Adaptive controller design for modified projective synchronization of Genesio–Tesi chaotic system with uncertain parameters

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Abstract

The paper addresses control problem for the modified projective synchronization of the Genesio–Tesi chaotic systems with three uncertain parameters. An adaptive control law is derived to make the states of two identical Genesio–Tesi systems asymptotically synchronized up to specific ratios. The stability analysis in the paper is proved using a well-known Lyapunov stability theory. A numerical simulation is presented to show the effectiveness of the proposed chaos synchronization scheme.

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1. Introduction

Chaos is very interesting nonlinear phenomenon and has been intensively studied in the last three decades. In particular, chaos synchronization, first described by Fujisaka and Yamada [1] in 1983, did not received great attention until 1990 [2]. From then on, chaos synchronization has been developed extensively due to its various applications [3–30]. The idea of synchronization is to use the output of the master system to control the slave system so that the output of the response system follows the output of the master system asymptotically. So various synchronization schemes, such as variable structure control [9], OGY method [3], parameters adaptive control [12–15,21], observer-based control [24], active control [22], time-delay feedback approach [7], backstepping design technique [16], and so on, have been successfully applied to the chaos synchronization. Using these methods, numerous works for the synchronization problem of well-known chaotic systems such as Lorenz, Chen, Li, and Rossler systems have been done by many scientists. However, most of research efforts mentioned above have concentrated on studying complete synchronization (CS), identical synchronization, or conventional synchronization, where two coupled chaotic systems exhibit identical oscillations. In the practical applications, CS only occurs at a certain point in the parameter space, and it is difficult to achieve CS except under ideal conditions. Recently, thus a more general form of synchronization scheme, called

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generalized synchronization (GS) has been extensively investigated [25–28], where the drive and response systems could be synchronized up to a scaling factor $\alpha$. It suggests that one can achieve control of this synchronization in general classes of chaotic systems including non-partially-linear systems. More recently, Li [28] consider a new GS method, called modified projective synchronization (MPS), where the responses of the synchronized dynamical states synchronize up to a constant scaling matrix.

In this paper, the problem of chaos synchronization to Genesio–Tesi system with three uncertain parameters is considered. For MPS of the system, a novel adaptive control scheme has been proposed. Then, the chaos synchronization of the system is proved by the Lyapunov’s direct method.

The organization of this paper is as follows. In Section 2, the problem statement and drive-response synchronization scheme are presented for the Genesio–Tesi system. In Section 3, a numerical example is given to demonstrate the effectiveness of the proposed method. Finally concluding remark is given.

2. Chaos synchronization of Genesio–Tesi system

The Genesio–Tesi system, proposed by Genesio and Tesi [31], is one of paradigms of chaos since it captures many features of chaotic systems. It includes a simple square part and three simple ordinary differential equations that depend on three positive real parameters. The dynamic equation of the system is as follows:

$$\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -cx - by - az + x^2,
\end{align*}$$  

(1)

where $x$, $y$, $z$ are state variables, and $a$, $b$ and $c$ are the positive real constants satisfying $ab < c$.

For instance, the system is chaotic for the parameters $a = 1.2$, $b = 2.92$, $c = 6$.

In order to observe the synchronization behavior in Genesio–Tesi system, when the parameters of the drive system are fully unknown and different with those of the response system, we assume that we have two Genesio–Tesi systems where the drive system with the subscript $m$ drives the response system having identical equations denoted by the subscript $s$. For the systems (1), the drive (or master) and response (or slave) systems are defined below, respectively,

$$\begin{align*}
\dot{x}_m &= y_m, \\
\dot{y}_m &= z_m, \\
\dot{z}_m &= -c x_m - b y_m - a z_m + x_m^2
\end{align*}$$

(2)

and

$$\begin{align*}
\dot{x}_s &= y_s + u_1, \\
\dot{y}_s &= z_s + u_2, \\
\dot{z}_s &= -c_1 x_s - b_1 y_s - a_1 z_s + x_s^2 + u_3,
\end{align*}$$

(3)

where $a_1$, $b_1$ and $c_1$ are parameters of the slave system which needs to be estimated, and $u_1$, $u_2$ and $u_3$ are the nonlinear controller such that two chaotic systems can be synchronized in the sense of MPS, i.e.,

$$\begin{align*}
\lim_{t \to \infty} \|x_m - a_1 x_s\| &= 0, \\
\lim_{t \to \infty} \|y_m - a_2 y_s\| &= 0, \\
\lim_{t \to \infty} \|z_m - a_3 z_s\| &= 0.
\end{align*}$$  

(4)

Note that when the parameter $x_i$ has same value for each $i$, it is called generalized projective synchronization. That is, all the dynamical states should be amplified or reduced synchronously.

Subtracting Eq. (2) from Eq. (3) yields error dynamical system between Eqs. (2) and (3)

$$\begin{align*}
\dot{e}_1(t) &= e_2 + (x_2 - x_1) y_s - a_1 u_1, \\
\dot{e}_2(t) &= e_3 + (x_3 - x_2) z_s - a_2 u_2, \\
\dot{e}_3(t) &= -c x_m + c_1 x_s - b y_m + b_1 x_3 y_s - a z_m + a_1 x_3 z_s + x_m^2 - a_3 x_s^2 - x_3 u_3
\end{align*}$$

(5)
where
\[
\begin{align*}
e_1(t) &= x_m(t) - x_i x_i(t), \\
e_2(t) &= y_m(t) - x_2 y_i(t), \\
e_3(t) &= z_m(t) - x_3 z_i(t).
\end{align*}
\]

Here, our goal is to make synchronization between two Genesio–Tesi systems by using adaptive control scheme \(u_i, i = 1, 2, 3\) when the parameters of the drive system are unknown and different with those of the response system, i.e.,
\[
\lim_{t \to \infty} \|e(t)\| = 0,
\]
where \(e = [e_1, e_2, e_3]^T\).

For two identical Genesio–Tesi systems without control \((u_i = 0, i = 1, 2, 3)\), if the initial condition \((x_m(0), y_m(0), z_m(0)) \neq (x_i(0), y_i(0), z_i(0))\), the trajectories of the two identical systems will quickly separate each other and become irrelevant. However, for the two controlled Genesio–Tesi systems, the two systems will approach synchronization for any initial condition by appropriate control gain. For this end, we propose the following control law for the system (3):
\[
\begin{align*}
u_1 &= \frac{1}{x_1}(k_1 e_1 + e_2 - c_1 e_3 + (x_2 - x_1) y_i), \\
u_2 &= \frac{1}{x_2}(k_2 e_2 + (1 - b_1) e_3 + (x_3 - x_2) z_i), \\
u_3 &= \frac{1}{x_3}[(k_3 - a_1) e_3 + (x_m + x_1 x_i) e_1 + b_1 (x_3 - x_2) y_i + c_1 (x_3 - x_1) x_i + (x_1^2 - x_3^2) x_i^2],
\end{align*}
\]
and the update rule for three unknown parameters \(a_1, b_1, c_1\)
\[
\begin{align*}
a_1 &= -z_m e_3, \\
b_1 &= -y_m e_3, \\
c_1 &= -x_m e_3,
\end{align*}
\]
where \(k_i, (i = 1, 2, 3) > 0\).

Then, we have the following main result.

**Theorem.** For given synchronization scaling factor \(x_i, (i = 1, 2, 3)\) and any initial condition \((x_m(0), y_m(0), z_m(0)) \neq (x_i(0), y_i(0), z_i(0))\), the modified projective synchronization between drive systems (2) and response system (3) will occur by the control law (7) and the update law (8).

**Proof.** Choose the following Lyapunov candidate:
\[
V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2)
\]
where \(e_a = a_1 - a, e_b = b_1 - b\) and \(e_c = c_1 - c\).

The differential of the Lyapunov function along the trajectory of error system (5) is:
\[
\begin{align*}
dV &= \dot{e}_1 e_1 + \dot{e}_2 e_2 + \dot{e}_3 e_3 + \dot{e}_a e_a + \dot{e}_b e_b + \dot{e}_c e_c \\
&= e_1 [e_2 + (x_2 - x_1) y_i - x_1 u_1] + e_2 [e_3 + (x_3 - x_2) z_i - x_2 u_2] \\
&+ e_3 [-c x_m + c_1 x_1 x_i - b y_m + b_1 x_3 y_i - a z_m + a_1 x_2 z_i + x_m^2 - x_3 x_1^2] \\
&+ \dot{a}_1 (a_1 - a) + \dot{b}_1 (b_1 - b) + \dot{c}_1 (c_1 - c).
\end{align*}
\]

By substituting Eqs. (7) and (8) into Eq. (10) and simple manipulation, we have
\[
\begin{align*}
dV/dt &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 = -e^T Ke
\end{align*}
\]
where
\[
e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}.
\]
Since $\dot{V}$ is negative semi-definite, we cannot immediately obtain that the origin of error system (6) is asymptotically stable. In fact, as $\dot{V} \leq 0$, then $e_1, e_2, e_3 \in \mathcal{L}_\infty$ and $e_a, e_b, e_c \in \mathcal{L}_\infty$. From the error system (5), we have $\dot{e}_1, \dot{e}_2, \dot{e}_3 \in \mathcal{L}_\infty$. Since $V = -e^T e$, then we have
\[
\int_0^t \|e\|^2 \, dt \leq \int_0^t e^T e \, dt \leq \int_0^t -V \, dt = V(0) - V(t) \leq V(0).
\]
Thus $e_1, e_2, e_3 \in \mathcal{L}_2$. According to the Barbalat’s lemma, we have $e_1(t), e_2(t), e_3(t) \to 0$ as $t \to \infty$, i.e., $\lim_{t \to \infty} \|e(t)\| = 0$. Therefore, the response system (3) synchronize the drive system (2) by the controller (7). This completes the proof.

**Remark 1.** When $a_i = 1$ ($i = 1, 2, 3$) and $a_i = \sqrt{-1}$ ($i = 1, 2, 3$), respectively, the modified projective synchronization becomes complete synchronization and anti-synchronization, respectively.

**Remark 2.** The convergence rate of error signals can be regulated by adjusting the control parameters $k_i$ ($i = 1, 2, 3$).

### 3. Numerical example

In this section, to verify and demonstrate the effectiveness of the proposed method, we discuss the simulation result for Genesio–Tesi system. In the numerical simulations, the fourth-order Runge–Kutta method is used to solve the systems with time step size 0.001.

For this numerical simulation, we assume that the initial condition, $(x_m(0), y_m(0), z_m(0)) = (2, -3, 1)$, and $(x_s(0), y_s(0), z_s(0)) = (-3, 5, -2)$ is employed. As a test for verification of modified projective synchronization of the system, let us take $x_1 = 1, x_2 = -1, \text{ and } x_3 = 3$. Hence the error system has the initial values $e_1(0) = 5, e_2(0) = 2$ and $e_3(0) = 7$. The three unknown parameters are chosen as $a = 1.2, b = 2.92$ and $c = 6$ in simulations so that the Genesio–Tesi system exhibits a chaotic behavior. Synchronization of the systems (2) and (3) via adaptive control law (7) and (8) with the initial estimated parameters $a_i(0) = 0.5, b_i(0) = \sqrt{-1}, \text{ and } c_i(0) = 4$ are shown in Figs. 1 and 2. Fig. 1 displays the error signals between drive and response systems.

Fig. 1. Error signals between drive and response systems.
synchronization errors of systems (2) and (3). Fig. 2 shows that the estimates $a_1(t)$, $b_1(t)$, $c_1(t)$ of the unknown parameters converges to $a = 1.2$, $b = 2.92$ and $c = 6$ as $t \to \infty$.

4. Concluding remark

In this paper, we investigate the synchronization of controlled Genesio–Tesi chaotic systems with three uncertain parameters. We have proposed a novel adaptive nonlinear control scheme for modified projective synchronization using the Lyapunov stability theory. Finally, a numerical simulation is given to illustrate the effectiveness of our method.

References