# Time Preference and the Distributions of Wealth and Income

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SUEN: TIME PREFERENCE AND INEQUALITY

**Abstract**

This paper examines the connection between time preference heterogeneity and economic inequality in a deterministic environment. Specifically, we extend the standard neoclassical growth model to allow for (i) heterogeneity in consumers' discount rates, (ii) direct preferences for wealth, and (iii) human capital formation. The second feature prevents the wealth distribution from collapsing into a degenerate distribution. The third feature generates a strong positive correlation between earnings and capital income across consumers. A calibrated version of the model is able to generate patterns of wealth and income inequality that are very similar to those observed in the United States.

*Keywords*: Inequality, Heterogeneity, Time Preference, Human Capital

*JEL classification*: D31, E21, O15.

*I would like to thank James Davies, Nezih Guner (the editor), Jang-Ting Guo, Karen Kopecky, Jim MacGee, Pierre-Daniel Sarte, Ping Wang, two anonymous referees, seminar participants at the University of Western Ontario, conference participants at the 2010 Midwest Macro Meetings and the 2010 CEA Annual Conference for helpful comments and suggestions.

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## I. INTRODUCTION

Empirical studies show that individuals do not discount future values at the same rate.[1] Since individuals' asset accumulation and schooling choices are strongly affected by the way they discount the future, this type of heterogeneity would naturally lead to cross-sectional differences in wealth and income. To examine the connection between time preference heterogeneity and economic inequality, this study develops a dynamic competitive equilibrium model in which consumers differ only in terms of their discount rates. It is shown that a calibrated version of the model can generate patterns of wealth and income inequality that are very similar to those observed in the United States.

The importance of time preference heterogeneity in explaining wealth inequality is well recognized in existing studies. There is now a vast literature in macroeconomics that uses the incomplete markets model of Huggett (1993, 1996) and Aiyagari (1994) to explain wealth and income inequality.[2] The standard incomplete markets model, however, has difficulty in explaining certain features of the wealth distribution in the United States. In particular, it fails to generate a high concentration of wealth at the top end of the wealth distribution.[3] Krusell and Smith (1998) show that introducing time preference heterogeneity can significantly improve the Aiyagari (1994) model in this regard. Similarly, Hendricks (2007) shows that introducing this type of heterogeneity into the life-cycle model of Huggett (1996) can improve the model's ability to account for wealth inequality.

In both Krusell and Smith (1998) and Hendricks (2007), cross-sectional variation in income is mainly driven by uninsurable idiosyncratic earnings risk, which is exogenous and independent of the heterogeneity in discount rates. These two sources of consumer heterogeneity are then used to account for the wide dispersion in wealth. This approach, however, ignores the effects of time preferences on lifetime earnings. Intuitively, more patient individuals are more willing to invest in financial assets and human capital than less patient ones. A higher level of human capital then leads to a higher level of lifetime earnings for those who are more patient. This intuition is consistent with empirical findings. Lawrance (1991) and Warner and Pleeter (2001) find that more-educated households and individuals tend to have lower discount rates than less-educated ones. This connection between patience and educational attainment suggests that human capital formation may provide an additional channel through which time preference heterogeneity can give rise to wealth and income inequality.

[1] A detailed review of these studies can be found in Frederick *et al.* (2002) Section 6.
[2] An excellent review of this literature can be found in Heathcote *et al.* (2009).
[3] See Castañeda *et al.* (2003) for a detailed discussion of this problem.

The main objective of this study is to explore the quantitative implications of this additional channel. To achieve this, we generalize the standard deterministic neoclassical growth model to allow for three additional features, namely (i) heterogeneity in time preference, (ii) human capital formation, and (iii) consumers' direct preferences for wealth. The assumption of direct wealth preference has long been used in economic studies. In an early paper, Kurz (1968) introduces wealth preference into the optimal growth model and explores the long-run properties of the model. Zou (1994) interprets this type of preference as reflecting the "capitalist spirit," or the tendency to treat wealth acquisition as an end in itself rather than a means of satisfying material needs. Cole *et al.* (1992) suggest that the inclusion of financial wealth in consumers' preferences can be viewed as a reduced-form specification to capture people's concern for their wealth-induced status within society. Subsequent studies have followed these traditions and interpreted this type of preference as either capturing the spirit of capitalism or reflecting the demand for wealth-induced status. In this paper, we refer to this feature simply as wealth preference. There is now a rapidly growing literature that explores the implications of wealth preference on a wide range of issues, such as asset pricing, economic growth, expectations-driven business cycles, effects of fiscal policy and wealth inequality.[4]

The main purpose of introducing wealth preference in our model is as follows. It is now well known that the standard neoclassical growth model has difficulty in generating realistic wealth distribution based on differences in discount rates alone. Becker (1980) shows that when consumers have time-additive separable preferences and different constant discount rates, all the wealth in the neoclassical world will eventually be concentrated in the hands of the most patient consumers. In other words, the wealth distribution is degenerate and extremely unequal in the long run. Several existing studies have identified conditions under which the long-run wealth distribution is non-degenerate.[5] In this study, we show that a non-degenerate wealth distribution can be obtained by assuming that consumers

[4] Studies that explore the implications of wealth preference on asset pricing include Bakshi and Chen (1996), and Boileau and Braeu (2007) among others. Studies on economic growth include Zou (1994) and Smith (1999) among others. Karnizova (2010) introduces this type of preference into a neoclassical growth model with capital adjustment costs and shows that the model can generate expectations-driven business cycles. Gong and Zou (2002) and Nakamoto (2009) examine the welfare implications of fiscal policy when consumers value wealth directly. Finally, Luo and Young (2009) explore the implications of wealth preference on wealth inequality. This study will be discussed in greater detail later.

[5] Boyd (1990) shows that Becker's result is no longer valid when consumers have recursive preferences. Sarte (1997) establishes the existence of a non-degenerate wealth distribution by introducing a progressive tax structure into Becker's model. Sorger (2002) shows that Becker's result cannot be extended to the case where consumers are strategic players, rather than price-takers, in the capital market. Espino (2005) establishes a non-degenerate wealth distribution by assuming that consumers have private information over an idiosyncratic preference shock. Except for Sarte (1997), none of these studies have explored the quantitative implications of their model. Sarte shows that a calibrated version of his model can replicate the income distribution in the United States. However, unlike the current study, he does not attempt to explain wealth and income inequality simultaneously.

have direct preferences for wealth. The intuition behind this result can be explained as follows. In the original Becker (1980) model where there is no direct wealth preference, a consumer will choose to hold a constant positive level of financial wealth only when the equilibrium interest rate is identical to his discount rate. Since there is only one interest rate in the neoclassical model, it is not possible for consumers with different discount rates to maintain constant positive levels of wealth simultaneously. In the long-run equilibrium, interest rate is equated to the lowest discount rate in the population. Thus, only the most patient consumers would have positive asset holdings. All other consumers with discount rate greater than the equilibrium interest rate will deplete their wealth until it reaches zero. Thus, the long-run wealth distribution in the Becker (1980) model is extremely unequal. Introducing direct preferences for wealth changes this result by creating some additional benefits of holding financial assets. Because of these additional benefits, consumers are now willing to maintain constant positive levels of wealth even if the interest rate is lower than their discount rates. These additional benefits also induce different types of consumers to hold different levels of wealth. This gives rise to a non-degenerate wealth distribution in the long-run equilibrium.

In the quantitative analysis, we find that consumers' direct preferences for wealth play a crucial role in explaining wealth inequality. In particular, our results show that a model with time preference heterogeneity and wealth preference *alone* can replicate some key features of the wealth distribution in the United States (see Section V). Such a model, however, cannot produce large variations in earnings across consumers. This type of variation is important in explaining income inequality because earnings account for a large fraction of individual income in the model economy. Consequently, a model with only time preference heterogeneity and wealth preference cannot explain the observed patterns of wealth *and* income inequality simultaneously. Introducing human capital formation helps improve this result in two ways. First, consumers' earnings are now tied to their discount rates through the investment in human capital. This provides a channel via which time preference heterogeneity can lead to significant variations in earnings across consumers. Second, introducing human capital helps create a strong positive correlation between earnings and capital income. This happens because more patient consumers have higher earnings and more financial wealth than less patient ones. A calibrated version of the model with all three features is able to replicate the observed patterns of wealth and income inequality in the United States.[6]

[6] We do not claim that other factors, such as life-cycle factors, income uncertainty, precautionary savings, redistributive taxation and transfer programs, are not important in understanding economic inequality. The main purpose of the numerical exercise is to illustrate the quantitative relevance of the mechanism captured by this model in explaining

The current study differs from Krusell and Smith (1998) in three important ways: First, the current study aims to explain *both* wealth and income inequality using only one source of consumer heterogeneity, namely differences in discount rates. Second, the current study takes into account the endogenous components of labor income, namely endogenous labor hours and human capital formation. Third, instead of assuming that individuals' discount rates are stochastic and idiosyncratic in nature, the current study focuses on fixed, predetermined differences in discount rates across individuals.[7]

This study is also close in spirit to Luo and Young (2009) in the sense that both studies analyze wealth and income inequality in the presence of wealth preference. There are two major differences between the two studies. First, the source of consumer heterogeneity is different in the two models. In Luo and Young (2009), consumers share the same discount rate but face idiosyncratic uncertainty in labor productivity as in the Aiyagari (1994) model. Thus, this study does not consider the effects of time preference heterogeneity on wealth and income inequality. Second, the earnings distribution in the two models are determined by different factors. In Luo and Young (2009), earnings are jointly determined by labor productivity shock and consumers' labor-leisure choices. Their model does not include human capital formation. Despite these differences in model specification, both studies find that wealth preference is a force that tends to *reduce* wealth inequality. In our model, this tendency is manifested in two ways. First, the equilibrium wealth distribution is no longer extremely unequal once we introduce wealth preference into Becker's model. Second, in the quantitative analysis, we find that the degree of wealth inequality decreases as we increase the coefficient that controls the strength of wealth preference. Similar results are also reported in Luo and Young (2009).

The rest of this paper is organized as follows. Section II describes the model environment. Section III describes the benchmark parameter values that we use in the quantitative exercise. Section IV presents the benchmark results. Section V discusses the main determinants of wealth and income inequality in the benchmark economy. This is followed by some concluding remarks in Section VI.

economic inequality.

[7]Existing studies show that predetermined factors (or ex ante heterogeneity) are at least as important as idiosyncratic shocks (or ex post heterogeneity) in explaining cross-sectional variation in lifetime utility. Keane and Wolpin (1997) argue that as much as 90 percent of the dispersion in lifetime utility can be attributed to predetermined, fixed factors. The remaining ten percent is attributed to exogenous idiosyncratic shocks. More recently, Huggett *et al.* (2011) find that predetermined factors are more important in explaining the dispersion in lifetime earnings and lifetime wealth than idiosyncratic shocks.

## II. MODEL

### *A. Consumers*

Consider an economy populated by $N > 1$ groups of infinitely-lived consumers. Each group is indexed by a subjective discount factor $\beta_i$, for $i \in \{1, 2, ..., N\}$. The discount factors can be ranked according to $0 < \beta_1 \leq \beta_2 \leq ... \leq \beta_N < 1$. Consumers within the same group are identical in all aspects. The share of type-$i$ consumers in the population is given by $\lambda_i \in (0, 1)$. The size of total population is constant and is normalized to one, hence $\sum_{i=1}^{N} \lambda_i = 1$.

There is a single commodity in this economy which can be used for consumption and investment. The consumers' preferences are represented by

$$\sum_{t=0}^{\infty} \beta_i^t u\left(c_{i,t}, k_{i,t}\right),$$

where $c_{i,t}$ is the consumption of a type-$i$ consumer at time $t$, and $k_{i,t}$ is the stock of physical capital owned by the consumer at the beginning of time $t$. The (period) utility function $u\left(c, k\right)$ is identical for all types of consumers and is given by

$$u\left(c, k\right) = \frac{c^{1-\sigma}}{1-\sigma} + \theta \frac{k^{1-\sigma}}{1-\sigma}, \tag{1}$$

with $\sigma > 0$ and $\theta \geq 0$.[8] The parameter $\theta$ captures the importance of wealth preference in the utility function. In particular, a higher value of $\theta$ means that the same increase in wealth would generate a larger gain in utility. If $\theta = 0$, then the consumers are said to have no direct preference for wealth.

In each period, each consumer is endowed with one unit of time which can be divided between market work and on-the-job training. Consider a type-$i$ consumer with human capital $h_{i,t}$ at the beginning of time $t$. If he spends a fraction $l_{i,t} \in [0, 1]$ of time on market work during the period, then his earnings are given by $w_t l_{i,t} h_{i,t}$. We refer to $l_{i,t} h_{i,t}$ as effective unit of labor hours. The variable $w_t$ is the market wage rate for an effective unit of labor. The consumer also receives $\varphi\left(1 - l_{i,t}\right)^{\epsilon} h_{i,t}^{\varsigma}$ units of newly produced human capital, where $\varphi > 0$, $\epsilon \in (0, 1)$ and $\varsigma \in (0, 1)$. Human capital at time $t + 1$ is then given by

$$h_{i,t+1} = \varphi\left(1 - l_{i,t}\right)^{\epsilon} h_{i,t}^{\varsigma} + \left(1 - \delta_h\right) h_{i,t}, \tag{2}$$

[8] This type of utility function is commonly used in the literature of wealth preference. See, for instance, Zou (1994), Gong and Zou (2002), and Luo and Young (2009) among others.

where $\delta_h \in (0,1)$ is the depreciation rate of human capital.

Besides labor income, consumers also receive interest income from their previous savings. All savings are held in the form of physical capital, which is the only tangible asset in this economy. The effective rate of return from savings is $(r_t - \delta_k)$, where $r_t$ is the rental rate of physical capital at time $t$, and $\delta_k \in (0,1)$ is the depreciation rate of physical capital. As in Becker (1980), consumers are not allowed to borrow in every period.

Given a sequence of wage rates and rental rates, the consumers' problem is to choose a sequence of consumption, labor hours, physical capital and human capital so as to maximize their lifetime utility, subject to the sequential budget constraints, borrowing constraints and the law of motion for human capital. For each type-$i$ consumer, this problem can be expressed as

$$\max_{\{c_{i,t},l_{i,t},k_{i,t+1},h_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t u\left(c_{i,t},k_{i,t}\right) \tag{P1}$$

subject to

$$c_{i,t} + k_{i,t+1} - \left(1-\delta_k\right) k_{i,t} = w_t l_{i,t} h_{i,t} + r_t k_{i,t}, \tag{3}$$

$$h_{i,t+1} = \varphi\left(1 - l_{i,t}\right)^{\epsilon} h_{i,t}^{\varsigma} + \left(1-\delta_h\right) h_{i,t},$$

$$k_{i,t+1} \geq 0, \qquad l_{i,t} \in [0,1],$$

and the initial conditions: $k_{i,0} > 0$ and $h_{i,0} > 0$.

The optimal choices are determined by (2), (3) and the first-order conditions, which can be expressed as

$$\left(c_{i,t}\right)^{-\sigma} \geq \beta_i \left[\theta\left(k_{i,t+1}\right)^{-\sigma} + \left(1 + r_{t+1} - \delta_k\right)\left(c_{i,t+1}\right)^{-\sigma}\right], \tag{4}$$

with equality holds if $k_{i,t+1} > 0$,

$$w_t h_{i,t}\left(c_{i,t}\right)^{-\sigma} = \chi_{i,t}\epsilon\varphi\left(1 - l_{i,t}\right)^{\epsilon-1} h_{i,t}^{\varsigma}, \tag{5}$$

$$\chi_{i,t} = \beta_i \chi_{i,t+1}\left\{\varphi\left(1 - l_{i,t+1}\right)^{\epsilon-1} h_{i,t+1}^{\varsigma-1}\left[\varsigma\left(1 - l_{i,t+1}\right) + \epsilon l_{i,t+1}\right] + \left(1-\delta_h\right)\right\}. \tag{6}$$

Equation (4) is the Euler equation for consumption. Introducing direct preferences for wealth creates some additional benefits for holding assets. These additional benefits are captured by the term

$\theta\,(k_{i,t+1})^{-\sigma} > 0$ in (4). If consumers have no direct preference for wealth (i.e., $\theta = 0$), then the Euler equation is identical to the one in Becker (1980). Equation (5) is the optimality condition for choosing labor hours. This equates the marginal benefits of working longer hours, $w_t h_{i,t}\,(c_{i,t})^{-\sigma}$, to the marginal costs, $\chi_{i,t}\epsilon\varphi\,(1 - l_{i,t})^{\epsilon-1}\,h_{i,t}^{\varsigma}$. The variable $\chi_{i,t}$ is the Lagrangian multiplier for (2), which is often interpreted as the shadow value of human capital. Equation (6) determines the dynamics of $\chi_{i,t}$.

### *B. Production*

Output is produced according to a Cobb-Douglas production function:

$$Y_t = K_t^{\alpha}\,(X_t L_t)^{1-\alpha}, \qquad \text{with } \alpha \in (0,1),$$

where $Y_t$ denote aggregate output at time $t$, $K_t$ is aggregate capital, $L_t$ is aggregate labor hours, and $X_t$ is the level of labor-augmenting technology. We will refer to $\widehat{L}_t \equiv X_t L_t$ as effective unit of aggregate labor. The technological factor is assumed to grow at a constant exogenous rate so that $X_t \equiv \gamma^t$ for all $t$, where $\gamma \geq 1$ is the exogenous growth factor and $X_0$ is normalized to one.[9]

Since the production function exhibits constant returns to scale, we can focus on a representative firm whose problem is given by

$$\max_{K_t, L_t} \left\{ K_t^{\alpha}\,(X_t L_t)^{1-\alpha} - w_t L_t - r_t K_t \right\},$$

for any $t \geq 0$. The solution of this problem is completely characterized by the first-order conditions:

$$w_t = (1-\alpha)\,X_t K_t^{\alpha}\,(X_t L_t)^{-\alpha}, \tag{7}$$

$$r_t = \alpha K_t^{\alpha-1}\,(X_t L_t)^{1-\alpha}. \tag{8}$$

### *C. Competitive Equilibrium*

Let $\mathbf{c}_t = (c_{1,t}, c_{2,t}, ..., c_{N,t})$ be a distribution of consumption across groups at time $t$. Similarly, define $\mathbf{k}_t$, $\mathbf{h}_t$ and $\mathbf{l}_t$ as the distributions of physical capital, human capital and labor hours at time $t$, respectively.

[9] Unlike the endogenous growth model considered in Lucas (1988), human capital accumulation does not serve as the engine of growth in this model. This is due to the condition $\varsigma \in (0,1)$, which implies diminishing marginal returns to $h_{i,t}$ in the production of human capital. The main purpose of introducing human capital in this study is to increase the cross-sectional variation in earnings.

Given the initial distributions $\mathbf{k}_0$ and $\mathbf{h}_0$, a competitive equilibrium for this economy consists of a sequence of distributions, $\{\mathbf{c}_t, \mathbf{k}_t, \mathbf{l}_t, \mathbf{h}_t\}_{t=0}^{\infty}$, a sequence of aggregate inputs, $\{K_t, L_t\}_{t=0}^{\infty}$, and a sequence of prices, $\{w_t, r_t\}_{t=0}^{\infty}$, so that

(i) Given the prices, the sequence $\{c_{i,t}, k_{i,t}, l_{i,t}, h_{i,t}\}_{t=0}^{\infty}$ solves each type-$i$ consumer's problem.

(ii) Given the prices, the sequence $\{K_t, L_t\}_{t=0}^{\infty}$ solves the representative firm's problem in each period, i.e., (7) and (8) are satisfied for all $t \geq 0$.

(iii) All markets clear in every period, i.e.,

$$K_t = \sum_{i=1}^{N} \lambda_i k_{i,t} \qquad \text{and} \qquad L_t = \sum_{i=1}^{N} \lambda_i l_{i,t} h_{i,t}, \qquad \text{for all } t \geq 0.$$

In the following analysis, we confine our attention to balanced-growth equilibria which are independent of the initial conditions. Thus, the initial distributions $\mathbf{k}_0$ and $\mathbf{h}_0$ are irrelevant to our analyses. A balanced-growth equilibrium is a sequence $\mathcal{S} = \{\mathbf{c}_t, \mathbf{k}_t, \mathbf{l}_t, \mathbf{h}_t, K_t, L_t, w_t, r_t\}_{t=0}^{\infty}$ such that

(i) $\mathcal{S}$ is a competitive equilibrium as defined above.

(ii) The rental rate of physical capital is stationary over time, i.e., $r_t = r^*$ for all $t$.

(iii) The distributions of labor hours and human capital are stationary over time.

(iv) Individual consumption and asset holdings, aggregate output, aggregate capital and wage rate are all growing at the same constant rate. In particular, the common growth factor is $\gamma \geq 1$.

Define $\widehat{k}_t^d \equiv K_t / (X_t L_t)$ as aggregate capital per effective unit of aggregate labor at time $t$. Using (8) we can express this as a function of $r$,

$$\widehat{k}^d(r) \equiv \left(\frac{\alpha}{r}\right)^{\frac{1}{1-\alpha}}. \tag{9}$$

Substituting (9) into (7) gives

$$\omega_t(r) \equiv (1-\alpha)\gamma^t \left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}}. \tag{10}$$

Intuitively, if $r$ is the equilibrium rental rate, then $\omega_t(r)$ is the corresponding equilibrium wage rate at time $t$. Define the transformed variables: $\widehat{c}_i \equiv c_{i,t}/\gamma^t$, $\widehat{k}_i \equiv k_{i,t}/\gamma^t$ and $\widehat{w}(r) \equiv \omega_t(r)/\gamma^t$. Along any balanced growth path, the values of $\left\{\widehat{c}_i, \widehat{k}_i, l_i, h_i\right\}_{i=1}^{N}$ and the rental rate $r^*$ are determined by

$$\widehat{c}_i = \widehat{w}\left(r^*\right) l_i h_i + \left(r^* - \widehat{\delta}_k\right) \widehat{k}_i, \tag{11}$$

$$\frac{\gamma^{\sigma}}{\beta_i} - (1 - \delta_k) - r^* \geq \theta \left(\frac{\widehat{c}_i}{\widehat{k}_i}\right)^{\sigma}, \tag{12}$$

which holds with equality if $\widehat{k}_i > 0$,

$$\frac{l_i}{1 - l_i} = \frac{1}{\epsilon} \left\{ \frac{1}{\delta_h} \left[ \frac{\gamma^{\sigma - 1}}{\beta_i} - (1 - \delta_h) \right] - \varsigma \right\}, \tag{13}$$

$$h_i = \left[ \frac{\varphi}{\delta_h} \left(1 - l_i\right)^{\epsilon} \right]^{\frac{1}{1-\varsigma}}, \tag{14}$$

$$\sum_{i=1}^{N} \lambda_i \widehat{k}_i = \left( \sum_{i=1}^{N} \lambda_i l_i h_i \right) \widehat{k}^d \left(r^*\right), \tag{15}$$

where $\widehat{\delta}_k \equiv \gamma - 1 + \delta_k$. Equations (11) and (12) can be obtained from (3) and (4), respectively, after imposing the balanced-growth conditions. Similarly, (14) can be obtained from (2) after imposing these conditions. Equation (13) then follows from (6) and (14). The details of this can be found in the appendix. Finally, (15) is the capital market clearing condition.

Under some mild regularity conditions, there exists a unique balanced-growth equilibrium for this economy. This unique equilibrium has two important properties. First, the borrowing constraint is not binding for all types of consumers. Thus, the Euler equation in (12) holds with equality for all $i$. Second, unlike the Becker (1980) model, the equilibrium wealth distribution in this economy is non-degenerate. The formal proof of these results are shown in the appendix. Here we focus on the intuitions behind the non-degenerate wealth distribution.

To explain the differences between our result and Becker's result, it suffice to contrast the Euler equations in the two models. Set $\gamma = 1$ for the moment. In the original Becker (1980) model, where the consumers have no direct preference for wealth, the Euler equation is given by

$$\rho_i \equiv \frac{1}{\beta_i} - 1 \geq r^* - \delta_k, \tag{16}$$

which holds with equality if $\widehat{k}_i > 0$. The parameter $\rho_i$ is the discount rate or rate of time preference for a type-$i$ consumer. This equation states that a consumer with no direct preference for wealth will invest according to the following rules: (i) accumulate assets indefinitely if the effective rate of return $(r^* - \delta_k)$ exceeds his rate of time preference, (ii) run down his assets to zero (the lower bound) if the effective rate of return is lower than his rate of time preference, and (iii) maintain a constant positive amount of assets if the two are equal. Since all consumers face the same effective rate of return from savings, it is not possible for consumers with different discount rates to maintain a constant positive amount of assets simultaneously. In addition, no one can accumulate assets indefinitely in a stationary equilibrium. Thus, the effective rate of return must be equated to the lowest rate of time preference in the population. In other words, only the most patient group of consumers will have positive asset holdings in a stationary equilibrium. All other groups of consumers will run down their wealth until it reaches zero.

Introducing direct preferences for wealth breaks this spell by creating some additional benefits of holding wealth. These additional benefits fundamentally change the consumers' saving behavior. In particular, a consumer is now willing to maintain a constant positive level of assets even if the effective rate of return is lower than his rate of time preference. This can be seen from the Euler equation in the current model. Substituting (11) into (12) gives

$$\rho_i - (r^* - \delta_k) = \theta \left[ \frac{\widehat{w}(r^*) \, l_i h_i}{\widehat{k}_i} + r^* - \widehat{\delta}_k \right]^{\sigma} > 0. \tag{17}$$

It is now possible to obtain a non-degenerate wealth distribution because different types of consumers will choose different values of $\widehat{k}_i$ according to (17).

By comparing (16) and (17), it is obvious that direct preferences for wealth (i.e., $\theta > 0$) is the main factor that prevents the wealth distribution from collapsing into a degenerate distribution. The accumulation of human capital is not essential for this result. In particular, a non-degenerate wealth distribution can be obtained in a model with $\theta > 0$ but without human capital formation.[10]

Before concluding this section, we want to highlight the key features of the distributions of labor hours and human capital. In the unique balanced-growth equilibrium, the values of $\{l_i, h_i\}_{i=1}^N$ can be obtained by solving (13) and (14). These equations show that the distributions of labor hours and human capital are non-degenerate, and are independent of the preference parameter $\theta$. Thus, changing this parameter value would have no impact on the distributions of labor hours, human capital and

[10] In the working paper version of this paper, we begin by introducing wealth preference into the original Becker (1980) model, where there is no human capital formation and labor supply is exogenous. It is shown that a unique balanced-growth equilibrium exists, and the equilibrium wealth distribution is non-degenerate.

earnings. The values of $\{l_i, h_i\}_{i=1}^N$ are also independent of the equilibrium rental rate $r^*$ and the consumers' asset holdings $\left\{\widehat{k}_i\right\}_{i=1}^N$. Thus, in the stationary equilibrium, the distribution of earnings is not affected by the consumers' savings decisions.

## III. Parameter Values

To examine the extent of income and wealth inequality that can be generated by the model, we solve it numerically using the parameter values indicated in Table 1. Most of these values are chosen based on empirical findings. Others are chosen to match certain real-world statistics. The details of this procedure are explained below.

One period in the model is a year. The share of capital income in total output ($\alpha$) is 0.33. The growth rate of per-capita variables ($\gamma - 1$) is 2.2 percent, which is the average annual growth rate of real per-capita GDP in the United States over the period 1950-2000. In the benchmark scenario, the parameter $\sigma$ in the utility function is set to one. Results obtained under different values of $\sigma$ are reported in Section V. The range of subjective discount factors is chosen based on the estimates in Lawrance (1991). Using data from the Panel Study of Income Dynamics over the period 1974-1982, Lawrance (1991) estimates that the average rate of time preference for households in the bottom five percent of the income distribution is 3.5 percent, after controlling for differences in age, educational level and race. This implies an average discount factor of 1/(1+0.035)=0.966. The estimated rate of time preference for the richest five percent is 0.8 percent, which gives a discount factor of 0.992.[11]

In the benchmark scenario, we consider a hypothetical population of one thousand groups of consumers and assume that the subjective discount factors are uniformly distributed between $\beta_{\min} = 0.966$ and $\beta_{\max} = 0.992$. In other words, we set $N = 1,000$ and $\lambda_i = 1/N$ for all $i$.[12] The mean discount factor is 0.979. The main ideas of using a uniform distribution are as follows. Take wealth inequality as an example. In the stationary equilibrium, wealth inequality is driven by two types of variations: (i) variations in population share across groups, captured by $\{\lambda_i\}_{i=1}^N$, and (ii) variations in the equilibrium level of asset holdings across groups, captured by $\left\{\widehat{k}_i\right\}_{i=1}^N$. By adopting a uniform distribution, we can rule out the first type of variations. Thus, wealth inequality in the benchmark results is entirely driven

[11] To obtain these results, Lawrance (1991) estimate the Euler equation for a model without direct preference for wealth. This range of values, however, encompasses the values of discount factors that are typically used in quantitative studies (with or without wealth preference). In Section V, we will examine the effects of changing these endpoint values on the benchmark results.

[12] The choice of $N$ is immaterial for our benchmark results. In particular, changing the number of groups to either 500 or 5,000 has virtually no impact on the benchmark results.

by the cross-sectional variations in asset holdings. The same argument applies to inequality in earnings and income. Our benchmark results then provide a clear illustration of how much inequality can be generated by the key features of the model, namely wealth preference, human capital accumulation and heterogeneity in discount factors. In Section V, we will examine the effects of relaxing the uniform distribution assumption and changing the values of $\beta_{\min}$ and $\beta_{\max}$.

[Please insert Table 1 around here.]

As for the parameter values in the human capital production function, we normalize $\varphi$ to unity and set the values of $\epsilon$ and $\varsigma$ according to the estimates reported in Heckman *et al.* (1998). Using data from the National Longitudinal Survey of Youth for the period 1979-1993, these authors find that the values of $\epsilon$ and $\varsigma$ for people who have completed at least one year of college education are 0.939 and 0.871, respectively. For those who do not have any college education, the corresponding values are 0.945 and 0.832. We use the first set of parameter values in the numerical analysis because workers with college education account for a larger share of the U.S. labor force than those without college education.[13] As for the depreciation rate of human capital, Heckman *et al.* (1998) assume that it is zero. Other studies in the existing literature typically find that this rate is greater than zero.[14] In the benchmark scenario, we set $\delta_h = 0.037$ which is consistent with the estimate reported in Heckman (1976).

The two remaining parameters, $\theta$ and $\delta_k$, are calibrated so that the model can match two real-world statistics. The first parameter plays a crucial role in explaining wealth inequality. This point is discussed fully in Section V. In the benchmark scenario, we choose the value of $\theta$ so that the Gini coefficient of wealth predicted by the model is 0.816, which matches the value reported in Díaz-Giménez *et al.* (2011).[15] The required value of $\theta$ is 0.01202. As explained earlier, the distribution of earnings is independent of the preference parameter $\theta$. Thus, the calibration of $\theta$ has no impact on the extent of earnings inequality generated by the model. The second parameter $\delta_k$ is calibrated so that the capital-output ratio generated by the model is 3.0.

[13] Over the past twenty years, workers with at least some college education have accounted for an increasingly larger share of the U.S. labor force. In 1992, this type of worker represented 51.8 percent of civilian labor force (over 25 years old). This increased to 62.1 percent by the year 2010. These figures are based on the data reported in the U.S. Statistical Abstract.

[14] See Browning *et al.* (1999) Table 2.3 for a summary of these studies.

[15] Similar calibration strategy is also used in Krusell and Smith (1998), Erosa and Koreshkova (2007), and Hendricks (2007) to determine the parameter values in the Markov process of the random discount factor. In both cases, a set of unobserved, undetermined parameters is chosen so that the model can match certain key features of the U.S. wealth distribution.

## IV. Benchmark Results

Table 2 summarizes the characteristics of the earnings, income and wealth distributions obtained under the benchmark parameter values. The first three columns show the Gini coefficients, the coefficients of variation and the mean-to-median ratios for the three variables. The mean-to-median ratio is intended to measure the degree of skewness in these distributions. The rest of Table 2 shows the share of earnings, income and wealth held by consumers in different percentiles of the corresponding distribution.

[Please insert Table 2 around here.]

Under the benchmark parameter values, the wealth distribution in the model economy is highly concentrated with a large group of wealth-poor consumers and a small group of extremely wealthy ones. For instance, the share of total wealth owned by consumers in the second quintile of the wealth distribution is merely 1.3 percent, whereas the share owned by the wealthiest five percent is 58.5 percent. These figures are very close to the actual values observed in the United States. As for the income distribution, the model is able to generate a Gini coefficient and a mean-to-median ratio that are similar to the observed values. It is also able to replicate reasonably well the share of aggregate income owned by different quintiles of the income distribution.

As for earnings, the model predicts a more equal distribution than that observed in the data. In the model economy, earnings-poor consumers own a larger share of total earnings than their real-world counterparts. Consequently, the Gini coefficient predicted by the model is much lower than the actual value.[16] The big difference between the model's prediction and the actual value can be explained by two factors. First, in the actual data, a large number of households have reported negative earnings. According to Díaz-Giménez *et al.* (2011), the average earnings of households in the bottom quintile of the U.S. earnings distribution are negative due to sizable business losses. In the model economy, earnings must be above zero. This restriction reduces the range and dispersion of the earnings distribution, which in turn lowers earnings inequality in the model. Second, and more importantly, almost all the households in the bottom quintile of the U.S. earnings distribution are not workers. As shown in Díaz-Giménez *et al.* (2011) Table 4, retirees and nonworkers represent 96.9 percent of these households, and

[16] Our results on earnings inequality, however, are comparable to those obtained by Pijoan-Mas (2006) and Erosa and Koreshkova (2007). In the benchmark model of Pijoan-Mas (2006), the Gini coefficient and the coefficient of variation for the earnings distribution are 0.33 and 0.65, respectively. In the benchmark model of Erosa and Koreshkova (2007), the Gini coefficient of earnings is 0.289.

labor income only account for 0.2 percent of their total income. If we consider only households headed by employed worker, then the Gini coefficient for earnings in the United States is 0.47. This value is much closer to the one predicted by the model which assumes that all consumers are employed.[17]

## V. Discussion

The benchmark results in Table 2 show that our model is able to generate realistic patterns of wealth and income inequality. To achieve this, we have extended the standard neoclassical growth model to allow for (i) direct preferences for wealth, (ii) human capital formation, and (iii) heterogeneity in subjective discount factors. In the benchmark scenario, we also assume that the utility function is logarithmic and additively separable, and the distribution of discount factors is uniform. In this section, we examine the significance of each of these features in explaining wealth and income inequality. The main objective of this exercise is to better understand the determinants of wealth and income inequality in our model.

### *A. Strength of Wealth Preference*

The purpose of this subsection is to illustrate the effects of wealth preference on wealth and income inequality. To achieve this, we compute a series of balanced-growth equilibria using different values of $\theta$ ranging from 0.005 to 0.5. For each value of $\theta$, the depreciation rate $\delta_k$ is recalibrated so that the capital-output ratio is maintained at 3.0. All other parameters are fixed at their benchmark values.

[Please insert Table 3 around here.]

The results of this exercise are shown in Panel A of Table 3.[18] These results show a strong negative relationship between wealth inequality and the value of $\theta$. As $\theta$ approaches zero, the Gini coefficient of wealth increases toward unity. This result is consistent with theoretical prediction: when $\theta = 0$, the wealth distribution is degenerate as in the Becker (1980) model. As the value of $\theta$ increases, wealth distribution becomes more and more equal. The intuition behind this result is as follows. An increase in $\theta$ means that the same increase in asset holdings would now generate a larger gain in utility. This has two opposing effects on wealth inequality. First, a stronger preference for wealth encourages all

[17] According to Díaz-Giménez *et al.* (2011), the Gini coefficients of income and wealth for households headed by employed workers are 0.48 and 0.78, respectively. These values are also close to the ones generated by the model.

[18] As explained earlier, the earnings distribution is independent of $\theta$. Thus, for all the cases considered in this panel, the earnings distribution is the same as in the benchmark scenario.

types of consumers to accumulate more assets. This effect is stronger for wealth-rich consumers than for wealth-poor ones. Thus, holding other things constant, an increase in $\theta$ would make the wealth distribution more unequal. Second, since aggregate savings increase as $\theta$ increases, the effective rate of return from savings $(r^* - \delta_k)$ needs to be adjusted downward in order to discourage savings and maintain the same capital-output ratio. This tends to have a larger effect on more patient consumers than on the less patient ones. As a result, the share of total wealth owned by the wealthiest consumers will be lowered and the wealth distribution will become more equal. The overall effect of $\theta$ on wealth inequality depends on the relative magnitude between these two forces. Our results show that the second effect dominates under the benchmark parameter values.

An increase in $\theta$ also makes the income distribution more equal. But the decline in income inequality is much smaller than the decline in wealth inequality. This happens because (i) consumers' earnings are not affected by the parameter $\theta$, and (ii) for most of the consumers in this economy, earnings account for a large fraction of their income.[19] Thus, changing $\theta$ has only a mild impact on the income distribution.

### *B. Importance of Human Capital Formation*

In this subsection, we present a simplified version of our model in which there is no human capital formation. The main objective of this exercise is to illustrate the importance of this feature in generating wealth and income inequality. Suppose now each consumer in the model economy is endowed with one unit of time which is supplied inelastically to the market. Labor income at time $t$ is identical for all types of consumers and is given by $w_t$. The consumer's problem is now given by

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t u\left(c_{i,t},k_{i,t}\right) \tag{P2}$$

subject to the sequential budget constraint,

$$c_{i,t} + k_{i,t+1} - \left(1-\delta_k\right) k_{i,t} = w_t + r_t k_{i,t},$$

the borrowing constraint, $k_{i,t+1} \geq 0$, and the initial condition, $k_{i,0} > 0$. The utility function is additively separable as in (1). The rest of the model is the same as in Section II.

A balanced-growth equilibrium for this economy can be defined similarly as in Section II. The details

[19]When $\theta$ is 0.05 or less, earnings represent more than 70 percent of income for those in the bottom four quintiles (i.e., the bottom 80 percent) of the wealth distribution.

of this can be found in Suen (2012). Here we focus on the quantitative implications of this simplified model. We use the same values of $\{\alpha, \sigma, \gamma, N, \beta_{\min}, \beta_{\max}\}$ as shown in Table 1. The distribution of discount factors is again assumed to be uniform. Similar to the previous subsection, we solve the model under different values of $\theta$ ranging from 0.005 to 0.5. For each value of $\theta$, the depreciation rate of physical capital is recalibrated so that the capital-output ratio is maintained at 3.0. The main findings of this exercise are summarized in Panel B of Table 3.

In the absence of human capital, our model can still generate a highly concentrated wealth distribution with a large group of wealth-poor consumers and a small group of extremely wealthy ones. When comparing the results in Panel A and Panel B, we can see that removing human capital formation from the model only lowers the Gini coefficient of wealth by 1.5 percent when $\theta = 0.01202$. In other words, wealth inequality in the benchmark results is mainly driven by wealth preference and the heterogeneity in discount factors. However, these two features alone cannot generate a substantial degree of income inequality. This happens because (i) earnings are identical for all consumers in the simplified model, and (ii) earnings represent a sizable portion of income for most of the consumers in this economy. Introducing human capital formation helps improve this result in two ways. First, the distribution of earnings is now endogenously determined and is closely tied to the distribution of discount factors. This introduces significant variations in earnings across consumers. Second, since more patient consumers tend to invest more in both physical and human capital than less patient ones, there exists a strong positive correlation between earnings and capital income in the benchmark economy. This positive correlation plays a crucial role in generating substantial income inequality in our model.[20]

### *C. Changing the Distribution of Discount Factors*

We now examine the effects of changing the distribution of discount factors. In particular, we consider two types of changes: relaxing the uniform distribution assumption and changing the values of $\beta_{\min}$ and $\beta_{\max}$.

[20] An alternative way to endogenize the distribution of earnings is to introduce labor-lesiure choice in (P2). In this setting, however, wealth-rich consumers would tend to work less than wealth-poor ones due to the wealth effect on labor supply. Consequently, earnings and capital income are negatively correlated in a model with labor-leisure choice but without human capital. Because of this negative correlation, introducing labor-leisure choice in (P2) would actually *lower* the Gini coefficients of income in Panel B of Table 3. The details of this can be found in the working paper version.

***Changing the Shape of the Distribution***

Suppose now the share of each group in the population is determined by

$$\lambda_i = \left(\frac{i}{N}\right)^{\frac{1}{\upsilon}} - \left(\frac{i-1}{N}\right)^{\frac{1}{\upsilon}}, \qquad \text{with } \upsilon > 0,$$

for $i \in \{1, 2, ..., N\}$. The endpoints of the distribution are fixed at their benchmark values, i.e., $\beta_{\min} = 0.966$ and $\beta_{\max} = 0.992$. The benchmark scenario then corresponds to the case when $\upsilon = 1$. When $\upsilon > 1$, the size of the most patient group is less than $1/N$ and the distribution of discount factors is more concentrated on low values of $\beta$. The opposite is true when $\upsilon \in (0, 1)$. Intuitively, a high value of $\upsilon$ represents an economy in which most of the consumers share similar values of discount factor that are close to $\beta_{\min}$, while a small group of consumers are relatively more patient. To better understand the effects of changing $\upsilon$, we consider two calibration exercises. In the first exercise, we examine the extent of economic inequality under different values of $\upsilon$. The results are shown in Panel A of Table 4. For each value of $\upsilon$, the depreciation rate of physical capital is adjusted so as to maintain the capital-output ratio at 3.0. All other parameters (including $\theta$) are fixed at their benchmark values. In the second exercise, both $\theta$ and $\delta_k$ are recalibrated in each case so that the two calibration targets (Gini coefficient of wealth and capital-output ratio) are the same as in the benchmark scenario. The results of the second experiment are summarized in Panel B of Table 4.

[Please insert Table 4 around here.]

We begin by summarizing the effects of changing $\upsilon$ on the distribution of discount factors. Increasing $\upsilon$ from 1.0 to 2.0 raises the size of the least patient group ($\lambda_1$) from 0.0010 to 0.0316, and reduces the size of the most patient group ($\lambda_N$) by half. Because of the skewness of the distribution, the mean value of $\beta$ is greater than the median value when $\upsilon > 1$. These results are identical for the two panels.

The results in Panel A show that the Gini coefficients produced by the model are rather robust to changes in the size of the most patient group ($\lambda_N$). More specifically, reducing $\lambda_N$ by half raises the Gini coefficients of earnings, income and wealth by 13.6 percent, 10.8 percent and 8.5 percent, respectively. The share of total wealth and total income owned by the richest consumers are more sensitive to this change. The intuitions behind these results are as follows. First, consider the increase in earnings inequality. In the stationary equilibrium, this type of inequality is driven by (i) cross-sectional variations in the population share, $\{\lambda_i\}_{=1}^{N}$, and (ii) cross-sectional variations in human capital and labor hours,

$\{h_i, l_i\}_{i=1}^{N}$. As shown in (13) and (14), the values of $\{h_i, l_i\}_{i=1}^{N}$ are independent of the equilibrium rate of return $(r^* - \delta_k)$ and the population shares. This means changing $\upsilon$ has no impact on the values of $\{h_i, l_i\}_{i=1}^{N}$. Thus, the increase in earnings inequality that we observed in Panel A is completely driven by the changes in $\{\lambda_i\}_{=1}^{N}$. In particular, an increase in $\upsilon$ lowers the share of very patient consumers in the population. Since these consumers tend to have more human capital and higher earnings than the less patient ones, a large portion of total earnings is now concentrated in the hands of fewer consumers. Thus, the earnings distribution becomes more unequal as $\upsilon$ increases.

An increase in $\upsilon$ has a similar effect on wealth inequality. Specifically, such an increase means that a large portion of total wealth is now concentrated in the hands of fewer consumers. This makes the wealth distribution more unequal. However, an increase in $\upsilon$ would also induce changes in the effective rate of return from savings. This creates a second effect on wealth inequality. More specifically, an increase in the share of less patient consumers leads to a decline in aggregate savings. In order to maintain the same capital-output ratio, we need to adjust the effective rate of return upward as $\upsilon$ increases. Since more patient consumers are more responsive to interest rate changes than less patient ones, this widens the differences in asset holdings across groups and further increases wealth inequality. As for income, since it is just the sum of earnings and capital income, income inequality increases as earnings and wealth inequality increase.

Next, we turn to the results in Panel B. Since adjusting $\theta$ has no effect on the earnings distribution, the Gini coefficients of earnings are the same as in Panel A. When the Gini coefficient of wealth is held constant, increasing $\upsilon$ from 1.0 to 2.0 raises the Gini coefficient of income by 6.5 percent, which is smaller than the increase in Panel A. The most significant difference between the two panels is that, when the Gini coefficient of wealth is held constant, an increase in $\upsilon$ would lower the share of total wealth and total income held by the richest consumers. This happens because we need to adjust $\theta$ upward as $\upsilon$ increases so as to maintain the Gini coefficient of wealth at the same level. As shown in Table 3, this tends to lower the share of total wealth and total income held by the richest consumers. The results in Panel B thus show that the qualitative effects of $\theta$ on wealth and income inequality are robust to changes in $\upsilon$.

***Changing the Range of Discount Factors***

We now examine the effects of changing the range of discount factors. We maintain the uniform distribution assumption as in the benchmark scenario, but consider five different combinations of endpoint values. In the first variation, the benchmark values are reduced by 0.01 so that $\beta_{\text{min}} = 0.956$ and $\beta_{\text{max}} = 0.982$. In the second variation, the benchmark values are reduced by 0.02. In these two experiments, the range $\triangle\beta \equiv |\beta_{\text{max}} - \beta_{\text{min}}|$ is the same as in the benchmark scenario. In the third and fourth experiments, this range is reduced by half. We consider the upper half in the third experiment, i.e., $\beta_{\text{min}} = 0.979$ and $\beta_{\text{max}} = 0.992$, and the lower half in the fourth one. In the final experiment, we extend the benchmark interval to the left by 50 percent, so that $\beta_{\text{min}} = 0.953$ and $\beta_{\text{max}} = 0.992$. Similar to the previous subsection, we report two sets of results for each experiment. Panel A of Table 5 reports the results obtained when the capital-output ratio is kept at 3.0 and $\theta$ is fixed at 0.01202. Panel B reports the results obtained when the two calibration targets are kept constant.

[Please insert Table 5 around here.]

Two main observations can be made from the results in Panel A. First, shifting the distribution of discount factors while leaving the range $\triangle\beta$ unchanged only has a mild impact on the Gini coefficients. For instance, reducing the benchmark endpoint values by 0.01 raises the Gini coefficients of earnings and income by 4.5 percent and 2.2 percent, respectively. The effect of this on the Gini coefficient of wealth is negligible. The share of total wealth and total income owned by the richest consumers is also quite robust to this change. The second observation is that inequality in all three variables are positively related to the size of $\triangle\beta$. This is evident from the results of the last three experiments. These results suggest that our model does not rely on large values of discount factor (i.e., very patient consumers) to generate a high concentration of wealth and income. Instead, inequality in our model is largely determined by the relative magnitude between $\beta_{\text{min}}$ and $\beta_{\text{max}}$.

Next, consider the results in Panel B. When the Gini coefficient of wealth is held constant, income inequality is less sensitive to changes in $\triangle\beta$. The main differences between the two panels are the effects of changing $\triangle\beta$ on the share of total wealth and total income held by the richest consumers. When the parameter $\theta$ is kept constant, these shares are positively related to the size of $\triangle\beta$. When the Gini coefficient of wealth is kept constant, these shares become negatively related to the size of $\triangle\beta$. This happens because, in order to maintain the same Gini coefficient of wealth, we need to adjust $\theta$ upward as the range of discount factors widens. This in turn lowers the share of total wealth and total income held by the richest consumers.

### *D. Changing the Intertemporal Elasticity of Substitution*

Table 6 reports the results obtained under different values of $\sigma$. Panel A shows that, when $\theta$ is held constant, an increase in $\sigma$ has only a mild impact on the Gini coefficients. In particular, increasing $\sigma$ from 1.0 to 1.8 lowers the Gini coefficient of earnings and income by 10.6 percent and 3.9 percent, and raises the Gini coefficient of wealth by 2.9 percent. The share of total wealth owned by the wealthiest agents, however, is rather sensitive to this change. The intuitions of this result are as follows. Similar to an increase in $\theta$, an increase in $\sigma$ would induce two opposing effects on wealth inequality. First, an increase in $\sigma$ lowers the intertemporal elasticity of substitution (IES) for consumption. Holding other things constant, every consumer would now prefer to have a flatter consumption profile and less savings. In particular, the reduction in savings tends to be larger for the wealthy consumers than for the poor ones. Thus, holding other things constant, this effect would make the wealth distribution more equal. Second, since aggregate savings decline as $\sigma$ increases, we need to adjust the effective rate of return from savings upward in order to maintain the same capital-output ratio. This induces a much larger increase in asset holdings for the wealthy consumers than for the poor ones, which in turn drives up the differences in wealth across groups. Hence, the second effect would make the wealth distribution more unequal and increase the share of wealth owned by the wealthiest consumers. The results in Panel A of Table 6 suggest that the second effect dominates under the benchmark parameter values.

[Please insert Table 6 around here.]

When the Gini coefficient of wealth is kept constant, the same increase in $\sigma$ now induces a smaller increase in the share of total wealth owned by the wealthiest consumers than in Panel A. This happens because we need to adjust $\theta$ upward as $\sigma$ increases so as to maintain the same Gini coefficient of wealth. As shown in Table 3, this tends to reduce the share of wealth owned by the wealthiest consumers, and thus partially offsets the effects of $\sigma$ on the top end of the wealth distribution.

## VI. Conclusion

This paper presents a highly tractable dynamic general equilibrium model that can generate patterns of wealth and income inequality that are very similar to those observed in the United States. To achieve this, we extend the standard deterministic neoclassical growth model to include three features: (i) consumer heterogeneity in time preference, (ii) direct preferences for wealth, and (iii) human capital formation. We show that a model with the first two features alone is able to replicate the patterns of wealth inequality observed in the United States. Such a model, however, cannot generate substantial degree of income inequality. Thus, we also need to introduce human capital in order to account for both wealth and income inequality.

Admittedly, the model considered in this study is rather stylized and has abstracted away a number of factors that are also relevant in explaining economic inequality. One possible extension is to introduce idiosyncratic uncertainty into the current framework. The extended model can then be used to evaluate the relative importance of predetermined factors and idiosyncratic shocks in explaining wealth and income inequality.

**TABLE 1**

Benchmark Parameters Values

| | | |
|---|---|---|
| $\alpha$ | Share of capital income in total output | 0.33 |
| $\gamma$ | Common growth factor | 1.022 |
| $\delta_k$ | Depreciation rate of physical capital | 0.08004* |
| $\sigma$ | Inverse of intertemporal elasticity of substitution | 1.0 |
| $\theta$ | Strength of wealth preference | 0.01202 |
| $\beta_{\min}$ | Minimum value of subjective discount factor | 0.966 |
| $\beta_{\max}$ | Maximum value of subjective discount factor | 0.992 |
| $N$ | Number of groups of consumers | 1,000 |
| $\varphi$ | Parameter in human capital production | 1.0 |
| $\epsilon$ | Parameter in human capital production | 0.939 |
| $\varsigma$ | Parameter in human capital production | 0.871 |
| $\delta_h$ | Depreciation rate of human capital | 0.037 |

Note: This figure has been rounded off to the fourth significant figure.

**TABLE 2**

Benchmark Results

| | | | | Share (%) Held by Consumers in Each Group | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Mean-to- | Bottom | | | Quintiles | | | | | Top | | |
| | Gini | C.V. | Median | 1% | 1-5% | 5-10% | 1st | 2nd | 3rd | 4th | 5th | 10% | 5% | 1% |
| **Earnings** | | | | | | | | | | | | | | |
| Model | 0.397 | 0.73 | 1.34 | 0.2 | 0.9 | 1.2 | 5.4 | 8.9 | 15.1 | 25.8 | 44.5 | 25.2 | 13.4 | 2.8 |
| Data | 0.636 | 3.60 | 1.72 | -0.1 | 0.0 | 0.0 | -0.1 | 4.2 | 11.7 | 20.8 | 63.5 | 47.0 | 35.3 | 18.7 |
| **Income** | | | | | | | | | | | | | | |
| Model | 0.536 | 1.39 | 1.82 | 0.1 | 0.6 | 0.9 | 3.8 | 6.4 | 11.1 | 20.2 | 58.2 | 41.1 | 28.3 | 10.5 |
| Data | 0.575 | 4.32 | 1.77 | -0.1 | 0.3 | 0.6 | 2.8 | 6.7 | 11.3 | 18.3 | 60.9 | 47.1 | 36.9 | 21.0 |
| **Wealth** | | | | | | | | | | | | | | |
| Model | 0.816 | 3.16 | 6.92 | 0.0 | 0.1 | 0.1 | 0.6 | 1.3 | 3.0 | 8.8 | 86.2 | 73.4 | 58.5 | 25.9 |
| Data | 0.816 | 6.02 | 4.61 | -0.1 | -0.1 | -0.0 | -0.2 | 1.1 | 4.5 | 11.2 | 83.4 | 71.4 | 60.3 | 33.6 |

Note: The data are taken from Díaz-Giménez *et al.* (2011)

**TABLE 3**

Changing the Strength of Wealth Preference

*Panel A: Model with Human Capital*

| | Gini Coeff. | | Share of Income (%) | | | | Share of Wealth (%) | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | Bottom | Top | | | Bottom | Top | | |
| $\theta$ | Income | Wealth | 40% | 10% | 5% | 1% | 40% | 10% | 5% | 1% |
| 0.005 | 0.569 | 0.918 | 9.9 | 46.1 | 36.0 | 22.9 | 0.8 | 88.5 | 81.8 | 63.8 |
| 0.010 | 0.544 | 0.842 | 10.2 | 42.4 | 30.3 | 13.0 | 1.5 | 77.5 | 64.5 | 33.6 |
| **0.01202** | **0.536** | **0.816** | **10.3** | **41.1** | **28.3** | **10.5** | **1.9** | **73.4** | **58.5** | **25.9** |
| 0.025 | 0.497 | 0.700 | 10.8 | 35.0 | 20.8 | 5.1 | 3.6 | 54.9 | 35.9 | 9.7 |
| 0.050 | 0.463 | 0.597 | 11.6 | 30.4 | 16.8 | 3.7 | 6.1 | 40.9 | 23.8 | 5.4 |
| 0.100 | 0.437 | 0.517 | 12.5 | 27.8 | 15.0 | 3.2 | 8.8 | 33.1 | 18.3 | 4.0 |
| 0.500 | 0.407 | 0.426 | 13.9 | 25.7 | 13.7 | 2.9 | 12.9 | 26.7 | 14.3 | 3.0 |
| Data | 0.575 | 0.816 | 9.5 | 47.1 | 36.9 | 21.0 | 0.9 | 71.4 | 60.3 | 33.6 |

*Panel B: Model without Human Capital*

| | Gini Coeff. | | Share of Income (%) | | | | Share of Wealth (%) | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | Bottom | Top | | | Bottom | Top | | |
| $\theta$ | Income | Wealth | 40% | 10% | 5% | 1% | 40% | 10% | 5% | 1% |
| 0.005 | 0.303 | 0.918 | 27.4 | 36.6 | 32.3 | 27.4 | 2.1 | 90.6 | 87.7 | 80.9 |
| 0.010 | 0.276 | 0.836 | 28.1 | 33.5 | 28.2 | 21.1 | 4.1 | 81.2 | 75.4 | 61.9 |
| 0.01202 | 0.265 | 0.804 | 28.4 | 32.2 | 26.6 | 18.6 | 5.0 | 77.4 | 70.5 | 54.4 |
| 0.025 | 0.201 | 0.608 | 30.1 | 24.6 | 17.0 | 6.3 | 10.2 | 54.3 | 41.4 | 17.1 |
| 0.050 | 0.124 | 0.375 | 32.8 | 16.4 | 9.1 | 2.0 | 18.4 | 29.5 | 17.4 | 4.1 |
| 0.100 | 0.066 | 0.201 | 35.6 | 12.5 | 6.4 | 1.3 | 26.9 | 17.6 | 9.3 | 1.9 |
| 0.500 | 0.014 | 0.041 | 38.9 | 10.4 | 5.2 | 1.0 | 37.0 | 11.2 | 5.6 | 1.1 |
| Data | 0.575 | 0.816 | 9.5 | 47.1 | 36.9 | 21.0 | 0.9 | 71.4 | 60.3 | 33.6 |

Note: Figures in bold are the benchmark results reported in Table 2. The data are taken from Díaz-Giménez *et al.* (2011).

**TABLE 4**

Changing the Distribution of Discount Factors

*Panel A: Holding capital-output ratio and $\theta = 0.01202$ constant*

| | Discount Factors | | | | | | | Share of Income (%) Held by | | | | Share of Wealth (%) Held by | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Share of Consumers with | | Gini Coeff. | | | Bottom | Top | | | Bottom | Top | | |
| $\upsilon$ | Mean | Median | $\beta = 0.966$ | $\beta = 0.992$ | Earnings | Income | Wealth | 40% | 10% | 5% | 1% | 40% | 10% | 5% | 1% |
| **1.0** | **0.9790** | **0.9790** | **0.0010** | **0.0010** | **0.397** | **0.536** | **0.816** | **10.3** | **41.1** | **28.3** | **10.5** | **1.9** | **73.4** | **58.5** | **25.9** |
| 1.2 | 0.9778 | 0.9773 | 0.0032 | 0.0008 | 0.417 | 0.556 | 0.838 | 12.6 | 39.9 | 28.0 | 11.0 | 2.2 | 73.8 | 59.7 | 28.0 |
| 1.5 | 0.9764 | 0.9752 | 0.0100 | 0.0007 | 0.436 | 0.576 | 0.861 | 15.7 | 38.7 | 27.7 | 11.8 | 2.7 | 74.4 | 61.2 | 30.9 |
| 2.0 | 0.9747 | 0.9725 | 0.0316 | 0.0005 | 0.451 | 0.594 | 0.885 | 20.0 | 37.4 | 27.6 | 13.0 | 3.4 | 75.4 | 63.5 | 35.1 |
| Data | – | – | – | – | 0.636 | 0.575 | 0.816 | 9.5 | 47.1 | 36.9 | 21.0 | 0.9 | 71.4 | 60.3 | 33.6 |

*Panel B: Holding capital-output ratio and Gini coefficient of wealth constant*

| | Discount Factors | | | | | | | Share of Income (%) Held by | | | | Share of Wealth (%) Held by | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Share of Consumers with | | Gini Coeff. | | | Bottom | Top | | | Bottom | Top | | |
| $\upsilon$ | Mean | Median | $\beta = 0.966$ | $\beta = 0.992$ | Earnings | Income | Wealth | 40% | 10% | 5% | 1% | 40% | 10% | 5% | 1% |
| **1.0** | **0.9790** | **0.9790** | **0.0010** | **0.0010** | **0.397** | **0.536** | **0.816** | **10.3** | **41.1** | **28.3** | **10.5** | **1.9** | **73.4** | **58.5** | **25.9** |
| 1.2 | 0.9778 | 0.9773 | 0.0032 | 0.0008 | 0.417 | 0.549 | 0.816 | 12.7 | 38.7 | 26.2 | 9.0 | 2.6 | 69.9 | 54.2 | 22.1 |
| 1.5 | 0.9764 | 0.9752 | 0.0100 | 0.0007 | 0.436 | 0.561 | 0.816 | 16.0 | 35.8 | 23.7 | 7.6 | 3.8 | 65.5 | 49.2 | 18.2 |
| 2.0 | 0.9747 | 0.9725 | 0.0316 | 0.0005 | 0.451 | 0.571 | 0.816 | 20.8 | 32.4 | 20.9 | 6.2 | 5.8 | 60.1 | 43.4 | 14.5 |
| Data | – | – | – | – | 0.636 | 0.575 | 0.816 | 9.5 | 47.1 | 36.9 | 21.0 | 0.9 | 71.4 | 60.3 | 33.6 |

Note: Figures in bold are the benchmark results reported in Table 2. The data are taken from Díaz-Giménez *et al.* (2011).

**TABLE 5**

Changing the Range of Discount Factors

*Panel A: Holding capital-output ratio and $\theta = 0.01202$ constant*

| | | | | | Share of Income (%) | | | | Share of Wealth (%) | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Gini Coeff. | | | Bottom | | Top | | Bottom | | Top | |
| $\beta_{\min}$ | $\beta_{\max}$ | Earnings | Income | Wealth | 40% | 10% | 5% | 1% | 40% | 10% | 5% | 1% |
| **0.966** | **0.992** | **0.397** | **0.536** | **0.816** | **10.3** | **41.1** | **28.3** | **10.5** | **1.9** | **73.4** | **58.5** | **25.9** |
| 0.956 | 0.982 | 0.379 | 0.524 | 0.818 | 11.1 | 40.8 | 28.5 | 10.9 | 2.0 | 74.0 | 59.7 | 27.5 |
| 0.946 | 0.972 | 0.353 | 0.507 | 0.819 | 12.1 | 40.2 | 28.4 | 11.6 | 2.1 | 74.6 | 61.0 | 29.7 |
| 0.979 | 0.992 | 0.218 | 0.363 | 0.657 | 18.8 | 29.1 | 18.2 | 5.0 | 6.2 | 53.8 | 37.3 | 11.5 |
| 0.966 | 0.979 | 0.206 | 0.356 | 0.662 | 19.4 | 29.2 | 18.5 | 5.2 | 6.3 | 54.8 | 38.5 | 12.3 |
| 0.953 | 0.992 | 0.527 | 0.642 | 0.876 | 5.7 | 49.4 | 34.9 | 14.3 | 7.0 | 81.8 | 68.6 | 35.3 |
| Data | | 0.636 | 0.575 | 0.816 | 9.5 | 47.1 | 36.9 | 21.0 | 0.9 | 71.4 | 60.3 | 33.6 |

*Panel B: Holding capital-output ratio and Gini coefficient of wealth constant*

| | | | | | Share of Income (%) | | | | Share of Wealth (%) | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Gini Coeff. | | | Bottom | | Top | | Bottom | | Top | |
| $\beta_{\min}$ | $\beta_{\max}$ | Earnings | Income | Wealth | 40% | 10% | 5% | 1% | 40% | 10% | 5% | 1% |
| **0.966** | **0.992** | **0.397** | **0.536** | **0.816** | **10.3** | **41.1** | **28.3** | **10.5** | **1.9** | **73.4** | **58.5** | **25.9** |
| 0.956 | 0.982 | 0.379 | 0.523 | 0.816 | 11.1 | 40.7 | 28.3 | 10.8 | 2.0 | 73.7 | 59.3 | 27.0 |
| 0.946 | 0.972 | 0.353 | 0.506 | 0.816 | 12.1 | 40.0 | 28.2 | 11.3 | 2.1 | 74.1 | 60.3 | 28.8 |
| 0.979 | 0.992 | 0.218 | 0.415 | 0.816 | 17.7 | 36.4 | 27.4 | 14.5 | 3.0 | 75.9 | 65.3 | 40.2 |
| 0.966 | 0.979 | 0.206 | 0.407 | 0.816 | 18.3 | 36.3 | 27.5 | 14.7 | 3.1 | 76.1 | 65.8 | 41.0 |
| 0.953 | 0.992 | 0.527 | 0.623 | 0.816 | 5.8 | 45.9 | 29.8 | 8.7 | 1.2 | 71.4 | 53.0 | 18.2 |
| Data | | 0.636 | 0.575 | 0.816 | 9.5 | 47.1 | 36.9 | 21.0 | 0.9 | 71.4 | 60.3 | 33.6 |

Note: Figures in bold are the benchmark results reported in Table 2. The data are taken from Díaz-Giménez *et al.* (2011).

**TABLE 6**

Changing the Intertemporal Elasticity of Substitution

*Panel A: Holding capital-output ratio and* $\theta = 0.01202$ *constant*

| | | | | Share of Income (%) | | | | Share of Wealth (%) | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Gini Coeff. | | | Bottom | Top | | | Bottom | Top | | |
| $\sigma$ | Earnings | Income | Wealth | 40% | 10% | 5% | 1% | 40% | 10% | 5% | 1% |
| **1.0** | **0.397** | **0.536** | **0.816** | **10.3** | **41.1** | **28.3** | **10.5** | **1.9** | **73.4** | **58.5** | **25.9** |
| 1.2 | 0.390 | 0.534 | 0.825 | 10.7 | 41.6 | 29.8 | 14.0 | 2.1 | 75.3 | 63.2 | 36.6 |
| 1.4 | 0.380 | 0.529 | 0.833 | 11.2 | 41.7 | 30.8 | 17.4 | 2.4 | 76.7 | 66.9 | 47.1 |
| 1.6 | 0.368 | 0.523 | 0.838 | 11.7 | 41.7 | 31.5 | 19.7 | 2.6 | 77.7 | 69.5 | 54.3 |
| 1.8 | 0.355 | 0.515 | 0.840 | 12.2 | 41.5 | 31.8 | 21.3 | 2.8 | 78.4 | 71.3 | 59.1 |
| Data | 0.636 | 0.575 | 0.816 | 9.5 | 47.1 | 36.9 | 21.0 | 0.9 | 71.4 | 60.3 | 33.6 |

*Panel B: Holding capital-output ratio and Gini coefficient of wealth constant*

| | | | | Share of Income (%) | | | | Share of Wealth (%) | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Gini Coeff. | | | Bottom | Top | | | Bottom | Top | | |
| $\sigma$ | Earnings | Income | Wealth | 40% | 10% | 5% | 1% | 40% | 10% | 5% | 1% |
| **1.0** | **0.397** | **0.536** | **0.816** | **10.3** | **41.1** | **28.3** | **10.5** | **1.9** | **73.4** | **58.5** | **25.9** |
| 1.2 | 0.390 | 0.530 | 0.816 | 10.7 | 41.1 | 29.1 | 13.0 | 2.2 | 73.9 | 61.2 | 33.5 |
| 1.4 | 0.380 | 0.524 | 0.816 | 11.3 | 40.9 | 29.7 | 15.5 | 2.6 | 74.3 | 63.4 | 41.5 |
| 1.6 | 0.368 | 0.516 | 0.816 | 11.8 | 40.7 | 30.1 | 17.6 | 2.9 | 74.7 | 65.3 | 48.0 |
| 1.8 | 0.355 | 0.507 | 0.816 | 12.4 | 40.4 | 30.4 | 19.1 | 3.2 | 75.1 | 66.9 | 52.7 |
| Data | 0.636 | 0.575 | 0.816 | 9.5 | 47.1 | 36.9 | 21.0 | 0.9 | 71.4 | 60.3 | 33.6 |

Note: Figures in bold are the benchmark results reported in Table 2. The source of data is Díaz-Giménez *et al.* (2011).

## Appendix

The main objective of this appendix is to establish the existence and uniqueness of balanced-growth equilibrium for the model described in Section II. These results are summarized in Theorem A1. In the following discussions, we also show that the borrowing constraint never binds in equilibrium. All the results presented in this appendix can be extended straightforwardly to a general neoclassical production technology and a more general class of utility functions. The details of this can be found in Suen (2012).

We begin with the mathematical derivations of equation (13). The first thing to note is that, in any balanced-growth equilibrium, the shadow price of human capital $\chi_{i,t}$ must be growing at a constant rate. To see this, substitute the balanced-growth conditions [i.e., $c_{i,t} = \gamma^t \widehat{c}_i$, $k_{i,t} = \gamma^t \widehat{k}_i$, $w_t = \gamma^t \widehat{w}(r)$, $l_{i,t} = l_i$ and $h_{i,t} = h_i$, for all $t$] into (5). This gives

$$(\widehat{c}_i)^{-\sigma} \widehat{w}(r) \gamma^{(1-\sigma)t} = \chi_{i,t} \epsilon \varphi (1 - l_i)^{\epsilon - 1} h_i^{\varsigma},$$

which implies $\chi_{i,t+1} = \gamma^{1-\sigma} \chi_{i,t}$, for all $t$. Substituting this and the balanced-growth conditions into (6) gives

$$1 = \beta_i \gamma^{1-\sigma} \left\{ \varphi (1 - l_i)^{\epsilon - 1} h_i^{\varsigma - 1} [\varsigma (1 - l_i) + \epsilon l_i] + (1 - \delta_h) \right\}. \tag{18}$$

Equation (13) can be obtained by combining (14) and (18).

We now turn to the existence and uniqueness of balanced-growth equilibrium. The main ideas of the proof are as follows. A balanced-growth equilibrium is characterized by a constant rental rate $r^*$ which clears the market for physical capital. Once this variable is determined, all other variables in a balanced-growth equilibrium can be uniquely determined. Thus, it suffices to establish the existence and uniqueness of $r^*$. To achieve this, we need to formulate the supply and demand of physical capital as functions of $r$.

Along any balanced-growth path with rental rate $r$, the quantity of physical capital demanded is given by $\left( \sum_{i=1}^{N} \lambda_i l_i h_i \right) \widehat{k}^d (r)$, where the aggregate labor input $\left( \sum_{i=1}^{N} \lambda_i l_i h_i \right)$ is independent of $r$, and $\widehat{k}^d (r)$ is determined by (9). The supply of physical capital is given by the sum of individual asset holdings. Along any balanced-growth path, the Euler equation for consumption is given by

$$\frac{\gamma^{\sigma}}{\beta_i} - (1 - \delta_k) - r \geq \theta \left( \frac{\widehat{c}_i}{\widehat{k}_i} \right)^{\sigma} \geq 0, \tag{19}$$

which holds with equality if $\widehat{k}_i > 0$. Since the ratio $\left(\widehat{c}_i/\widehat{k}_i\right)$ must be non-negative in equilibrium, the Euler equation is valid only for $r \leq \widehat{r}_i$, where $\widehat{r}_i \equiv \gamma^{\sigma}/\beta_i - (1-\delta_k) > 0$. This essentially imposes an upper bound on the equilibrium rental rate, which is $\min_i \{\widehat{r}_i\} = \widehat{r}_N$.[21] For any $r \in (0, \widehat{r}_N)$, it is never optimal for any type of consumer to choose $\widehat{k}_i = 0$. To see this, suppose the contrary that a type-$i$ consumer chooses $\widehat{k}_i = 0$ in a balanced-growth equilibrium with rental rate $r$. Then the right-hand side of (19) is infinite, which clearly exceeds the left-hand side of the inequality for any $r \in (0, \widehat{r}_N)$. This gives rise to a contradiction. Hence, each $\widehat{k}_i$ must be strictly positive and the Euler equation will always hold with equality. Using (17), we can obtain

$$\widehat{k}_i = g_i(r) \equiv \frac{\widehat{w}(r)\, l_i h_i}{\Theta_i(r) - \left(r - \widehat{\delta}_k\right)}, \tag{20}$$

where

$$\Theta_i(r) \equiv \left\{\frac{1}{\theta}\left[\frac{\gamma^{\sigma}}{\beta_i} - (1-\delta_k) - r\right]\right\}^{\frac{1}{\sigma}}.$$

The function $g_i(\cdot)$ in (20) is continuously differentiable on $(0, \widehat{r}_i)$. Denote by $\widehat{k}^s(r)$ the aggregate supply of physical capital when the rental rate is $r \in (0, \widehat{r}_N)$. Formally, this is defined as $\widehat{k}^s(r) \equiv \sum_{i=1}^{N} \lambda_i g_i(r)$. Since each $g_i(r)$ is continuously differentiable on $(0, \widehat{r}_N)$, the function $\widehat{k}^s(r)$ is also continuously differentiable on this range. A unique balanced-growth equilibrium exists if there exists a unique value $r^*$, within the range $(0, \widehat{r}_N)$, that clears the market for physical capital. Theorem A1 provides the conditions under which a unique value of $r^*$ exists.

**Theorem A1** *Suppose $\beta_i \gamma^{1-\sigma} < 1$ for all $i \in \{1, ..., N\}$, and*

$$\left(\sum_{i=1}^{N} \lambda_i l_i h_i\right) \widehat{k}^d\left(\widehat{\delta}_k\right) > \widehat{k}^s\left(\widehat{\delta}_k\right). \tag{21}$$

*Then there exists a unique balanced-growth equilibrium. In the unique equilibrium, all consumers hold a strictly positive amount of physical capital.*

[21] More specifically, if $r > \widehat{r}_N$ then the Euler equation will not be satisfied for some consumers and so $r$ cannot be an equilibrium rental rate.

**Proof of Theorem A1**

The proof of this theorem is divided into two main steps. First, it is shown that there exists a rental rate $\widetilde{r}_N > \widehat{\delta}_k$ such that $\widehat{k}^s(r) \to \infty$ as $r$ approaches $\widetilde{r}_N$ from the left. Since $\widetilde{r}_N < \infty$, it follows from (9) that $\widehat{k}^d(\widetilde{r}_N) < \infty$. Hence, we have $\lim_{r \to \widetilde{r}_N} \widehat{k}^s(r) = \infty > \left(\sum_{i=1}^N \lambda_i l_i h_i\right) \widehat{k}^d(\widetilde{r}_N)$ and $\left(\sum_{i=1}^N \lambda_i l_i h_i\right) \widehat{k}^d\left(\widehat{\delta}_k\right) > \widehat{k}^s\left(\widehat{\delta}_k\right)$. Since both $\widehat{k}^s(r)$ and $\widehat{k}^d(r)$ are continuous on $\left(\widehat{\delta}_k, \widetilde{r}_N\right)$, it follows from the intermediate value theorem that there exists at least one $r^* \in \left(\widehat{\delta}_k, \widetilde{r}_N\right)$ that clears the market for physical capital. The second step is to show that there exists at most one such solution in the interval $(0, \widetilde{r}_N)$.

**Step 1** First, consider the following equation

$$\Theta_i(r) \equiv \left\{\frac{1}{\theta}\left[\frac{\gamma^\sigma}{\beta_i} - (1 - \delta_k) - r\right]\right\}^{\frac{1}{\sigma}} = r - \widehat{\delta}_k. \tag{22}$$

Note that $\Theta_i(r)$ is continuous and strictly decreasing with $\Theta_i\left(\widehat{\delta}_k\right) = \left[\frac{1}{\theta}\left(\frac{\gamma^\sigma}{\beta_i} - \gamma\right)\right]^{\frac{1}{\sigma}} > 0$, for all $i$. This expression is strictly positive due to the assumption that $\beta_i \gamma^{1-\sigma} < 1$, for all $i$. Thus, there exists a unique $\widetilde{r}_i \in \left(\widehat{\delta}_k, \widehat{r}_i\right)$ that solves equation (22). Given the ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_N$, it is straightforward to see that $\widehat{\delta}_k < \widetilde{r}_N \leq \widetilde{r}_{N-1} \leq \ldots \leq \widetilde{r}_1$.

From (20), it is obvious that $g_i(r) \to \infty$ as $r$ approaches $\widetilde{r}_i$ from the left. Thus, when $r$ approaches $\widetilde{r}_N$ from the left, we have $g_N(r) \to \infty$ and $g_i(r) > 0$ for all $i \neq N$. Hence, $\widehat{k}^s(r) = \sum_{i=1}^N \lambda_i g_i(r) \to \infty$ as $r$ approaches $\widetilde{r}_N$.

**Step 2** To establish the uniqueness of $r^*$, we first consider the derivative of $g_i(r)$. Differentiating (17) with respect to $r$ gives

$$g_i'(r) = \frac{1}{\widehat{w}(r)}\left\{\frac{[g_i(r)]^2}{l_i h_i} + \widehat{w}'(r) g_i(r) + \frac{[g_i(r)]^2}{l_i h_i \Phi'(z_i(r))}\right\},$$

where $\Phi(z) \equiv \theta z^\sigma$, $z_i(r) \equiv \frac{\widehat{w}(r) l_i h_i}{g_i(r)} + r - \widehat{\delta}_k$ and $\widehat{w}'(r) = -\widehat{k}^d(r)$. Thus, the derivative of $\widehat{k}^s(r)$ is

$$\frac{d}{dr}\left[\widehat{k}^s(r)\right] = \frac{1}{\widehat{w}(r)}\left\{\sum_{i=1}^N \lambda_i \frac{[g_i(r)]^2}{l_i h_i} - \widehat{k}^d(r)\widehat{k}^s(r) + \sum_{i=1}^N \lambda_i \frac{[g_i(r)]^2}{l_i h_i \Phi'(z_i(r))}\right\},$$

Let $r^*$ be any value that solves $\left(\sum_{i=1}^N \lambda_i l_i h_i\right) \widehat{k}^d(r) = \widehat{k}^s(r)$. The derivative of $\widehat{k}^s(r)$ at $r = r^*$ is

$$\frac{1}{\widehat{w}(r^*)} \left\{ \sum_{i=1}^N \lambda_i \frac{[g_i(r^*)]^2}{l_i h_i} - \frac{\left[\sum_{i=1}^N \lambda_i g_i(r^*)\right]^2}{\sum_{i=1}^N \lambda_i l_i h_i} + \sum_{i=1}^N \lambda_i \frac{[g_i(r^*)]^2}{l_i h_i \Phi'(z_i(r^*))} \right\}.$$

This expression is strictly positive because, by the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^N \lambda_i \frac{[g_i(r^*)]^2}{l_i h_i}\right)\left(\sum_{i=1}^N \lambda_i l_i h_i\right) \geq \left(\sum_{i=1}^N \sqrt{\frac{\lambda_i}{l_i h_i}} g_i(r^*) \cdot \sqrt{\lambda_i l_i h_i}\right)^2 = \left[\sum_{i=1}^N \lambda_i g_i(r^*)\right]^2,$$

which implies

$$\sum_{i=1}^N \lambda_i \frac{[g_i(r^*)]^2}{l_i h_i} - \frac{\left[\sum_{i=1}^N \lambda_i g_i(r^*)\right]^2}{\sum_{i=1}^N \lambda_i l_i h_i} \geq 0.$$

Since $\widehat{k}^d(r)$ is monotonically decreasing, this result implies that $\widehat{k}^s(r)$ must be cutting $\left(\sum_{i=1}^N \lambda_i l_i h_i\right) \widehat{k}^d(r)$ from below *at every intersection point*. Since both $\widehat{k}^d(r)$ and $\widehat{k}^s(r)$ are continuous, if there are more than one $r^* \in \left(\widehat{\delta}_k, \widetilde{r}_N\right)$ that solve $\left(\sum_{i=1}^N \lambda_i l_i h_i\right) \widehat{k}^d(r) = \widehat{k}^s(r)$ then at least one of them must have $\widehat{k}^s(r)$ cutting $\left(\sum_{i=1}^N \lambda_i l_i h_i\right) \widehat{k}^d(r)$ from above. This gives rise to a contradiction and hence establishes the uniqueness of $r^*$. This completes the proof of the theorem. ■

To establish the results in Theorem A1, we have to impose two mild regularity conditions. The first condition requires $\beta_i \gamma^{1-\sigma} < 1$ for all $i \in \{1, ..., N\}$. This condition is both necessary and sufficient to ensure that the lifetime utility for all types of consumers is finite along the balanced growth path.[22] The second condition, stated in (21), ensures that the unique equilibrium rental rate $r^*$ is greater than $\widehat{\delta}_k$. It is important to point out that condition (21) can be checked before solving for the equilibrium rental rate $r^*$. Using (9) and (20), we can obtain $\widehat{k}^d\left(\widehat{\delta}_k\right) \equiv \left(\frac{\alpha}{\widehat{\delta}_k}\right)^{\frac{1}{1-\alpha}}$, and

$$\widehat{k}^s\left(\widehat{\delta}_k\right) = \widehat{w}\left(\widehat{\delta}_k\right) \sum_{i=1}^N \lambda_i l_i h_i \left[\frac{1}{\theta}\left(\frac{\gamma^\sigma}{\beta_i} - \gamma\right)\right]^{-\frac{1}{\sigma}}.$$

Hence, condition (21) holds if and only if

$$\left(\sum_{i=1}^N \lambda_i l_i h_i\right) \frac{\alpha}{\widehat{\delta}_k} > (1-\alpha)\, \theta^{\frac{1}{\sigma}} \left[\sum_{i=1}^N \lambda_i l_i h_i \left(\frac{\gamma^\sigma}{\beta_i} - \gamma\right)^{-\frac{1}{\sigma}}\right].$$

[22] This condition is commonly used in models that allow for perpetual growth in per-capita consumption. See, for instance, King, Plosser and Rebelo (1988) p.203.

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