of multibody robotic systems just by using Milli-CaTraSys and a suitable formulation.

Fig.1 A prototype of CATRASYS (Cassino Tracking System) at LARM in Cassino.

II. FORMULATION AND OUTLINE OF A PROCEDURE

One can define a 6x1 vector of the external wrench $\mathbf{W}$ and a 6x1 vector of the compliant displacements $\Delta \mathbf{S}$ that are due to the external wrench $\mathbf{W}$ through their components in the form

$$\Delta \mathbf{S} = (\Delta x, \Delta y, \Delta z, \Delta \phi, \Delta \psi, \Delta \theta)^T$$

$$\mathbf{W} = (F_x, F_y, F_z, T_x, T_y, T_z)^T$$

where $T$ is the transpose operator. In the case that the compliant displacements are small enough, one can write the relationship between the vectors $\mathbf{W}$ and $\Delta \mathbf{S}$ as

$$\mathbf{W} = \mathbf{K} \Delta \mathbf{S}$$

where $\mathbf{K}$ is called as Cartesian Stiffness Matrix.

One should note that the stiffness matrix $\mathbf{K}$ can be symmetric if and only if some conditions are satisfied on the external wrench and choice of affine connection as demonstrated for example in [14,15]. It is worth noting that
given a curve \( \gamma(t) \), an affine connection specifies how an element of the tangent space \( \gamma'(t_1) \) can be mapped at a different point \( \gamma(t_2) \), [14]. Therefore, the computation of the 6x6 Cartesian stiffness matrix requires the definition of 36 coefficients in the most general case.

The experimental evaluation of stiffness performances requires the measurement of the compliant displacements due to a known external wrench applied in static conditions. Nevertheless, the computation of 36 parameters cannot be defined through a single experiment. In fact, each experiment can measure a maximum of 6 compliant displacements. Thus, a maximum of 6 equations can be written for each experiment so that at least six experiments with different wrenches should be considered for computing all the 36 unknown coefficients of the stiffness matrix. Moreover, the computation of the 36 unknowns requires an experimental measure also of the 3 angular components of the vector of compliant displacements. This can be very difficult to achieve through commercially available sensors.

A possible procedure for overcoming the problem of lack of suitable sensors can be based on the use of a system suitable for computing the angular compliant displacements by measuring only linear distances. This can be achieved by using the Milli-CaTraSys wire tracking system. This system can be used also for measuring the applied wrench without the need of any additional force sensor. In fact, known masses (or forces) can be applied to the free ends of its wires and the wrench applied to the robot end-effector (due to these masses) can be computed by knowing the configuration assumed by Milli-CaTraSys as reported in the following.

### III. MILLI-CATRASYS AND TRILATERATION TECHNIQUE

The Milli-CATRASYS system is able to measure position and orientation of robot end-effector through reference points H and F on it, Fig.2.

The position of point H is obtained by means of the trilateration technique. Referring to Fig.2 the measured distances \( u_i \) (i=1,…,6) are used as radii of arcs from corresponding center points \( O_i \). The position of H on the robot end-effector is defined as the position of the point at which three arcs intersect. It is function of coordinates \( x_{O_i}, y_{O_i}, z_{O_i} \) of the points \( O_i \), (i=1,2,3), which are the source of the \( u_i \) wire lengths. Figure 3 shows a simplified prototype of Milli-CATRASYS composed of only three LVDTs that has been built at LARM.

In order to determine the robot end-effector position and orientation it is necessary to consider six parameters or know the position of two different points H and F of the robot end-effector, as shown in Fig.4. Using three measurements \( u_1, u_2 \) and \( u_3 \) it is possible to determine the position of H. With two more measures, \( u_4 \) and \( u_5 \), the position of the point F can be determined by measuring the lengths of three edges of a tetrahedron, whose three vertices of the triangular base are \( O_4, O_5 \) and H, and the fourth vertex is F. Thus, it is possible to determine the robot end-effector orientation through the orientation angles \( \alpha \) and \( \beta \) or \( \gamma \) and \( \delta \), as shown in Fig.4. Angle \( \psi \) gives the rotation about HF direction and can represent the roll of the robot end-effector.

![Fig.2](image1.png)  
**Fig.2** A scheme of Milli-CaTraSys showing the reference frame, LVDTs and masses \( m_i \), applied to the wires \( u_i \) (i=1,…,6).

![Fig.3](image2.png)  
**Fig.3** A set-up for experimental tests with the Milli-CATRASYS system by using only 3 LVDT sensors.

![Fig.4](image3.png)  
**Fig.4** A scheme showing how the orientation of the end-effector is defined through the angles \( \alpha \) and \( \beta \) or \( \gamma \) and \( \delta \).
Referring to Fig.2, the linear compliant displacements of point H can be determined from equations

\[ u_i^2 = (x_H - x_{oi})^2 + (y_H - y_{oi})^2 + (z_H - z_{oi})^2 \]  

(3)

where \( i = 1,2,3 \) and \( x_{oi}, y_{oi} \) and \( z_{oi} \) are the Cartesian coordinates of point H with respect to the fixed frame OXYZ.

After suitable algebraic manipulations, Eq.(3) gives

\[
\begin{align*}
    x_H &= H_x = -B_2 - \sqrt{B_2^2 - 4B_3B_1} + E_x \\
    y_H &= H_y = -B_2 - \sqrt{B_2^2 - 4B_3B_1} + E_y \\
    z_H &= H_z = -B_2 - \sqrt{B_2^2 - 4B_1B_3} + E_y
\end{align*}
\]

(4)

where the coefficients can be computed as

\[
\begin{align*}
    B_1 &= H_x^2 + H_y^2 + 1 \\
    B_2 &= 2H_x(E_x - x_3) + 2H_y(E_y - y_3) - 2z_3 \\
    B_3 &= E_x^2 + E_y^2 + A_2^2 - 2E_x x_3 - 2E_y y_3 \\
    A_1 &= u_i^2 - x_i^2 - y_i^2 - z_i^2 \\
    A_2 &= x_{oi} - y_{oi} - z_{oi} \\
    H_x &= \frac{\Delta Z_{21} + H_y \Delta Y_{21} - \Delta X_{21}}{\Delta Z_{21}} \\
    H_y &= \frac{\Delta Z_{21} \Delta X_{32} - \Delta Z_{32} \Delta X_{21}}{\Delta Y_{21} \Delta X_{32}} \\
    H_z &= \frac{\Delta Z_{21}}{\Delta X_{32}} \\
    E_x &= \frac{(A_2 - A_1) \Delta X_{32}}{\Delta Y_{21} \Delta X_{32}} \\
    E_y &= \frac{(A_2 - A_1) \Delta X_{32}}{\Delta Y_{21} \Delta X_{32}}
\end{align*}
\]

(5)

with \( \Delta X_{ij} = x_i - x_j \), \( \Delta Y_{ij} = y_i - y_j \) e \( \Delta Z_{ij} = z_i - z_j \) (\( i,j = 1,2,3 \)). Equations (3) to (5) express a general formulation of the trilateration scheme of Fig.2 when the origin O of the reference frame XYZ does not coincide with any point \( O \). Thus, the linear compliant displacements \( \Delta H_i, \Delta y_i, \Delta z_i \) can be computed by using Eqs.(4) and (5) as differences of the coordinates \( x_H, y_H, z_H \) before and after applying an external wrench.

Using expressions similar to Eqs.(3) to (5) for the measured lengths \( u_i, u_4, u_6 \) and HF, it is possible to determine position of point F through its components \( x_F, y_F, z_F \). Then, the orientation of the segment HF can be usefully described in the space through two angular coordinates \( \alpha \) and \( \beta \), which are evaluated from the geometry of Fig.4 as

\[
\begin{align*}
    \alpha &= \tan^{-1}\left(\frac{(y_H - y_F)}{(x_H - x_F)}\right) \\
    \beta &= \tan^{-1}\left(\frac{z_H - z_F}{\sqrt{(y_H - y_F)^2 + (x_H - x_F)^2}}\right)
\end{align*}
\]

(6)

Referring to Fig.4 the orientation given by angular coordinate \( \beta \) can be better described by means of angles \( \gamma = \tan^{-1}\left(\frac{z_H - z_F}{x_H - x_F}\right) \) and \( \delta = \tan^{-1}\left(\frac{z_H - z_F}{x_H - x_F}\right) \), which represent the rotations of the segment HF about the axes X and Y, respectively, in the form

\[
\begin{align*}
    \gamma &= \tan^{-1}\left(\frac{z_H - z_F}{x_H - x_F}\right) \\
    \delta &= \tan^{-1}\left(\frac{z_H - z_F}{x_H - x_F}\right)
\end{align*}
\]

(7)

An additional measure \( u_6 \) must be used together with known distances FQ and HQ, Fig.4, in order to compute the position of another point Q by using expressions, which are similar to Eqs.(3) to (5). Thus, the \( \psi \) angle about HF can be computed as

\[
\psi = \tan^{-1}\left(\frac{z_Q - z_S}{\sqrt{(y_Q - y_S)^2 + (x_Q - x_S)^2}}\right)
\]

(8)

where point S is the middle point of the HF segment and it can be evaluated by using positions of H and F points.

One can compute the angular compliant displacements \( \Delta \gamma, \Delta \delta, \Delta \psi \) as differences of the angles \( \gamma, \delta, \psi \) before and after applying the external wrench. In fact, wire transducers of LVDT type can be conveniently used in order to obtain distances \( u_i, u_4, u_6, \) \( i = 4,5 \) and \( u_6 \) that are needed to compute Eqs.(3) to (8).

The trilateration method allows also the computation of the angles between the forces applied by each wire. In fact, referring to the schemes in Figs.2 and 5 one can write the angles between the force applied by the wires \( u_i, u_4, u_6 \) as

\[
\begin{align*}
    \alpha_{ui} &= \sin^{-1}\left(\frac{x_H - x_{Oi}}{\sqrt{(x_H - x_{Oi})^2 + (y_H - y_{Oi})^2}}\right) \\
    \delta_{ui} &= \sin^{-1}\left(\frac{z_H - z_{Oi}}{\sqrt{(x_H - x_{Oi})^2 + (y_H - y_{Oi})^2 + (z_H - z_{Oi})^2}}\right)
\end{align*}
\]

(9)

\[
\begin{align*}
    \alpha_{u4} &= \sin^{-1}\left(\frac{x_F - x_{Oj}}{\sqrt{(x_F - x_{Oj})^2 + (y_F - y_{Oj})^2}}\right) \\
    \delta_{u4} &= \sin^{-1}\left(\frac{z_F - z_{Oj}}{\sqrt{(x_F - x_{Oj})^2 + (y_F - y_{Oj})^2 + (z_F - z_{Oj})^2}}\right)
\end{align*}
\]

(10)

where \( i = 1, ..., 3 \).

The angles between the forces applied by the wires \( u_4, u_6 \) can be similarly derived in the form

\[
\begin{align*}
    \alpha_{u6} &= \sin^{-1}\left(\frac{x_F - x_{Oj}}{\sqrt{(x_F - x_{Oj})^2 + (y_F - y_{Oj})^2}}\right) \\
    \delta_{u6} &= \sin^{-1}\left(\frac{z_F - z_{Oj}}{\sqrt{(x_F - x_{Oj})^2 + (y_F - y_{Oj})^2 + (z_F - z_{Oj})^2}}\right)
\end{align*}
\]
The angles between the forces applied by the wire $u_6$ can be also similarly derived in the form:

$$\alpha_{u6} = \sin^{-1}\left(\frac{x_Q - x_{Oh}}{\sqrt{(x_Q - x_{Oh})^2 + (y_Q - y_{Oh})^2}}\right)$$

$$\delta_{u6} = \sin^{-1}\left(\frac{z_H - z_{Oh}}{\sqrt{(x_Q - x_{Oh})^2 + (y_Q - y_{Oh})^2 + (z_Q - z_{Oh})^2}}\right)$$

The angles computed through Eqs.(9) to (11) can be used for computing the wrench applied to the end-effector $W_H=(F_{tx}, F_{ty}, F_{tz}, N_{tx}, N_{ty}, N_{tz})^T$ without requiring any additional sensor. In fact, one can write the relationship between the vector $W_H$ of masses $m_i$ to $m_k$ that are applied to the wires 1 to 6 and the wrench $W_H$ that is due to these masses in the form:

$$W_H = A M_W$$

where the 6x6 matrix $A$ can be written as:

$$A = \begin{bmatrix}
\cos\alpha_{u1} & \cos\alpha_{u2} & \cos\alpha_{u3} & \cos\alpha_{u4} & \cos\alpha_{u5} & \cos\alpha_{u6} \\
\sin\alpha_{u1} & \sin\alpha_{u2} & \sin\alpha_{u3} & \sin\alpha_{u4} & \sin\alpha_{u5} & \sin\alpha_{u6} \\
\sin\delta_{u1} & \sin\delta_{u2} & \sin\delta_{u3} & \sin\delta_{u4} & \sin\delta_{u5} & \sin\delta_{u6} \\
0 & 0 & 0 & \delta_{u1} & \delta_{u2} & \delta_{u3} \\
0 & 0 & 0 & \delta_{u4} & \delta_{u5} & \delta_{u6} \\
0 & 0 & 0 & \delta_{u4} & \delta_{u5} & \delta_{u6}
\end{bmatrix}$$

where $c$ stands for cosine and $s$ for sine; $g$ is the acceleration of gravity and the terms $A_{u1}$ to $A_{u6}$ can be written in the form:

$$A_{u_k} = [(y_Q - y_{Oh})\sin\delta_{u_k} - (z_Q - z_{Oh})\cos\delta_{u_k}] (k=4,5)$$

$$A_{u_k} = [(x_Q - x_{Oh})\cos\delta_{u_k} \sin\alpha_{u_k} - (x_{Oh} - x_{Oh})\sin\delta_{u_k}]$$

$$A_{u_k} = [(x_Q - x_{Oh})\cos\delta_{u_k} \cos\alpha_{u_k} - (y_Q - y_{Oh})\cos\delta_{u_k}]$$

$$A_{u_k} = [(y_Q - y_{Oh})\sin\delta_{u_k} - (z_Q - z_{Oh})\cos\delta_{u_k} \cos\alpha_{u_k}]$$

$$A_{u_k} = [(x_Q - x_{Oh})\cos\delta_{u_k} \cos\alpha_{u_k} - (y_Q - y_{Oh})\cos\delta_{u_k} \sin\alpha_{u_k}]$$

$$A_{u_k} = [(x_Q - x_{Oh})\cos\delta_{u_k} \cos\alpha_{u_k} - (y_Q - y_{Oh})\cos\delta_{u_k} \sin\alpha_{u_k}]$$

It is worth noting that the masses that are needed for applying a desired wrench can be computed by using the inverse of Eq.(12). Thus, the use of Milli-CATRASYS and Eqs.(3) to (14) allow the computation of linear and angular compliant displacements and the applied wrench at the same time. Therefore, the Milli-CATRASYS can be used for estimating the Cartesian stiffness matrix.

IV. THE STIFFNESS MATRIX AS FUNCTION OF EXPERIMENTS

The computation of a 6x6 Cartesian stiffness matrix $K$ can be achieved by considering the experimental results of at least 6 experiments by writing the equations given by Eq.(2) as:

$$\begin{bmatrix}
\delta_{u1} \\
\delta_{u2} \\
\delta_{u3} \\
\delta_{u4} \\
\delta_{u5} \\
\delta_{u6}
\end{bmatrix} = \begin{bmatrix}
\delta_{u1} \\
\delta_{u2} \\
\delta_{u3} \\
\delta_{u4} \\
\delta_{u5} \\
\delta_{u6}
\end{bmatrix} \cdot \begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\
k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\
k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\
k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\
k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\
k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66}
\end{bmatrix}$$

where the $k_{ij}$ coefficients refer to the following stiffness matrix:

$$K = \begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\
k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\
k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\
k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\
k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\
k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66}
\end{bmatrix}$$

The numerical solution of Eq.(15) provides the required values of the 36 coefficients of the stiffness matrix in Eq.(16) once the wrenches (due to known masses) and compliant displacements (due to those wrenches) that have been measured in 6 experiments are available for a given
configuration. Moreover, it is possible to solve Eq.(15) in a symbolic manner in order to obtain the expressions of the 36 coefficients as function of the measured wrenches and compliant displacements as reported in [8].

In some cases the computation of the angular compliant displacements can be neglected. In these cases a simplified approach can be developed by considering only the linear compliant displacements and linear components of the wrenches. In this case, the stiffness matrix $K$ is only 3x3 and it is necessary to measure only the displacements of 3 wires of Milli-CaTraSys. Moreover, the unknown coefficients of the stiffness matrix are 9 only and a minimum number of 3 experiments with different wrenches for a given configuration should be considered in order to compute these unknowns. In fact, Eq.(15) can be simplified as

$$
\begin{bmatrix}
1\Delta x & 1\Delta y & 1\Delta z \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
k'_{11} \\
k'_{12} \\
k'_{13}
\end{bmatrix}
= \begin{bmatrix}
F_x \\
F_y \\
F_z
\end{bmatrix}
= 0
$$

(17)

where the $k'_{ij}$ coefficients refer to the stiffness matrix in the form

$$
K' = \begin{bmatrix}
k'_{11} & k'_{12} & k'_{13} \\
k'_{21} & k'_{22} & k'_{23} \\
k'_{31} & k'_{32} & k'_{33}
\end{bmatrix}
$$

(18)

By solving Eq.(17) in a symbolic manner one can obtain the stiffness parameters as reported in Eq.(A.1) in the appendix.

It is worth noting that it is necessary that the determinant of matrix in Eq.(15) or Eq.(17) is not zero in order to have a number of equations equal to the unknowns. Moreover, it is necessary that the determinant of the stiffness matrix is not zero in order to avoid the computation of a singular stiffness matrix. In fact, a computed value of the determinant of the stiffness matrix $K$ equal to zero means that the rank of the matrix is less than six for the case of Eq.(15) or less then three for the case of Eq.(17). In this configuration one cannot find a unique vector of the compliant displacements for a given static wrench. Therefore, this is a critical situation to avoid since it can correspond to an infinitesimal self motion of a robotic system.

The computation of the 9 coefficients of the stiffness matrix through Eq.(17) (in the more general case, through 36 coefficients of the stiffness matrix reported in Eq.(15)) requires to carry out some experimental tests in which compliant displacements and wrenches should be measured. Then, the comparison of the stiffness matrices that are obtained from experimental tests with the results of a numerical simulation with a stiffness model based on lumped stiffness parameters can validate the proposed model. Moreover, the comparison of the stiffness matrices obtained from the experimental tests and a validated stiffness model can be also used for parameter identification purposes. In fact, the stiffness model can be used in order to compute the stiffness matrix for a given configuration of the robotic system. It is worth noting that one can obtain a system of equations where the only unknowns are the lumped stiffness parameters by comparing this matrix with the one that is obtained from the experimental tests for the same configuration provides.

V. A CASE OF STUDY

Experimental tests have been carried out on a parallel manipulator named as CaPaMan 2 (Cassino Parallel Manipulator 2) that is shown in Fig.6. As shown in the kinematic scheme of Fig.6b), each leg of CaPaMan2 is composed of a four-bar linkage, a revolute joint RJ on the midpoint of the coupler link. Attached to RJ there is a connecting bar CB that is connected to the mobile platform MP by means of a spherical joint BJ. Each four-bar linkage lies on a plane AP that is perpendicular to the fixed platform FP, and forms an angle of $\pi/3$ with each of its neighbors. This manipulator has been already tested as part of a macro-milli serial-parallel system for surgery tasks as shown in Fig. 7 and reported in [16]. A preliminary stiffness analysis of this system has been also reported in [17].

![Fig.6 CaPaMan 2 parallel manipulator: a) a prototype; b) a kinematic scheme.](image-url)
The parallel architecture and small size of this prototype make more difficult to experimentally measure the compliant displacements. Therefore, it has been proposed to use the Milli-CATRASYS system for experimental tests as shown in Fig. 3. In these preliminary tests only three LVDT sensors and three masses $m_1$, $m_2$, $m_3$ have been used in order to apply a wrench having $F_{Hx}$, $F_{Hy}$, $F_{Hz}$ components and experimentally measure the linear components of the compliant displacement vector only. The set-up for the operation of Milli-CATRASYS is shown in the scheme of Fig. 8. In particular, an AT-MIO-16-E-2 Acquisition Card, [18] and a suitable virtual instrument developed in Labview environment, [19], have been used for managing the LVDT sensors of Milli-CATRASYS.

In the set-up of Figs.3 and 9, the use of Eqs.(4-5) and (9-14) becomes fairly simple. In fact, one can write Eqs.(4) and (5) in the form

$$\begin{align*}
x_1 & = 0.01 - 16.67u_2 + 8.34u_3 + 8.33u_1 - 0.0067\sqrt{DD} \\
y_1 & = 0.01 - 8.34u_3 + 16.67u_1 - 8.33u_2 + 0.0067\sqrt{DD} \\
z_1 & = -0.01 + 8.33u_2 - 16.67u_3 + 8.33u_1 - 0.0067\sqrt{DD} \\
D & = 1 + 2.50 \times 10^{-3}(u_1 + u_2 + u_3) + \\
& + 0.31 \times 10^{-7}(u_1u_2 + u_2u_1 + u_1u_3 - u_1^2 - u_2^2 - u_3^2) \\
\end{align*}$$

(19)

The linear compliant displacements can be also computed by using Eq.(19). In fact, the linear compliant displacements $\Delta x_H$, $\Delta y_H$, $\Delta z_H$ are the differences of coordinates $x_H$, $y_H$, $z_H$ before and after applying an external wrench, respectively.

It is worth noting that in Eq.(19) the origin of the world reference frame has been assumed as coincident with origin of the wire $O_1$ of the first LVDT as shown in the scheme of Fig.9. Moreover, referring to the set-up of Fig. 9 the distances $Lx$, $Ly$ and $Lz$ have been assumed equal to 0.02 m, 0.02 m and 0.02 m, respectively. These, dimensions have been chosen in order to have angles $\delta_i$ and $\alpha_i$ as close as possible to 0 or 90 degrees with $i=1,...,3$.

Referring to the set-up shown in Fig.3, Eq.(12) can be simplified by computing only the linear components of the wrench applied on the point $H$ of the movable plate of CaPaMan2 in the form

$$W = \begin{bmatrix} F_{Hx} \\ F_{Hy} \\ F_{Hz} \end{bmatrix} = \begin{bmatrix} \cos \alpha_1 \cos \delta_1 \\ \cos \alpha_2 \cos \delta_2 \\ \cos \alpha_3 \cos \delta_3 \end{bmatrix} \begin{bmatrix} g \delta_1 \\ g \delta_2 \\ g \delta_3 \end{bmatrix}$$

(20)

It is worth noting that in the initial configuration the angles $\alpha_1$, $\delta_1$, and $\delta_2$ are equal to zero, the angles $\alpha_2$, and $\alpha_3$ are close to 90 deg., the angle $\alpha_3$ is assumed to be equal to 0 deg.
In any other configuration the angles $\alpha_{u1}$, $\alpha_{u2}$, $\alpha_{u3}$, $\delta_{u1}$, $\delta_{u2}$ and $\delta_{u3}$ can be computed in the form

$$\alpha_{u1} = \sin^{-1}\left(\frac{x_H}{\sqrt{x_H^2 + y_H^2}}\right)$$
$$\delta_{u1} = \sin^{-1}\left(\frac{z_H}{\sqrt{x_H^2 + y_H^2 + z_H^2}}\right)$$
$$\alpha_{u2} = \sin^{-1}\left(\frac{0.02 - x_H}{\sqrt{(0.02 - x_H)^2 + (0.02 - y_H)^2}}\right)$$
$$\delta_{u2} = \sin^{-1}\left(\frac{z_H}{\sqrt{(0.02 - x_H)^2 + (0.02 - y_H)^2 + z_H^2}}\right)$$
$$\alpha_{u3} = \sin^{-1}\left(\frac{x_H}{\sqrt{x_H^2 + (0.02 - y_H)^2}}\right)$$
$$\delta_{u3} = \sin^{-1}\left(\frac{0.02 - z_H}{\sqrt{x_H^2 + (0.02 - y_H)^2 + (0.02 - z_H)^2}}\right)$$

Equations (21) and (20) can be easily computed by using the values of the coordinates $x_H$, $y_H$, $z_H$ that are obtained through Eq.(19) after applying any set of external masses $m_1$, $m_2$ and $m_3$.

By using the set-up of Milli-CATRASYS proposed in Figs.3 and 9 it has been possible to measure the linear compliant displacements and applied wrenches at the same time. In particular, Fig.10 shows the measured compliant displacements for $m_1=0.1$ Kg, $m_2=0$ Kg, $m_3=0$ Kg when CaPaMan2 is in its vertical configuration. The maximum compliant displacement in X direction $\Delta x_H$ is about 0.28 mm while the other components can be considered as negligible. In this case, the maximum component of the wrench computed through Eq.(20) is $F_{Hx}=0.98$ N while the other components can be considered as negligible. Figure 11 shows the compliant displacements for $m_1=0$ Kg, $m_2=0.1$ Kg, $m_3=0$ Kg. The maximum compliant displacement in Y direction $\Delta y_H$ is about 0.28 mm while the other components can be considered as negligible. In this case, the maximum component of the wrench computed through Eq.(20) is $F_{Hy}=0.98$ N while the other components can be considered as negligible. Figure 12 shows the compliant displacements for $m_1=0$ Kg, $m_2=0$ Kg, $m_3=0.1$ Kg. The maximum compliant displacement in Z direction $\Delta z_H$ is about 0.50 mm while the other components can be considered as negligible. In this case, the maximum component of the wrench computed through Eq.(20) is $F_{Hz}=0.98$ N while the other components can be considered as negligible. It is worth noting that the plots of Figs.10 to 12 show the measured compliant displacements versus time. This is necessary in order to find the stationary value of the measured compliant displacements after applying the external wrench.

By using the results of the experimental tests together with Eq.(18) and Eq.(A.1) in the appendix one can compute the 3x3 stiffness matrix of CaPaMan2 in its vertical configuration as

$$K = \begin{pmatrix} 3500 & 0 & 0 \\ 0 & 3500 & 0 \\ 0 & 0 & 1960 \end{pmatrix}$$

A 3x3 stiffness matrix can be similarly derived for any other configuration without any change in the set-up of Milli-CATRASYS. This is one of the main advantages of the proposed procedure and formulation. In fact, it is possible to evaluate the stiffness matrix for several different configurations just by moving the robot in those configurations.

Fig.10 Measured compliant displacements for a wrench equal to $m_1=0.1$ Kg, $m_2=0$ Kg; $m_3=0$ Kg when CaPaMan2 is in its vertical configuration: a) $\Delta x$; b) $\Delta y$; c) $\Delta z$. 
Fig. 11 Measured compliant displacements for a wrench equal to $m_1=0$ Kg, $m_2=0.1$ Kg; $m_3=0$ Kg when CaPaMan2 is in its vertical configuration: a) $\Delta x$; b) $\Delta y$; c) $\Delta z$.

Fig. 12 Measured compliant displacements for a wrench equal to $m_1=0$ Kg, $m_2=0$ Kg; $m_3=0.1$ Kg when CaPaMan2 is in its vertical configuration: a) $\Delta x$; b) $\Delta y$; c) $\Delta z$. 
In this paper a procedure has been presented for the experimental evaluation of the stiffness matrix by means of the new Milli-CATRASYS tracking system and a trilateration technique. One of the main advantages of the proposed procedure is the evaluation of the stiffness matrix of a robotic system for several different configurations by using the same set-up. Moreover, Milli-CATRASYS gives the possibility of computing small linear and angular compliant displacements by measuring changes of distances only. The effectiveness of the proposed procedure has been preliminary tested on a built prototype of the parallel manipulator named as CaPaMan2.

REFERENCES