Generalized inexact proximal algorithms: habit’s formation with resistance to change, following worthwhile changes.

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Abstract

This paper shows how, in a quasi metric space, an inexact proximal algorithm with a generalized perturbation term appears to be a nice tool for Behavioral Sciences (Psychology, Economics, Management, Game theory, . . . ). More precisely, the new perturbation term represents an index of resistance to change, defined as a “curved enough” function of the quasi distance between two successive iterates. Using this behavioral point of view, the present paper shows how such a generalized inexact proximal algorithm can modelize habit’s formation in a striking way. This idea comes from a recent “variational rationality approach” of human behavior which unifies a lot of different theories of stability (habits, routines, equilibrium, traps, . . . ) and changes (creations, innovations, learning and destructions, . . . ) in Behavioral Sciences and a lot of concepts and algorithms in Variational Analysis. In this variational context, the perturbation term represents a specific instance of the very general concept of resistance to change, which is the desutility of some costs to change. Central to the analysis are the original variational concepts of “worthwhile changes” and “marginal worthwhile stays”. At the behavioral level, this paper advocates that proximal algorithms

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are well suited to modelize the emergence of routinized human behaviors. We show when, and at which speed, a “worthwhile to change” process converges to a behavioral trap.

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1 Introduction

The main message of this paper is that, using the behavioral context of a recent “Variational rationality approach” [1][2], a generalized proximal algorithm can modelize fairly well an habituation process as described in Psychology. This is the case even when resistance to change (inertia) is weak (our paper). This opens the door to a new vision of proximal algorithms. They are not only very nice mathematical tools in Optimization theory, with striking computational aspects. They can also be nice tools to modelize the dynamics of human behaviors.

Recent progress have been made in the modelization of the theories of stability (habits, routines, equilibrium, traps, . . . ). and change (creations, destructions, learning, innovation, attitudes as well as beliefs formation and revision, self regulation problems including goal setting, goal striving and goal revision, habit formation, breaking and forming habits. . . ). In the interdisciplinary context which characterize all these theories in Behavioral Sciences, the “Variational rationality approach” (see [1][2]), shows how to modelize the course of human activities as a succession of actions balancing, each step, between motivation to change (the utility of advantages to change) and resistance to change (the desutility of costs to be able to change). This very simple idea has allowed to see proximal algorithms as an essential tool to modelize the human course of actions, where the perturbation term of a proximal algorithm can be seen as a crude formulation of the complex concept of resistance to change, while the utility generated by a change in the objective function can represent a crude formulation of the motivation to change concept. This variational approach have provided an extra motivation to develop further the study of proximal algorithms in a nonconvex and possibly
nonsmooth setting. Among other recent applications of this simple idea, see Attouch and Soubeyran [11] for local search proximal algorithms, Flores-Bazan et al. [6] for worthwhile to change games, alternating inertial games with costs to move, Attouch et al. [7] and Cruz Neto et al. [8] for the “how to play Nash” problem, . . . . In all these papers the perturbation term of the usual proximal point algorithm is a linear or a quadratic function of the distance or quasi distance between two successive iterates. They modelize the case of “strong enough resistance” to change. Our paper examines the opposite case of “weak enough” resistance to change where the perturbation term modelizes the difficulty (relative resistance) to be able to change as a “curved enough” function of the quasi distance between two successive iterates. A quasi distance modelizes costs to be able to change as an index of dissimilarity between actions where the cost to be able to change from an action to an other one is not the same as the cost to be able to change in the other way. In this paper, we show when, in a quasi metric space, an inexact proximal algorithm with such a generalized perturbation term converges to a critical point, depending of the curvature of the perturbation term and of the Kurdyka-Lojasiewicz property associated to the objective function. In the context of the “Variational rationality” approach, we give sufficient conditions for this critical point to be a variational trap, “easy enough to reach” (that the agent can reach using a succession of worthwhile changes), and “difficult enough to leave” (such that, being there, it is not worthwhile to move from there). Central to the analysis are the original concepts of “worthwhile changes” and “non marginal worthwhile changes”, the pillar of the Variational rationality approach; see [1, 2]). These worthwhile and non marginal worthwhile changes represent sufficient descent conditions as well as an internal stopping rule, each round. Doing so, this paper extends the convergence result to a critical point of Attouch and Bolte [3], Attouch et al. [4] and Moreno et al. [5], using a fairly general “convex enough” perturbation term. As an application, we show how such a generalized inexact proximal algorithm can help to modelize habit formation, in a way close to Psychologists do verbally (see the appendix). Then, at the behavioral level, the main message of this paper is to advocate that our generalized proximal algorithm is well suited to modelize the formation of routinized human behaviors. We also examine the speed of convergence to a behavioral trap, in relation to the strength of resistance to change. We obtain a striking result for
Behavioral Sciences, related to the famous “loss aversion effect”, showing that higher relative aversion to change (a higher resistance to change with respect to motivation to change) favors convergence.

Our paper is organized as follows. Section 2 lists the main variational tools necessary to define the central concept of “worthwhile change” for behavioral applications. Section 3 examines a generalized inexact proximal algorithm which converges to a critical point when the objective function satisfies a Kurdyka-Lojasiewicz inequality. This section considers also the speed convergence of this process. Section 4 proposes and studies a worthwhile to change process which converges to a critical point which is also a weak trap. Finally, Section 5 shows how our generalized inexact proximal algorithm can modelize habit formation in Psychology. The conclusion follows. An annex gives a very short survey of habit formation in Psychology, compared to Economics and Management.

## 2 Worthwhile Changes

A recent variational rationality approach, see [1,2], unified a lot of theories of change in Behavioral Sciences (Psychology, Economics, Management, Decision theory, Philosophy, Game theory, Artificial Intelligence...) using as a central building bloc the concepts of “worthwhile change” and “marginal worthwhile change”. The main idea is quite evident. If a behavioral theory wants to explain “why, how and when” agents perform actions and change, this theory must define, each period, along a path of change, why the agent have, first, an incentive to do some steps away from his current position and, then, an incentive to stop changing one step more within this period. First, this is the case when this change is “worthwhile”, i.e, when his ex ante motivation to change is sufficiently higher than his ex ante resistance to change. Then, this change is desirable and feasible enough, i.e, acceptable or satisficing, improving with no too high costs to improve. Second, in the same period, the agent must also have to know when he must stop changing. This is the case when one step more is not worthwhile. More formally, this change is not “marginaly worthwhile” when the ex ante marginal motivation to change is sufficiently lower than the ex ante marginal resistance to change. In this case the agent does not regret ex ante to do not go one step further. The motivation to change again next period
comes from residual unsatisfied needs or variable preferences.

In this section we list, very briefly, the main variational tools (elementary concepts) which are necessary to define the main and complex concept of “worthwhile change”. To save space, for much more comments, and a more complete formulation of each of these variational concepts, with references to a lot of different discipliness in Behavioral Sciences which help to justify their unifying power; see [1][2]. This being done, we can easily show how an inexact proximal algorithm represents a nice benchmark process of step by step worthwhile changes in term of sufficient descent conditions.

2.1 Variational concepts

Let $X$ be a action space (or, depending on the context, position or state space), where, in this paper, $X$ is the Euclidean space $\mathbb{R}^n$.

1) **Actions:** A complex action, let us say an action, $x \in X$, transforms some state $s \in S$ into some state $s' \in S : s \sim^x s'$. Such an action is defined as a succession of elementary actions (defined as elementary transformations or operations)

$$x = \{ x_0, x_1, ..., x_{n-1} \}$$

which aim to transform state $s = s_0$ into state $s' = s_n$, where the elementary action $x_i$ transforms $s_i$ into $s_{i+1}, i = 0, 1, ... n - 1$. The state space $S$ can be anything.

2) **Behavioral Changes:** They represent moves a move $x \sim y$, where an agent changes from doing an action $x \in X$ to do a different action $y \in X - \{ x \}$. Stays, $x \sim x$, mean to do again the same action $y = x$.

3) **Payoff Functions:** They can be either a gain $g : X \to \mathbb{R} \cup \{-\infty\}$ which must be improved (performance, revenue, profit) or an unsatisfied need $f : X \to \mathbb{R} \cup \{+\infty\}$ to be decreased. The link between these two dual concepts works as follow. Let $\bar{g} = \sup \{ g(y) : y \in X \} < +\infty$ be the highest finite payoff that the agent can hope to reach; doing some action $y \in X$. Suppose that the agent knows this so called hope or aspiration level. Then, starting from the current state defined as “being able to do an action $x$”, the unsatisfied need of the agent will be $f(x) = \bar{g}(x) - g(x) \geq 0$.

4) **Advantages to Changes:** The advantage to change, which we denote by $A$, modelizes how much an agent gains
to move from $x \in X$ to $y \in X$. For example, in terms of payoff functions, $A_+(x,y) = g(y) - g(x) \geq 0$ can be the advantage to change function, the improvement in payoff from moving from $x$ to $y$ instead of staying at $x$ and $A_-(x,y) = f(x) - f(y) \geq 0$ can be the advantage to change function, the decrease in unsatisfied needs to move from $x$ to $y$, instead of staying at $x$. These advantages to change function coincide,

$$A_-(x,y) = [g - g(x)] - [g - g(y)] = g(y) - g(x) = A_+(x,y) = A(x,y).$$

5) **Motivation to Change:** Let $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an utility function where $U[A]$ is the utility of the advantage to change $A = A(x,y)$ with $U$ being invertible and $U[0] = 0$. As in Tversky and Kahneman [9], even in a context of riskless choice, the argument of the utility function are gains or losses $A \leq 0$. The motivation to change from the status quo $x$ to the new position (action) $y$ instead of staying at $x$ is given by

$$M(x,y) := U[A(x,y)]. \quad (2.1)$$

The motivation of the agent is either to increase his payoff or to decrease his unsatisfied needs.

6) **Costs to be Able to Change:** They are quite difficult to modelize and very different from the traditional “costs to do” (or “execution costs”), which are embedded in the payoff function $g(y/x)$. They require to modelize actions and the capabilities to do them; see [1]. These costs $C(x,y) \geq 0$ represent, either i) “having done action $x$ just before” (the gross formulation), or ii) “being able to do action $x$ right now” (the net formulation), the costs to be able to do a new action $y$ (or the same action $x$, i.e. doing $x$ again). To save space, consider the net formulation, where, each change, the agent moves from being able to do an old action $x$ to the ability to do a new action $y$, or to the ability to repeat the same action $x$. These ability costs to move have two components: knowledge costs and power costs to have the ability to move.

A) “Costs to have the ability to know what, how and when to change” from $x$ to $y$. They may be the sum of:

i) costs to know what means to delete, to conserve and to acquire;
ii) costs to know how and when to delete some means, and how and when to use the other means (old means to be used again and new means to be used for the first time).

B) Costs to have the power to change modelize the costs to have (own), available and ready for use, the means to do a new action. They represent the costs to delete and stop to use old means which will not be used to do the new action, the costs to regenerate and use the other old means which will be used again to do the new action, and the costs to acquire and use new means to do the new action.

Notice that there exist a huge variety of costs to change, depending of the activity. For example reactivity costs which depend of the speed of moving, fatigue costs which vary with the amount of effort done before, costs to alternate between different activities within a goal system, costs to self regulate own activities (goal setting, goal striving, goal revision and goal pursuit), emotional costs…. To save space (this is a quite complicated topic), consider only the net costs $C(x, y) \geq 0$ to be able to change. By definition $C(x, x) = 0$, for all $x \in X$, because the agent pays no costs to move from being able to do an action $x$ to be able to do this action $x$. Notice that for gross costs to be able to change, $C(x, x) \geq 0$ can be strictly positive for some $x \in X$. This is because, moving from having done the previous action $x$, it can be costly to be able to do it again (repetition). For costs to have the ability to know “what, how and when to change” from $x$ to $y = x$, these costs can be very low because, having done action $x$ the last period, the agent is supposed, for example, to know, at the beginning of the new period, what means to conserve and to use again, if he has in mind to do again action $x$. A natural assumption is that $C(x, y) > 0$ for $y \neq x$. This is true for gross and net costs to be able to do a new action, having done before an old action, or being able to do it. Costs to be able to change from $y$ to $x$ are different from costs to be able to change from $x$ to $y$. Costs to be able to change are supposed to satisfy the triangular inequality, $C(x, z) \leq C(x, y) + C(y, z)$, for all $x, y, z \in X$. In the context of the “variational rationality” approach, where paths of exploration and paths of change play a major role, this is a natural assumption; see [1].

Summarizing, $C : X \times X \to \mathbb{R}_+$ is a quasi-distance (not necessarily symmetric). For an explicit example where
the costs to be able to change is a quasi-distance, see Moreno et al. [5].

7) **Inconvenients to change:** The inconveniente to change is defined relative to cost to be able to change as follows:

\[ I(x,y) := C(x,y) - C(x,x), \quad x \in X. \]

8) **Resistance to Change:** Let \( D : \mathbb{R}_+ \to \mathbb{R}_+ \) be the desutility of costs to be able to change, where \( D \) is strictly increasing and \( D[0] = 0 \). The resistance to change from \( x \) to \( y \) is given by \( R(x,y) := D[I(x,y)] \). As we have supposed \( C(x,x) = 0 \), for all \( x \in X \), it reduces to

\[ R(x,y) = D[C(x,y)]. \quad (2.2) \]

It modelizes inertia, difficulties and barriers to change, depending of the context.

9) **Worthwhile Changes:** Worthwhile changes are those changes such that motivation to change is higher than some fraction of resistance to change, namely,

\[ M(x,y) \geq \xi(x)R(x,y). \]

The number \( \xi(x) > 0 \) modelizes “how much worthwhile” this change can be (more or less acceptable, desirable and feasible). The higher is this ratio, the more worthwhile it is to change and the more the agent will have the intention to change. It can be related to a reference dependent satisficing rate. [12], offering an original theory of satisficing. [10], when inertia matters. If \( 0 < \xi(x) < 1 \), the agent will accept to change, in spite of some possible temporary sacrifices \( (1 - \xi(x))R(x,y) > 0 \) if \( y \neq x \). If \( \xi(x) \geq 1 \), the agent will accept to change if he can hope the net temporary gain \( M(x,y) - R(x,y) \geq (\xi(x) - 1)R(x,y) \geq 0 \). Attouch and Soubeyran [11] examined a simpler modelization of worthwhile changes, where \( U[A] = A \) and \( D[C] = C \), to modelize a prototype version of local search proximal algorithms. The worthwhile to change set at \( x \in X \) is given by

\[ W(x) := \{ y \in X : M(x,y) \geq \xi(x)R(x,y) \}. \quad (2.3) \]

Since \( A(x,x) = C(x,x) = U[0] = D[0] = 0 \), from (2.1) and (2.2) it follows immediately that \( x \in W(x) \), for all \( x \in X \).
10) **Aversion to Change (inertia):** The variational rationality approach defines aversion to small change (a kind of inertia) where advantages to change $A > 0$ loom lower than the same size costs to be able to change $C = A = Z > 0$, this common size $Z$ being small enough. This means that if $Z > 0$, then $U[Z] \leq \delta D[Z]$ for some $\delta < 1$, for $Z$ small enough. Let us define the degree of aversion to change as $\delta[Z] = U[Z]/D[Z]$ for $Z > 0$. Then, there is aversion to change when $\delta[Z] \leq \bar{\delta} < 1$. In the last section we will compare “aversion to change” to the famous “loss aversion” concept (see [9], for our case of riskless choices, and Kahneman and Tversky [12] for the case of risky choices). For example, if $U[A] = A^\lambda, \lambda > 0$, the lower $\lambda$, the more concave is the utility function, and, if $D[C] = C^\mu, \mu > 0$, the higher $\mu$ the more convex is the cost to change function. Then, $\delta[Z] = Z^{(\lambda-\mu)} \leq \bar{\delta} < 1 \iff 0 < Z \leq (\bar{\delta})^{1/(\lambda-\mu)}$ for $\lambda > \mu$.

11) **Relative Resistance to Change:** Taking into account that the utility function is invertible, let us define the relative resistance to change function as

$$\Gamma[C(x,y)] = U^{-1}[D[C(x,y)]].$$  \hspace{1cm} (2.4)

For example, if $U[A] = A^\lambda, \lambda > 0$ and $D[C] = C^\mu, \mu > 0$, then

$$\Gamma[C(x,y)] = [C(x,y)^\mu]^{1/\lambda} = C(x,y)^\alpha,$$

where $\alpha = \mu/\lambda > 0$. Then, the shape of the relative resistance to change function $\Gamma[\cdot]$ depends of how much the utility and desutility functions are concave or convex.

12) **A Path of Worthwhile Changes:** A sequence $\{x^k\} \subset X$ is a path of worthwhile changes iff

$$x^{k+1} \in W(x^k), \quad k \in \mathbb{N}.$$ 

13) **An Habituation Process:** It is such that, step by step, gradually, the agent carries out a more and more similar action. This is equivalent to say than the quasi distance $C(x^k,x^{k+1})$ converges to zero as $k$ goes to infinite.

14) **Variational Traps and Habit Formation:** A given action $x^* \in X$ is a global variational trap if it is worthwhile to
stay there, i.e.,

\[ M(x^*, y) < \xi(x^*) R(x^*, y), \quad y \neq x^*. \]

Note that this condition means that the agent strictly prefers to stay at \( x^* \) than to move away from \( x^* \) to any other action \( y \). When a worthwhile to change process converges to a variational trap, this variational formulation offers a model of trap as the end point of a path of worthwhile changes.

### 2.2 Proximal Formulation of Worthwhile Changes

Suppose that:

**H1.** Costs to be able to change \( C \) are modeled as quasi-distances and we use the following notation: \( C := q \);

**H2.** Unsatisfied needs \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is a proper lower semicontinuous function;

**H3.** Advantages to change are \( A(x, y) = \rho(x) [f(x) - f(y)] \), where \( \rho(x) = 1/\lambda(x) > 0 \) is a weight attached to these advantages;

**H4.** The degree \( \xi(x) > 0 \) which modelizes how much a change can be worthwhile is constant, \( \xi(x) = 1 \) for all \( x \in X \);

**H5.** The utility function \( U[\cdot] \) is invertible with \( U[0] = 0 \).

Considering the assumptions H1, H3, H4 and H5, from (2.1), (2.2) and (2.4), it is easy to see that \( y \in W(x) \) iff

\[ f(x) - f(y) \geq \lambda(x) \Gamma[q(x, y)]. \]

Hence, the worthwhile to change set can be rewritten as follows:

\[ W_{\lambda(x)}(x) = \{ y \in X, f(x) - f(y) \geq \lambda(x) \Gamma[q(x, y)] \}. \]

Note that for different choices of \( \Gamma \) and \( q \), the worthwhile to change condition which defines the worthwhile to change set is a descent condition; see, for instance, [3,5]. This variational approach provides us an extra motivation to develop
further the study of proximal algorithms in a nonconvex and possibly nonsmooth setting where the perturbation term of the usual proximal point algorithm becomes a “curved enough” function of the quasi distance between two successive iterates. More precisely, given $x^0 \in \mathbb{R}^n$ and a bounded sequence of positive real numbers $\{\lambda_k\}$ (called regularization parameters), the next step is such that

$$x^{k+1} \in \arg\min_{y \in \mathbb{R}^n} \left\{ f(y) + \lambda_k \Gamma[q(x^k, y)] \right\}. \quad (2.5)$$

It is easy to see that any sequence generated from (2.5) is a path of worthwhile changes, namely,

$$x^{k+1} \in W_{\lambda_k}(x^k), \quad k \in \mathbb{N}. \quad \text{(2.6)}$$

In the remainder of this paper we assume that the assumptions H1-H5 hold and $\Gamma$ is a twice differentiable function such that:

$$\Gamma[0] = \Gamma'[0] = 0, \quad \Gamma'[q] > 0, \quad \Gamma''[q] > 0, \quad q > 0, \quad \text{and} \quad 0 < q \leq \bar{q}. \quad (2.7)$$

Let us consider a generalized rate of curvature of $\Gamma$ given by:

$$\rho_{\Gamma}(q, r) := \frac{\Gamma'[q/r]}{\Gamma'[q]/q}, \quad 0 < q \leq \bar{q}. \quad \text{(2.8)}$$

In the particular case $r = 1$, (2.8) represents, in Economics, the elasticity of the desutility curve $\Gamma$; see, for instance, [1, 2]. From (2.8), condition (2.7) is equivalent to the condition:

$$\bar{\rho}_{\Gamma}(r) = \sup\{\rho_{\Gamma}(q, r) : 0 < q < \bar{q}\} < +\infty, \quad r \in (0, 1) \quad \text{fixed}. \quad (2.9)$$

Let us consider, for each $\alpha > 1$ fixed, the function $\Gamma[q] := q^\alpha$. It is easy to see that, in this case, $\bar{\rho}_{\Gamma}(r) \in [\alpha r^{1-\alpha}, +\infty)$. In particular, we can take

$$\bar{\rho}_{\Gamma}(q, r) = \alpha r^{1-\alpha} = \bar{\rho}_{\Gamma}(r) < +\infty. \quad (2.9)$$
More accurately, for each $\alpha > 1$, $\Gamma[q] = q^\alpha$ represents a desutility of costs to change. It is strictly increasing and satisfies (2.6) and (2.7). Note that, when $q$ is the Euclidean distance and $\Gamma[q] = q^2/2$, then (2.5) retrieves the classical proximal method whose origins can be traced back to the 1960s associated to convex optimization problems, see Moreau [13] and Martinet [14], as well as in the study of variational inequalities associated to maximal monotone operators; see Rockafellar [15]. The Method (2.5) is new and more adapted for applications in Behavioral Sciences. Moreover, it retrieves recent approaches of the proximal method for nonconvex functions; see [3, 5]. For other approaches of the proximal method for nonconvex functions see, for instance, Fukushima and Mine [16], Kaplan and Tichatschke [17], Spingarn and Jonathan [18], Pennanen [19], Iusem et al. [20], Combettes and Pennanen [21], Garcia-Otero and Iusem [22].

2.3 Marginally Worthwhile Changes

Let $x = x^k \rightharpoonup y = x^{k+1}$ be a worthwhile change from $x^k$ to $x^{k+1} \in W_{\lambda_k}(x^k)$ and let $x^{k+1} \rightharpoonup z \in \mathcal{M}(x^{k+1}) \subset X$ be a marginal change, where $\mathcal{M}(x^{k+1})$ is a small neighborhood of $x^{k+1}$ in the quasi-metric space $X$. Then, at each period $k$, the agent who has done the worthwhile change $y = x^{k+1} \in W_{\lambda_k}(x^k)$ will stop to prolonge this change if, doing one step more this period $k$, from $x^{k+1}$ to $z \in \mathcal{M}(x^{k+1})$, this marginal change is not worthwhile, i.e., $z \notin W_{\lambda_{k+1}}(x^{k+1})$. This is a generalized stopping rule, a “not worthwhile marginal change” condition, that will be used later in the context of proximal algorithms; see condition (3.13).

2.4 Diminishing Sensitivity to Change

The variational stopping rule condition raises the following question: when, marginally, a change stops to be worthwhile? This strongly depends on the shapes of the utility and desutility functions $U[A]$ and $D[C]$, evaluated at the change $x \rightharpoonup y$, where $x$ is the current action and $y$ is the future action. Let $A = A(x,y)$ and $C = C(x,y)$ be a couple of advantages to change and costs to be able change from $x$ to $y$. This change is worthwhile if $U(A) \geq \xi D(C)$, $\xi > 0$. Let
\[ \Delta A = A(y, z) \text{ and } \Delta C = C(y, z), z \in \mathcal{M}(y) \] be a couple of marginal advantages to change and costs to be able to change from \( y \) to the marginal position \( z \in \mathcal{M}(y) \) with respect to action \( y \). This marginal change (starting from \( y \)) will not be worthwhile if \( U'(A)\Delta A \leq b \Gamma'(C)\Delta C \) where \( b > 0 \) is a rate of acceptability (how worthwhile this marginal change must be to continue to change this period). In this context, the size of the marginal utility \( U'(A) \) and the marginal desutility \( D'(C) \) relative to the change \( x \sim y \) matters much. There is strict diminishing sensitivity to change when \( U''(A) < 0 \) and \( D''(C) < 0 \). We will see that this aspect is also strongly related to prospect theory, in the case of riskless choice; see [9].

### 3 An Inexact Descent Method for KL Functions: Convergence to a Critical Point

One usual criticism for exact proximal algorithms which is even stronger for behavioral applications (in the context of applications, agents usually are not interested in optimizing in each step, since this would be too expensive), the analysis of an inexact version is very important. In this section, following the ideas presented in [4], we propose and study an inexact version of the proximal method (2.5) whose full convergence is assured for objective functions that satisfy the Kurdyka-Lojasiewicz inequality. This approach includes an inexact version of the proximal point method studied in [5].

#### 3.1 Some Definitions from Subdifferential Calculus

In this section some elements concerning the subdifferential calculus are recalled; see, for instance, [23, 24].

From the assumption H1., \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is a proper lower semicontinuous function. The domain of \( f \), which we denote by \( \text{dom} f \), is the subset of \( \mathbb{R}^n \) on which \( f \) is finite-valued. Since \( f \) is proper, then \( \text{dom} f \neq \emptyset \).

**Definition 3.1.**
1) The Fréchet subdifferential of $f$ at $x \in \mathbb{R}^n$, denoted by $\hat{\partial} f(x)$, is the set given by:

$$\hat{\partial} f(x) = \begin{cases} 
\{ x^* \in \mathbb{R}^n : \liminf_{y \to x, y \neq x} \frac{1}{\|x - y\|} (f(y) - f(x) - \langle x^*, y - x \rangle) \geq 0 \}, & \text{if } x \in \text{dom } f, \\
\emptyset, & \text{if } x \notin \text{dom } f.
\end{cases}$$

ii) The limiting Fréchet subdifferential (or simply subdifferential) of $f$ at $x \in \mathbb{R}^n$, denoted by $\partial f(x)$, is the set given by:

$$\partial f(x) = \begin{cases} 
x^* \in \mathbb{R}^n | \exists x_n \to x, f(x_n) \to f(x), x_n^* \in \hat{\partial} f(x_n); x_n^* \to x^* \}, & \text{if } x \in \text{dom } f, \\
\emptyset, & \text{if } x \notin \text{dom } f.
\end{cases}$$

It is immediate, from the last definition, that $\hat{\partial} f(x) \subset \partial f(x)$, for each $x \in \mathbb{R}^n$. Moreover, for each $x \in \text{dom } f$, $\hat{\partial} f(x) = \{ \nabla f(x) \}$ (resp. $\partial f(x) = \{ \nabla f(x) \}$) when $f$ is differentiable at $x$ (resp. when $f$ is continuously differentiable at $x$), where $\nabla f$ represents the gradient of $f$.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a point-to-set mapping. Its graph is denoted by:

$$\text{Gr}(T) = \{ (x, u) \in \mathbb{R}^n \times \mathbb{R}^n : u \in T(x) \}.$$

In our convergence analysis, as well as in any limiting processes used in an algorithmic context, we need consider a subdifferential $T$ of $f$ that satisfies the following closedness property:

**Property 1.** Let $\{(x^k, v^k)\}$ be a sequence in $\text{Gr}(T)$. If $(x^k, v^k)$ converges to $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ and $f(x^k)$ converges to $f(x)$, then $(x, v) \in \text{Gr}(T)$.

In this sense, throughout the paper we consider the subdifferential $\partial f$. Considering the following function:

$$f(x) = \begin{cases} 
x^2 \sin(1/x), & \text{if } x \neq 0, \\
0, & \text{if } x = 0.
\end{cases} \quad (3.1)$$

it is easy to see that $\hat{\partial} f$ don’t satisfy Property 1.

From this generalized notion of differentiation a necessary condition for a given point $x \in \mathbb{R}^n$ to be a minimizer of $f$ is

$$0 \in \partial f(x). \quad (3.2)$$

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The function defined in (3.1) provides an example ensuring that, unless \( f \) is convex, (3.2) is not a sufficient condition. In the remainder, a point that satisfies (3.2) is called limiting-critical or simply critical.

**Proposition 3.1.** Let \( f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \) be functions such that \( f_1 \) is locally Lipschitz continuous at \( \bar{x} \in \mathbb{R}^n \) while \( f_2 \) is proper lower semicontinuous, with \( f_2(\bar{x}) \) finite. Then,

\[
\partial (f_1 + f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x}).
\]

*Proof.* See [23, pages 350,431].

**Proposition 3.2.** Let \( f_1 : \mathbb{R}_+ \to \mathbb{R}_+ \) be continuously differentiable, \( f_2 : \mathbb{R}^n \to \mathbb{R}_+ \) a locally Lipschitz function at \( \bar{x} \in \mathbb{R}^n \). Then,

\[
\partial (f_1 \circ f_2)(\bar{x}) = f'_1(f_2(\bar{x}))\partial f_2(\bar{x}), \quad x \in [0 < f_2 < 1].
\]

*Proof.* See [25, Lemma 43].

### 3.2 The algorithm

Let us denote by \( B_q(x, \varepsilon) \) the open ball, with respect to the quasi distance \( q \), of center \( x \in X \) and radius \( \varepsilon > 0 \), defined as follows:

\[
B_q(x, \varepsilon) = \{ y \in M : q(x, y) < \varepsilon \}.
\]

In particular, if \( q \) is the Euclidean distance, \( B_q(x, \varepsilon) \) will be denoted \( B(x, \varepsilon) \).

**Assumption 3.1.** There exist \( \beta_1, \beta_2 \in \mathbb{R}_{++} \) such that:

\[
\beta_1 \|x - y\| \leq q(x, y) \leq \beta_2 \|x - y\|, \quad x, y \in \mathbb{R}^n.
\]

For an explicit example where the costs to be able to change is a quasi-distance satisfying Assumption 3.1 see Moreno et al. [5].
Let us consider the proximal point method (2.5) and assume that \( f \) is bounded below. Since \( f \) is proper lower semicontinuous, from the first inequality in (2.6) and (2.7) combined with Assumption 3.1, it is easy to see that this method is well-defined. In particular, \( x^{k+1} \in \text{dom} f \), for all \( k \in \mathbb{N} \). From the definition of \( x^{k+1} \), we have

\[
f(x^{k+1}) + \lambda_k \Gamma[q(x^k, x^{k+1})] \leq f(x^k), \quad k = 0, 1, \ldots.
\] (3.3)

Let \( \{x^k\} \) be a sequence generated by (2.5) and assume that:

\[
\{x^k\} \subset \text{dom} f, \quad x^k \neq x^{k+1}, \quad 0 < \underline{\lambda} \leq \lambda_k \leq \bar{\lambda}, \quad k \in \mathbb{N}.
\]

As \( x^k \neq x^{k+1}, k \in \mathbb{N} \), combining definition of \( q \) with definition of \( \Gamma \), it follows that

\[
\Gamma[q(x^k, x^{k+1})] > 0, \quad k = 0, 1, \ldots
\]

Hence, the inequality (3.3) implies that \( \{f(x^k)\} \) is strictly decreasing. Moreover, since \( \underline{\lambda} \geq \bar{\lambda} > 0 \) and \( f \) is bounded below, using again inequality (3.3), it is easy to see that \( \{\Gamma[q(x^k, x^{k+1})]\} \) is a summable sequence. In particular, taking into account that \( \Gamma \) is a continuous function, we conclude that there exists \( k_0 \in \mathbb{N} \) sufficiently large such that

\[
q(x^k, x^{k+1}) < 1, \quad k \geq k_0.
\] (3.4)

On the other hand, from the definition of \( x^{k+1} \) together with optimality condition (3.2), it follows that

\[
0 \in \partial f(x^{k+1}) + \lambda_k \partial \Gamma[q(x^k, \cdot)](x^{k+1}), \quad k = 0, 1, \ldots
\] (3.5)

Note that \( \Gamma \) is a continuously differentiable function and \( q(x^k, \cdot) \) is a Lipschitz function, for each \( k \in \mathbb{N} \) (see [5, Proposition 3.6]). In particular, \( \Gamma[q(x^k, \cdot)] \) is a locally Lipschitz function. So, applying Proposition 3.1 with \( f_1 = \Gamma[q(x^k, \cdot)] \) and \( f_2 = f \), from inclusion (3.5), we obtain:

\[
0 \in \partial f(x^{k+1}) + \lambda_k \partial \Gamma[q(x^k, \cdot)](x^{k+1}).
\] (3.6)

Now, since there exists \( k_0 \in \mathbb{N} \) such that (3.4) holds, using Proposition 3.2 with \( f_1 = \Gamma \) and \( f_2 = q(x^k, \cdot) \), we conclude that there exist \( w^{k+1} \in \partial f(x^{k+1}) \) and \( v^{k+1} \in \partial q(x^k, \cdot)(x^{k+1}) \) such that:

\[
0 = w^{k+1} + \lambda_k \Gamma'[q(x^k, x^{k+1})]v^{k+1}, \quad k \geq k_0.
\] (3.7)
As mentioned previously, this version of the proximal point method is new and generalizes the methods considered in [3, 5]. However, here we are interested in an inexact version of (2.5) characterized from the two conditions (3.3) and (3.7). For inexact versions in the convex case see, for instance, Rockafellar [15], Solodov and Svaiter [26, 27] and references therein. In the particular case where $\Gamma[q] = q^2/2$, $q(x, y) = \|x - y\|$ and $f$ is convex, in [27] the authors consider that $(x^{k+1}, w^{k+1}) \in \mathbb{R}^n \times \mathbb{R}^n$ is an inexact solution, with tolerance $\sigma \in [0, 1)$, for the subproblem (3.6) if:

$$w^{k+1} \in \partial f(x^{k+1}), \quad \tilde{e}^k = w^{k+1} + \lambda_k (x^{k+1} - x^k),$$

(3.8)

and

$$\varepsilon_k := \|\tilde{e}^k\| \leq \sigma \max\{|\|w^{k+1}\|, \lambda_k \|x^{k+1} - x^k\|\}.$$  

(3.9)

Note that (3.9) implies the following weaker condition:

$$\exists \ b > 0 : \ |w^{k+1}| \leq b\|x^{k+1} - x^k\|, \quad (3.10)$$

which coincides with the inexact optimality condition proposes in [4].

Next, we introduce an inexact version of the proximal point method (2.5):

**Algorithm 3.1.** Take $x^0 \in \text{dom } f$, $0 < \bar{\lambda} \leq \lambda < +\infty$, $\sigma \in [0, 1)$ and $b > 0$. For each $k = 0, 1, \ldots$, choose $\lambda_k \in [\bar{\lambda}, \hat{\lambda}]$ and find $(x^{k+1}, w^{k+1}, v^{k+1}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ such that:

$$f(x^k) - f(x^{k+1}) \geq \lambda_k (1 - \sigma) \Gamma[q(x^k, x^{k+1})], \quad (3.11)$$

$$w^{k+1} \in \partial f(x^{k+1}), \quad v^{k+1} \in \partial q(x^k, \cdot)(x^{k+1}), \quad (3.12)$$

$$|w^{k+1}| \leq b \Gamma'(q(x^k, x^{k+1}))(v^{k+1}), \quad (3.13)$$

**Remark 3.1.** Note that Algorithm [2.7] is in fact an inexact version of the proximal method (2.5) that retrieves the inexact algorithm proposed in [4, Algorithm 2] in the particular case $\Gamma[q] = q^2/2$, $q(x, y) = \|x - y\|$ and $1 - \sigma = \theta$. 
If \( \{x^k\} \) is a sequence finitely generated by Algorithm 1, from the definition of \( q \) together with the first inequality in (2.6) and (3.13), it is easy to see that it terminates at a critical point. Unless stated to the contrary, in the remainder of this paper we assume that \( \{x^k\} \) is an infinite sequence generated by Algorithm 1 and \( f \) is bounded from below and continuous on \( \text{dom} f \).

### 3.3 Convergence and speed of convergence

Next we present a partial convergence result for Algorithm 1. We observe that in [5, Lemma 5.1] the proof presented by the authors holds for any bounded sequence \( \{y^k\} \) and any function \( \Psi_k \) which is locally Lipschitz for each \( k \in \mathbb{N} \). Hence, taking into account that \( q(x^k, \cdot) \) is a locally Lipschitz function, for each \( k \in \mathbb{N} \) (see [5, Proposition 3.6]), in the particular case where \( \{x^k\} \) is a bounded sequence and \( \Psi_k = q(x^k, \cdot) \), it follows that the sequence \( \{v^k\} \) defined in Algorithm 1 is bounded. In the remainder of this paper we assume that there exists \( L > 0 \) such that

\[
\|v^k\| \leq L, \quad k \in \mathbb{N}. \tag{3.14}
\]

**Proposition 3.3.** The following statements hold:

i) The sequence \( \{f(x^k)\} \) is strictly increasing;

ii) \( \sum_{k=0}^{+\infty} \Gamma[q(x^k, x^{k+1})] < +\infty; \)

iii) \( \lim_{k \to +\infty} q(x^k, x^{k+1}) = 0; \)

iv) Each accumulation point of the sequence \( \{x^k\} \), if any, is a critical point of \( f \).

**Proof.** The proof of the items i), ii) and iii) are of simple verification (see what was presented at the beginning of Section 3.2). Let us deal with item iv). Suppose that \( \bar{x} \in \mathbb{R}^n \) be an accumulation point of \( \{x^k\} \) and let \( \{x^{j_k}\} \) be a subsequence converging to \( \bar{x} \). The item i) implies that \( \bar{x} \in \text{dom} f \) and \( f(x^{j_k}) \) converges to \( f(\bar{x}) \). Since \( \{x^k\} \) is generated by Algorithm 1, there exist sequences \( \{w^k\} \) and \( \{v^k\} \) such that \( w^{k+1} \in \partial f(x^{k+1}) \) and \( v^{k+1} \in \partial (q(x^k, \cdot)(x^{k+1}) \) satisfying
Now, $\{v^k\}$ being bounded, take $\bar{v} \in \mathbb{R}^n$ and assume that $\{v^k\}$ is a subsequence of $\{v^k\}$ converging to $\bar{v}$. Hence, taking into account that $\{\lambda_k\}$ is bounded, inequality (3.13) allows us to conclude that $\{w^k\}$ has a subsequence converging to zero. Without loss of generality we can assume that $\{w^k\}$ converges to zero. Therefore, as $\partial f$ satisfies Property 1, the proof is complete.

\[\square\]

\textbf{Remark 3.2.} From the viewpoint of applications, item iii) of the last proposition tell us that, Algorithm 2.1 is an habituation process, i.e., step by step, gradually, the agent carries out a more and more similar action.

As in [3–5], our main convergence result is restricted to functions that satisfy the so-called Kurdyka-Lojasiewicz inequality. This was first introduced by Lojasiewicz [28], to real analytic functions, and extended by Kurdyka [30] to differentiable definable functions in an o-minimal structure (for a detailed discussion on o-minimal structures see, for example, Dries and Miller [29]) through of the following result:

Given an open bounded set $U \subset \mathbb{R}^n$ and $g : U \rightarrow \mathbb{R}_+$ a differentiable definable function in an o-minimal structure, there exist $c, \eta > 0$ and a positive increasing definable function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ continuously differentiable such that

\[\|\nabla (\varphi \circ g)(x)\| \geq c, \quad x \in U \cap g^{-1}(0, \eta). \tag{3.15}\]

Note that, taking $\varphi(t) = t^{1-\theta}$, $\theta \in [0, 1)$, the inequality (3.15) becomes:

\[\|\nabla g(x)\| \geq cg(x)^\theta, \quad c = 1/(1-\theta), \tag{3.16}\]

which is the famous Lojasiewicz inequality. For extensions of the Kurdyka-Lojasiewicz inequality, in the Euclidean context, to the class of nonsmooth functions see Bolte et al. [31], Bolte et al. [32] and Attouch et al. [33]. For extensions of the Kurdyka-Lojasiewicz property to functions defined on non-linear spaces, see Kurdyka et al. [34], Lageman [35], Bolte et al. [25] and Bento et al. [36]. Next formal definition of the Kurdyka-Lojasiewicz inequality can be finding in [32], where it is also possible to find several examples and a good discussion over important classes of functions which satisfy the mentioned property.

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Definition 3.2. A proper lower semicontinuous function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) is said to have the Kurdyka-Lojasiewicz property at \( \bar{x} \in \text{dom } \partial f \) if there exists \( \eta \in (0, +\infty) \), a neighborhood \( U \) of \( \bar{x} \) and a continuous concave function \( \varphi : [0, \eta) \rightarrow \mathbb{R}_+ \) such that:

\[
\varphi(s) = 0, \quad \varphi'(s) > 0, \quad s \in (0, \eta); \quad (3.17)
\]

\[
\varphi'(f(x) - f(\bar{x})) \text{dist}(0, \partial f(x)) \geq 1, \quad x \in U \cap [f(\bar{x}) < f(x) + \eta], \quad (3.18)
\]

- \( \text{dist}(0, \partial f(x)) := \inf \{ \|v\| : v \in \partial f(x) \} \),
- \( [\eta_1 < f < \eta_2] := \{ x \in M : \eta_1 < f(x) < \eta_2 \}, \quad \eta_1 < \eta_2. \)

In the remainder of this paper we assume that \( f \) is a KL function, i.e., a function which satisfies the Kurdyka-Lojasiewicz inequality at each point of \( \text{dom } \partial f \).

Theorem 3.1. Assume that \( \{x^k\} \) is bounded, \( \bar{x} \in \mathbb{R}^n \) is an accumulation point of \( \{x^k\} \) and Assumption 3.1 holds. Let \( U \subset \mathbb{R}^n \) be a neighborhood of \( \bar{x} \), \( \eta \in (0, +\infty) \) and \( \varphi : [0, \eta) \rightarrow \mathbb{R}_+ \) a continuous concave function such that (3.17) and (3.18) hold. If \( \delta \in (0, \bar{q}) \) (see condition (2.7)) and \( r \in (0, 1) \) are fixed constants, \( B(\bar{x}, \delta / \beta_1) \subset U \), \( a := \tilde{\lambda}(1 - \sigma) \) and \( M := \frac{L a}{\tilde{\lambda}} \), then there exists \( k_0 \in \mathbb{N} \) such that:

\[
f(\bar{x}) < f(x^k) < f(\bar{x}) + \eta, \quad k \geq k_0, \quad (3.19)
\]

\[
q(\bar{x}, x^k) + \frac{1}{1 - r} \gamma^{-1} \left[ \frac{f(x^k) - f(\bar{x})}{a} \right] + \frac{r}{1 - r} \gamma^{-1} \left[ \frac{f(x^{k-1}) - f(\bar{x})}{a} \right] + \tilde{\beta} r M \left[ \varphi(f(x^k)) - \varphi(f(\bar{x})) \right] < \delta, \quad (3.20)
\]

\[
x^{k_0} + j \in B(\bar{x}, \bar{q}), \quad j = 1, 2, \ldots. \quad (3.21)
\]

Moreover,

\[
\sum_{k=0}^{+\infty} q(x^k, x^{k+1}) < +\infty. \quad (3.22)
\]

In particular, the whole sequence \( \{x^k\} \) converges to \( \bar{x} \) and it is a critical point of \( f \).
Proof. From item i) of Proposition 3.3, it follows that $x^k, \tilde{x} \in \text{dom} f$, for all $k \in \mathbb{N}$. In particular, the sequence $\{f(x^k)\}$ is well defined and converges to $\inf_{k \geq 0} f(x^k) = f(\tilde{x})$. Changing $f$ into $f - \inf_{k \geq 0} f(x^k)$ we can assume, without loss of generality, that $f(\tilde{x}) = 0$. Since $\{f(x^k)\}$ is a decreasing sequence converging to 0, we have

$$0 < f(x^k), \quad k \in \mathbb{N}.$$  \hfill (3.23)

In particular, there exists $N \in \mathbb{N}$ such that

$$0 < f(x^k) < \eta, \quad k \geq N. \hfill (3.24)$$

Since (3.23) holds, let us define the sequence $\{b_k\}$ given by

$$b_k = q(x^k, \tilde{x}) + \frac{1}{1-r} \Gamma^{-1} \left[ \frac{f(x^k)}{a} \right] + \frac{r}{1-r} \Gamma^{-1} \left[ \frac{f(x^{k-1})}{a} \right] + \frac{\partial r(r) M}{1-r} \varphi(f(x^k)).$$

As $q(\cdot, \tilde{x})$, $\Gamma^{-1}$ and $\varphi$ are continuous functions, $\varphi(0) = 0$ and $\{f(x^k)\}$ converges to 0, it follows that 0 is an accumulation point of the sequence $\{b_k\}$. Hence, there exists $k_0 := k_{j_0} > N$ such that (3.20) holds. In particular, as $k_0 > N$, from (3.24) it also holds (3.19).

Note that (3.20) implies $x^{k_0} \in B_q(\tilde{x}, \delta)$. On the other hand, from Assumption 3.1 there exists $\beta_1 > 0$ such that $q(\tilde{x}, x^{k_0}) \geq \beta_1 \|x^{k_0} - \tilde{x}\|$. This tells us that $x^{k_0} \in B(\tilde{x}, \delta/\beta_1) \subset U$ (latter inclusion follows by hypothesis). Hence, from (3.19), we have

$$x^{k_0} \in U \cap [0 < f < \eta].$$

Then, since $\tilde{x}$ is a point where $f$ satisfies the Kurdyka-Lojasiewicz inequality it follows that $0 \notin \partial f(x^{k_0})$. Moreover, conditions (3.14) and (3.13) combined with the definition of $\text{dist}(0, \partial f(x^k))$, yield

$$b\Gamma'[q(x^{k-1}, x^k)]L \geq b \Gamma'[q(x^{k-1}, x^k)]\|v^k\| \geq \|w^k\| \geq \text{dist}(0, \partial f(x^k)), \quad k = 1, 2, \ldots.$$  \hfill (3.25)

Hence, again from the Kurdyka-Lojasiewicz inequality of $f$ at $\tilde{x}$, it follows that

$$\varphi'(f(x^{k_0})) \geq \frac{1}{b \Gamma'[q(x^{k-1}, x^k)]}.$$
On the other hand, the concavity of the function $\varphi$ implies that

$$\varphi(f(x^{k_0}))-\varphi(f(x^{k_0+1})) \geq \varphi'(f(x^{k_0}))(f(x^{k_0}) - f(x^{k_0+1})),$$

which, combined with $\varphi' > 0$ and condition (3.11) (taking into account that $\lambda_k \geq \bar{\lambda} = \frac{a}{1-\sigma} \in \mathbb{N}$) yields

$$\varphi(f(x^{k_0}))-\varphi(f(x^{k_0+1})) \geq \varphi'(f(x^{k_0}))a\Gamma[q(x^{k_0},x^{k_0+1})].$$

(3.26)

So, using the inequalities (3.26) and (3.25), we get

$$M \left[ \varphi(f(x^{k_0})) - \varphi(f(x^{k_0+1})) \right] \geq \frac{\Gamma[q(x^{k_0},x^{k_0+1})]}{\Gamma[q(x^{k_0-1},x^{k_0})]},$$

(3.27)

We state that

$$q(x^k,x^{k+1}) \leq rq(x^{k-1},x^k) + \bar{\rho}(r)M(\varphi(f(x^k)) - \varphi(f(x^{k+1}))),$$

(3.28)

holds for $k = k_0$. Indeed, we have two possibilities to consider:

a) $q(x^{k_0},x^{k_0+1}) \geq rq(x^{k_0-1},x^{k_0})$;

b) $q(x^{k_0-1},x^{k_0}) > rq(x^{k_0},x^{k_0+1})$.

Let us suppose that holds a). Then, $q(x^{k_0-1},x^{k_0}) \leq q(x^{k_0},x^{k_0+1})/r$. Hence, from the last inequality in (2.6), we obtain

$$\Gamma'[q(x^{k_0-1},x^{k_0})] \leq \Gamma'[q(x^{k_0},x^{k_0+1})] \leq \bar{\rho}(r) \left[ \Gamma[q(x^{k_0},x^{k_0+1})] / q(x^{k_0},x^{k_0+1}) \right],$$

where the last inequality follows from condition (2.7). Therefore, (3.28) follows by combining last inequality with (3.27), taking into consideration that $rq(x^{k_0},x^{k_0+1}) \geq 0$. If happens item b) the statement is immediately verified.

Let us prove (3.21) by induction on $j$. Suppose that $j = 1$. Since $\Gamma$ is invertible, $f(x^k) > 0$, $\lambda_k \geq \bar{\lambda} = \frac{a}{1-\sigma} \in \mathbb{N}$ and $r \in (0,1)$, condition (3.11) implies that

$$q(x^{k_0},x^{k_0+1}) \leq \Gamma^{-1} \left[ \frac{f(x^{k_0})}{a} \right] \leq \frac{1}{1-r} \Gamma^{-1} \left[ \frac{f(x^{k_0})}{a} \right].$$

(3.29)
On the other hand, combining the triangle inequality with last inequality and condition (3.20) we obtain, from (3.20):

\[ q(\hat{x},x^{k_0+1}) \leq q(\hat{x},x^{k_0}) + q(x^{k_0},x^{k_0+1}) \leq q(\hat{x},x^{k_0}) + \frac{1}{1-r} \Gamma^{-1} \left[ \frac{f(x^{k_0})}{a} \right] < \rho. \]

But this tells us that (3.21) holds with \( j = 1 \). Take \( j \geq 1 \) and let us suppose that (3.21) holds for all \( k = k_0 + 1, \ldots, k_0 + j - 1 \). In this case, (3.28) holds for \( k = k_0 + 1, \ldots, k_0 + j - 1 \) and, hence, we get

\[ \sum_{i=0}^{j-1} q(x^{k_0+i},x^{k_0+i+1}) \leq \frac{r}{1-r} q(x^{k_0-1},x^{k_0}) + \frac{\hat{p}_r(r)M}{1-r} \left[ \phi(f(x^{k_0})) - \phi(f(x^{k_0+j})) \right]. \] (3.30)

From the triangle inequality, we have:

\[ q(\hat{x},x^{k_0+j}) \leq q(\hat{x},x^{k_0}) + q(x^{k_0},x^{k_0+1}) + \sum_{i=0}^{j-1} q(x^{k_0+i},x^{k_0+i+1}). \]

Thus, combining these two last inequalities with (3.29) and taking into account the inequality \(-\phi(f(x^{k_0+j})) < 0\), we have

\[ q(\hat{x},x^{k_0+j}) \leq q(\hat{x},x^{k_0}) + \frac{1}{1-r} \Gamma^{-1} \left[ \frac{f(x^{k_0})}{a} \right] + \frac{r}{1-r} \Gamma^{-1} -1 \left[ \frac{f(x^{k_0-1})}{a} \right] + \frac{\hat{p}_r(r)M}{1-r} \left[ \phi(f(x^{k_0})) \right], \]

which, from (3.20), allows us to conclude the induction proof.

Note that (3.22) follows immediately from (3.30). Now, combining Assumption 3.1 with (3.22), we conclude that \( \{x^k\} \) is a Cauchy sequence and hence converges. The conclusion of the proof follows from item iii) of Proposition 3.3.

\[ \square \]

Under the conditions of the last theorem, for \( k \) sufficiently large \( (k \geq k_0) \), we have

\[ \sum_{p=k}^{N} q(x^p,x^{p+1}) \leq \frac{r}{1-r} q(x^{k-1},x^k) + \frac{\hat{p}_r(r)M}{1-r} \left[ \phi(f(x^k)) - \phi(f(x^{N+1})) \right], \quad N \geq k. \] (3.31)

Letting \( N \) goes to infinity and taking into account that \( f(x^k) \) decreases to zero and \( \phi(0) = 0 \), last inequality yields

\[ \sum_{p=k}^{\infty} q(x^p,x^{p+1}) \leq \frac{r}{1-r} q(x^{k-1},x^k) + \frac{\hat{p}_r(r)M}{1-r} \phi(f(x^k)), \quad k \geq k_0. \] (3.32)
Let us suppose that \( q(s) = s^{1-\theta} \), \( \theta \in [0, 1) \) and, for simplicity of notation, define \( \Delta_k := \sum_{k=k_0}^{\infty} q(x^k, x^{k+1}) \). So, considering that \( q(x^{k-1}, x^k) = \Delta_{k-1} - \Delta_k \), we can rewrite inequality (3.32) as it follows:

\[
\Delta_k \leq \frac{r}{1-r} (\Delta_{k-1} - \Delta_k) + \frac{\hat{\beta} r}{1-r} (f(x^k))^1-\theta, \quad k \geq k_0.
\]  

(3.33)

Since \( f(x) = 0 \) and \( x^k \in U \cap [0 < f < \eta] \), for \( k \geq k_0 \), combining (3.18) with (3.33), we obtain

\[
\Delta_k \leq \frac{r}{1-r} (\Delta_{k-1} - \Delta_k) + \frac{\hat{\beta} r}{1-r} \left[ (1-\theta) \text{dist}(0, \partial f(x^k)) \right]^\frac{1-\theta}{\theta}, \quad k \geq k_0.
\]  

(3.34)

Now, from the definition of the sequence \( \{x^k\} \), there exist \( b > 0 \), \( w^k \in \partial f(x^k) \), \( v^k \in \partial q(x^{k-1}, \cdot)(x^k) \) such that

\[
\|w^k\| \leq b \Gamma[q(x^{k-1}, x^k)] \|v^k\| \leq b L \Gamma[q(x^{k-1}, x^k)], \quad k \in \mathbb{N},
\]

where the last inequality follows from (3.14). As \( \text{dist}(0, \partial f(x^k)) \leq \|w^k\| \), combining last inequality with (3.34) and taking into account that \( q(x^{k-1}, x^k) = \Delta_{k-1} - \Delta_k \), we get

\[
\Delta_k \leq \frac{r}{1-r} (\Delta_{k-1} - \Delta_k) + \frac{\hat{\beta} r}{1-r} \left[ (1-\theta) b L \Gamma[q(x^{k-1}, x^k)] \left(\Delta_{k-1} - \Delta_k\right) \right]^\frac{1-\theta}{\theta}, \quad k \geq k_0.
\]  

(3.35)

**Assumption 3.2.** For each \( \bar{\beta} > 0 \) fixed, there exist positive constants \( \gamma = \gamma(\bar{\beta}) \) and \( h = h(\bar{\beta}) \) such that

\[
\Gamma[q] \leq \gamma q^\bar{\beta}, \quad 0 < q < \bar{q} < 1, \quad \bar{q} \text{ fixed.}
\]

If \( \Gamma \) is a desutility of costs to change satisfying last assumption with \( \bar{\beta} = \frac{\theta}{1-\theta} \), then there exist \( \gamma(\theta), h(\theta) > 0 \) such that (3.35) becomes

\[
\Delta_k \leq \frac{r}{1-r} (\Delta_{k-1} - \Delta_k) + \frac{\hat{\beta} r}{1-r} \left[ (1-(\theta) \left(\Delta_{k-1} - \Delta_k\right)^{h(\theta)}, \quad k \geq k_0,
\]  

(3.36)

where \( \bar{C} = (1-\theta) b L \gamma(\theta) \).

**Remark 3.3.**

a) Given \( \bar{\beta} > 0 \), the desutility of costs to change, defined by \( \Gamma[q] = q^\alpha \), satisfies Assumption 3.2 for \( \gamma = \alpha \) and \( h = \frac{\alpha-1}{\bar{\beta}} \).

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b) Note that (3.31) (resp. (3.36)) is a generalization of (10) (resp. (12)) in [3]. Indeed, this can be easily observed when \( \Gamma[d] = d^2 \), \( q(x,y) = \|y - x\| \) and \( \varphi(s) = s^{1-\theta} \) together with (2.5) (\( \alpha = 2 \)).

Next we present an analysis of the rate convergence in the particular case where the desutility of costs to change \( D \) satisfies Assumption 3.2 and \( \varphi(s) = s^{1-\theta}, \theta \in [0,1) \). This resulted is a generalization of [3, Theorem 2] and its proof follows the same idea presented in the refereed paper.

**Theorem 3.2.** Let \( \Gamma \) be a relative resistance to change function satisfying Assumption 3.2 and \( \varphi(s) = s^{1-\theta}, \theta \in [0,1) \). Assume also that all assumptions of the previous theorem hold. Let \( \bar{x} \) be the limit point of the sequence \( \{x^k\} \). Then,

i) If \( \theta = 0 \), the sequence \( \{x^k\} \) converges in a finite number of steps;

ii) If \( h(\theta) \geq 1 \), then, there exist \( c_1 > 0, Q \in (0,1) \) and \( k_1 \in \mathbb{N} \) such that

\[
q(\bar{x}, x^k) \leq c_1 Q^k, \quad k \geq k_1;
\]

iii) If \( h(\theta) \in (0,1) \), then, there exists \( c_2 > 0 \) such that

\[
q(\bar{x}, x^k) \leq c_2 k^{\frac{h(\theta)}{1-h(\theta)}}, \quad k \geq k_1.
\]

**Proof.** First, let us suppose that \( h(\theta) \in (0,1) \). Since \( \Delta_k \) converges to zero as \( k \) goes to infinity, there exists \( k_1 \geq k_0 \) such that

\[
\Delta_{k+1} - \Delta_k \leq \|\Delta_{k+1} - \Delta_k\|^{h(\theta)}, \quad k \geq k_1.
\]

Hence, (3.36) shows that there exists a positive constant \( \mathcal{C} = \left[ \frac{r + \tilde{p}(r)MCN(\theta)}{1-r} \right]^{\frac{1}{1-h(\theta)}} \), such that

\[
\Delta_k^{\frac{1}{1-h(\theta)}} \leq \mathcal{C}(\Delta_{k-1} - \Delta_k), \quad k \geq k_1.
\]

Define \( \psi : (0, +\infty) \to \mathbb{R} \) given by \( \psi(r) = r^{-\frac{1}{h(\theta)}} \) and take \( R \in (1, +\infty) \). For each \( k \geq k_1 \) fixed, we state that there exist \( \mu > 0 \) and \( \nu < 0 \) such that

\[
\Delta_k^{\nu} - \Delta_{k-1}^{\nu} \geq \mu.
\]
Indeed, we have two cases to consider:

a) $\psi(\Delta_k) \leq R\psi(\Delta_{k-1})$;

b) $\psi(\Delta_k) > R\psi(\Delta_{k-1})$.

Let us suppose firstly that $a)$ holds. Then, from (3.37) together with definition of the function $h$, it follows that

$$1 \leq CR(\Delta_{k-1} - \Delta_k)\psi(\Delta_{k-1}).$$

Now, since $\psi(\Delta_{k-1}) = \min_{\Delta_k \leq t \leq \Delta_k-1} \psi(t)$, last inequality implies

$$1 \leq CR \int_{\Delta_k}^{\Delta_{k-1}} \psi(t) dt,$$

and, consequently, we have

$$1 \leq CR \frac{h(\theta)}{h(\theta)-1} \left[ \frac{h(\theta)-1}{\Delta_{k-1}} - \frac{h(\theta)-1}{\Delta_k} \right].$$

Note that $\frac{h(\theta)-1}{h(\theta)} < 0$ and, hence, $\frac{1-h(\theta)}{h(\theta)} > 0$. Therefore, the desired resulted follows from the last inequality with

$$\mu = \frac{1-h(\theta)}{R\epsilon h(\theta)} \quad \text{and} \quad \nu := \frac{h(\theta)-1}{h(\theta)}. \quad \text{(3.39)}$$

Let us suppose now that holds $b)$ and define $q = \left(\frac{1}{\pi}\right)^{h(\theta)}$. Then, from the definition of $h$, it follows that $\Delta_k \leq q\Delta_{k-1}$ and, taking into account that $\nu < 0$, we get

$$\Delta_k^\nu \geq q^\nu \Delta_{k-1}^\nu.$$

Hence, subtracting $\Delta_{k-1}^\nu$ from both sides of the last inequality, we have

$$\Delta_k^\nu - \Delta_{k-1}^\nu \geq (q^\nu - 1)\Delta_{k-1}^\nu.$$

Recall that $\Delta_p$ converges to zero as $p$ goes to infinity. Without loss of generality, we can suppose that $\Delta_k \in (0,1)$ for $k \geq k_1$ and $\Delta_k^\nu \geq \mu k_1$ (this is possible because $\nu < 0$). Since $q \in (0,1)$ and $\nu < 0$, then, there exists $\tilde{\mu} > 0$ such that $(q^\nu - 1)\Delta_{k-1} > \tilde{\mu}$. This tell us that the desired resulted holds with $\mu = \tilde{\mu}$ and $\nu$ as defined in (3.39). Therefore, the statement is proved.
Take $k \in \mathbb{N}$ greater that $k_1$. By summing the inequality (3.38) from $k_1$ to $k$, we obtain
\[ \Delta^\nu_k - \Delta^\nu_{k_1} \geq \mu (k - k_1), \]
and, hence (since $\nu < 0$),
\[ \Delta_k \leq \left[ \Delta^\nu_{k_1} + \mu (k - k_1) \right]^\frac{1}{\nu}. \]
Using again that $\nu < 0$, it follows from the last inequality that there exists a positive constant $\omega_1$, for example, $\omega_1 = \mu^1/\nu$, such that
\[ \Delta_k \leq \omega_1 k^\frac{\mu(\theta)}{\nu}. \] (3.40)
On the other hand, combining the first inequality in Assumption 3.1 with definition of $\Delta_k$, we have
\[ \Delta_k \geq \beta_1 \lim_{N \to +\infty} \sum_{p=k}^N \|x^{p+1} - x^p\| \geq \beta_1 \lim_{N \to +\infty} \sum_{p=k}^N [\|x^p - \tilde{x}^p\| - \|x^{p+1} - \tilde{x}\|] = \beta_1 \|x^k - \tilde{x}\|, \] (3.41)
where the second inequality and equality follow, respectively, from the triangle inequality and definition of $\tilde{x}$. Thus, item iii) follows by combining inequalities (3.40), (3.41) and the second inequality in Assumption 3.1 with $c_2 = (\beta_2 \omega_1)/\beta_1$.

Suppose now that $h(\theta) \geq 1$. From (3.36), we get
\[ \Delta_k \leq \mathcal{C}_1 (\Delta_{k-1} - \Delta_k), \quad k \geq k_1 \geq k_0, \]
where $\mathcal{C}_1 = \mathcal{C}^{h(\theta)}$. So, last inequality implies $\Delta_k \leq \frac{\mathcal{C}_1}{1+\mathcal{C}_1} \Delta_{k-1}$, for $k \geq k_0$ and, hence,
\[ \Delta_k \leq \omega_2 Q^k, \] (3.42)
where $Q = \frac{\mathcal{C}_1}{1+\mathcal{C}_1} \in (0, 1)$ and $\omega_2 = Q^{k_0} \Delta_{k_0}$. Using again (3.41), item ii) follows with $c_1 = (\beta_2 \omega_2)/\beta_1$.

Now, assume that $\theta = 0$ and suppose, for contradiction, that the sequence $\{x^k\}$ is infinitely generated. Take $k_0 \in \mathbb{N}$ sufficiently large such that $x^k \in B_q(\tilde{x}, \delta) \cup [0 < f < \eta], k \geq k_0$, (we are assuming that $f(\tilde{x}) = 0$). Since $\varphi(s) = s$, from (3.18), we obtain
\[ 1 \leq \text{dist}(0, \partial f(x^k)), \quad k \geq k_0, \] (3.43)
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On the other hand, from the definition of the sequence \( \{x^k\} \), for each \( k \in \mathbb{N} \) there exist \( w_k \in \partial f(x^k) \) and \( v_k \in \partial q(x^{k-1}, \cdot)(x^k) \) such that

\[
\|w_k\| \leq b \Gamma'[q(x^{k-1}, x^k)]\|v_k\| \leq bL \Gamma'[q(x^{k-1}, x^k)],
\]

where the last inequality follows from (3.14). Taking into account that \( D \) satisfies (2.6), it is easy to see that \( \Gamma'[q(x^{k-1}, x^k)] \) converges to zero as \( k \) goes to \( +\infty \). Hence, combining (3.43) with (3.44) and letting \( k \) goes to \( +\infty \), we obtain a contradiction. This proves item i) and conclude the proof.

Remark 3.4. The constants \( Q, c_1 \) and \( c_2 \), present in Theorem 3.2, may be made explicit as follows:

\[
c_1 = \frac{\beta_2}{\beta_1} Q^{k_0} \Delta_{k_0}, \quad \text{where} \quad Q = \frac{r + \tilde{\rho}_T(r)M[(1 - \theta)bL\gamma(\theta)]h(\theta)}{1 + \tilde{\rho}_T(r)M[(1 - \theta)bL\gamma(\theta)]h(\theta)},
\]

\[
c_2 = \frac{\beta_2}{\beta_1} \frac{(1 - h(\theta))^{\frac{h(\theta)}{h(\theta) - 1}} (1 - \rho)^{\frac{1}{h(\theta) - 1}}}{[R\theta(\theta)]^{\frac{h(\theta)}{h(\theta) - 1}} [r + \tilde{\rho}_T(r)M[(1 - \theta)bL\gamma(\theta)]h(\theta)]^{\frac{1}{h(\theta) - 1}}}
\]

Note that if \( \Gamma[q] = q^\alpha, \alpha > 1 \), from item a) of Remark 3.3 it is easy to see that the last theorem holds with \( h(\theta) = \frac{(1 - \theta)(\alpha - 1)}{\theta} \). In particular, [3, Theorem 2] is obtained for \( \alpha = 2 \).

4 An Inexact Proximal Point Algorithm: Convergence to a Weak or Strong Trap

In terms of the “Variational rationality approach” which considers, as a central dynamical concept, worthwhile to change processes (see Algorithm 3.1), we have shown, under the conditions of Theorem 3.1, that: starting from a given point \( x^0 \in \mathbb{R}^n \) (initial action), an infinite bounded sequence of worthwhile changes, such that, at each step, it is marginally worthwhile to stop (before starting to do an other worthwhile to change step) converges to a critical point of \( f \). However, this important result in Applied Mathematics is not very useful, from the viewpoint of the our applications to Behavioral Sciences, unless we can show that this critical point is a variational trap (strong or weak) where the agent will prefer to stay than to move, because his motivation to change is strictly or weakly lower than his
resistance to change. Then, we want to know when this limit critical point is a variational trap (say a trap). This section presents, under the conditions of Theorem 3.1, a worthwhile to change process which converges to a critical point of \( f \) which is a weak trap (compare, below, with the definition of a strong global trap). To give a complete argument, following [1, 2], the last section and the annex will show, very briefly, why a strong or weak trap is a good model for permanent habits and routines. Then, start with the general definition of a weak global trap instead of a strong one.

**Definition 4.1.** Let \( x \in X \) be a given action and \( \xi = \xi(x) > 0 \) be a satisficing rate of change chosen by the agent. Let

\[
W_\xi(x) = \{ y \in X, M(x,y) \geq \xi \cdot R(x,y) \}
\]

be his worthwhile to change set, starting from \( x \in X \). Then, starting from \( x^* \in X \) with a given satisficing rate \( \xi^* > 0 \), a strong global trap \( x^* \in X \) is such that motivation to change is strictly lower than resistance to change, \( M(x^*,y) < \xi^* \cdot R(x^*,y) \) for all \( y \neq x^* \in X \). A weak global trap is such that \( M(x^*,y) \leq \xi^* \cdot R(x^*,y) \), for all \( y \in X \).

**Remark 4.1.**

a) Notice that a strong global trap is such that \( W_{\xi^*}(x^*) = \{ x^* \} \) and a weak global trap is such that \( W_{\xi^*}(x^*) = \{ y \in X, M(x^*,y) = \xi \cdot R(x^*,y) \} \). At a strong (weak) global trap, the agent strictly (weakly) prefer to stay than to move;

b) As observed in Section 2 page 11, in the specific context of this paper, we have

\[
M(x,y) = U[A(x,y)] = U[\rho(x)[f(x) - f(y)]], \quad \rho(x) = 1/\lambda(x),
\]

\[
R(x,y) = D[C(x,y)], \quad \Gamma[q(x,y)] = U^{-1}[D[q(x,y)]], \quad \xi(x) = 1.
\]

c) For a weak local variational trap (see 5), we need only the conditions \( M(x^*,y) \leq \xi \cdot R(x^*,y) \) for all \( y \in B_q(x^*,r) \).

**Remark 4.2.**
a) Assuming that $\{\lambda_k\}$ converges to $\lambda_\infty$, our sufficient condition proposes an algorithm which, following a succession of worthwhile changes $x^{k+1} \in W_{\lambda_k}(x^k)$, $k \in \mathbb{N}$, converges to a weak global trap $x^*$ such that $W_{\lambda_\infty}(x^*) = \{y \in X : M(x^*, y) = \lambda_\infty R(x^*, y)\}$. Since the agent is free to choose all his satisficing rates $\lambda_k$ in an adaptive way, this will show that the agent, choosing at the limit point $x^*$ a satisficing rate $\lambda_* > \lambda_\infty$, ends in a strong global trap $x^*$, because $M(x^*, y) = \lambda_\infty R(x^*, y) < \lambda_* R(x^*, y)$, for all $y \in X$.

b) When a process of worthwhile changes converges to a strong global trap, this variational formulation defines a trap as the end point of a path of worthwhile changes, worthwhile to approach, but not worthwhile to leave. This because, starting from there, there is no way to do any other worthwhile change, except repetitions.

4.1 The Algorithm

Working in the inexact maximizing context, Attouch and Soubeyran [11] consider the following new class of algorithms (the “epsilon inexact proximal” algorithm). Starting from the current position $x^k$, let us define the next iterate $x^{k+1}$ as follows:

$$x^{k+1} \in \varepsilon_k \arg\max\{g(y) - \theta_k c(x^k, y) : y \in E(x^k, r_k)\},$$

where $c : X \times X \to \mathbb{R}_+$ is a distance which represents a cost-to-move function, $E(x^k, r_k) \subset X$ is a neighborhood of $x^k$ and $r_k, \theta_k > 0$ and $\varepsilon \geq 0$ are parameters reflecting various behavioral aspects of an given agent. In our global context, assuming that $E(x^k, r_k) = X$ for all $k \in \mathbb{N}$, a similar approach helps us to consider the following iterative process: from the current position $x^k$ define the next iterate $x^{k+1}$ such that:

$$f(x^{k+1}) + \lambda_k \Gamma[q(x^k, x^{k+1})] \leq f(y) + \lambda_k \Gamma[q(x^k, y)] + \varepsilon_k, \quad y \in X,$$

where $\varepsilon_k \geq 0$, $k \in \mathbb{N}$ and $q(x, y)$ is a quasi distance. In the particular case where $\Gamma[q(x, y)] = 2^{-1}d(x, y)$ and $d$ is a distance instead of a quasi distance, this iterative process coincides with the case considered by Zaslavski [37].

Next, we show that the limit point of any convergent sequence generated from the iterative process (4.1) is a weak
Proposition 4.1. Let \( \{x^k\} \) be a sequence generated from the iterative process (4.1) in the particular case where 
\[
\epsilon_k = \lambda_k \sigma \Gamma q(x^k, x^{k+1}) \quad (\sigma \in [0, 1]).
\]
Assume that the sequences \( \{\epsilon_k\}, \{\lambda_k\} \subset \mathbb{R}^+ \) and \( \{x^k\} \) are such that 
\[
\lim_{k \to +\infty} \lambda_k = \lambda_\infty \in (0, +\infty) \quad \lim_{k \to +\infty} x^k = x^\star.
\] (4.2)
Then, \( x^\star \) is a weak global trap relative to the end worthwhile to change set \( W_{\lambda_\infty} \). In particular, if, relative to the sequence of satisficing rate of change \( \{\lambda_k\} \), the agent chooses an end satisficing rate \( \lambda_\ast > \lambda_\infty \), then \( x^\star \) is a strong global trap relative to the end worthwhile to change set \( W_{\lambda_\ast} \).

Proof. Without loss of generality we can take \( x^0 \in \text{dom} f \). Since \( \{x^k\} \) follows the iterative process (4.1), taking \( y = x^k \) and using that \( \epsilon_k = \lambda_k \sigma \Gamma q(x^k, x^{k+1}) \quad (\sigma \in [0, 1]) \), we have
\[
f(x^{k+1}) + \lambda_k (1 - \sigma) \Gamma[q(x^k, x^{k+1})] \leq f(x^k).
\]
This tells us that \( \{f(x^k)\} \) is a non-increasing sequence and that \( \epsilon_k \) converges to zero as \( k \) goes to infinity. As \( x^0 \in \text{dom} f \), it follows that the whole sequence \( \{x^k\} \) as well as \( x^\star \) are in \( \text{dom} f \). Now, given that \( q(\cdot, y) \) is continuous for each \( y \in X \) (see [5]), \( \Gamma \) is continuous and \( f \) is continuous in \( \text{dom} f \), taking the limit in (4.1) as \( k \) goes to infinity, from (4.2), we get:
\[
f(x^\star) \leq f(y) + \lambda_\ast \Gamma[q(x^\star, y)], \quad y \in X,
\]
and the first part of the proposition is proved. The second part follows from Remark 4.2 and this ends the proof.

Comparing Algorithm 3.1 with the iterative process (4.1), we observe the following:

i) on one side, the iterative process (4.1) is much more specific than our Algorithm 3.1. Indeed the weak “worthwhile to change” condition (3.11) is replaced by the much stronger condition (4.1).

ii) on the other side, the iterative process (4.1) does not impose the “not worthwhile marginal change condition” (3.13) as the Algorithm 3.1 does.
Next we propose a new inexact proximal algorithm, combining \((4.1)\) (for a particular choice of the sequence \(\{\varepsilon_k\}\), namely, \(\varepsilon_k = \lambda_k \sigma \Gamma q(x^k, x^{k+1}), \sigma \in (0, 1)\)) with the stopping rule \((3.13)\).

**Algorithm 4.1.** Take \(x^0 \in \text{dom} f\), \(0 < \bar{\lambda} \leq \tilde{\lambda} < +\infty\), \(\sigma \in (0, 1)\) and \(b > 0\). For each \(k = 0, 1, \ldots\), choose \(\lambda_k \in [\bar{\lambda}, \tilde{\lambda}]\) and find \((x^{k+1}, w^{k+1}, v^{k+1}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\) such that:

\[
    f(x^k) - f(x^{k+1}) \geq \lambda_k (1 - \sigma) \Gamma[q(x^k, x^{k+1})] - \Gamma[q(x^k, y)], \quad y \in X, \tag{4.3}
\]

\[
    w^{k+1} \in \partial f(x^{k+1}), \quad v^{k+1} \in \partial q(x^k, \cdot)(x^{k+1}), \tag{4.4}
\]

\[
    \|w^{k+1}\| \leq b \Gamma'[q(x^k, x^{k+1})]\|v^{k+1}\|. \tag{4.5}
\]

**Remark 4.3.** Note that the exact proximal algorithm \((2.5)\) is a specific case of our new algorithm (it holds by taking \(\sigma = 0\)).

**Theorem 4.1.** Let \(\{x^k\}\) be a bounded sequence generated by Algorithm \((4.1)\) and assume that all the assumptions of Theorem \((3.1)\) hold. Then, the whole sequence \(\{x^k\}\) converges to some critical point of \(f\) which is a strong global trap \(x^s\), relative to the worthwhile to change set \(W_{\lambda^*}(x^s)\), for any choice of the final satisfying rate \(\lambda_\infty > \lambda_\infty\).

**Proof.** The first part of the theorem is immediate because any sequence, generated from Algorithm \((4.1)\) satisfy the conditions \((3.11)\) and \((3.13)\) of Algorithm \((3.1)\). Let \(x^s\) be the limit point of the sequence \(\{x^k\}\). Since the sequence \(\{\lambda_k\} \subset [\bar{\lambda}, \tilde{\lambda}], 0 < \bar{\lambda} \leq \tilde{\lambda} < +\infty\), taking a subsequence, if necessary, we can assume \(\lambda_k\) converges to a certain \(\lambda_\infty \in (0, +\infty)\). Therefore, the second part of theorem follows from Proposition \((4.1)\) which completes the proof.

5 Habit’s Formation: Inexact Proximal Processes and Weak Resistance to Change

Comparing worthwhile changes and stays processes, inexact proximal algorithms and habituation processes.
As an application we will consider habit formation as an inexact proximal algorithm in the context of weak resistance to change. Because of its strongly interdisciplinary aspect (Mathematics, Psychology, Economics, Management), to be carefully justified, this application needs several steps (an a short annex relative to the vast litterature on habit formation). This, because we need to compare three differsents processes. First, a general worthwhile changes and stay process. Then, as two specific instances, an inexact proximal algorithm and an habituation process. The comparison must consider three aspects; a dynamical system (which one ?), which converges (how ?), to an end point (which one ?).

**Step.1.** At the mathematical level, an exact proximal algorithm represents a dynamical system, which, each step, minimizes a proximal payoff \( f + \lambda_k \left[ q(x^k, \cdot) \right]^2 \) over the whole space \( X \). The perturbation term is \( \lambda_k \left[ q(x^k, y) \right]^2, \lambda_k > 0, \) \( y \in X \), while the payoff function is \( f \). An inexact proximal algorithm represents a dynamical process which, each step, uses a descent condition and a temporary stopping rule. In both cases, the problem is to give conditions under which this process converges to a limit point. Then, an exact or inexact proximal algorithm considers three points:

i) a dynamical process. It represents the proximal sub-problem which can be the minimization of the current proximal payoff (for an exact proximal algorithm), or a descent condition, i.e., a sufficient decrease of the proximal payoff (in the inexact case);

ii) the existence of some end points, which can be, or not, a critical point, a local, or global minimum of \( f \);

iii) and the convergence of the process towards an end point. This convergence can be linear, quadratic . . .

**Step.2.** At the behavioral level, in the context of his “Variational rationality approach”, Soubeyran [1,2] has examined “worthwhile changes and worthwhile stays” processes. This dynamical process is a sucession of worthwhile changes and worthwhile stays. The end points are variational traps. The convergence of the process materializes in small steps, whose length goes to zero. Let us remind that (see Section 2), in this behavioral context, end points represent traps (reachable, i.e, more or less easy to reach, but difficult to leave). Worthwhile changes balance, each step,
motivation and resistance to change forces. The motivation force is the utility \( U[A(x,y)] \) of the advantage to change function \( A(x,y) = f(x) - f(y) \) where \( f \) represents an unsatisfied need to be minimized. The resistance to change force represents the desutility \( D[C(x,y)] \) of the costs to be able to change \( C(x,y) = q(x,y) \), where \( q(x,y) \) is a quasi distance.

In the context of this paper, the variational approach considers the relative resistance to change or aversion to change function \( \Gamma[q(x,y)] = U - 1[D[q(x,y)]] \). This shows that the perturbation term of an exact or inexact proximal algorithm is a specific instance of a relative resistance to change function \( \Gamma[q(x,y)] \). Moreno et al. [5] have considered the specific quadratic case of weak resistance to change, where \( \Gamma[q(x,y)] = q(x,y)^2 \). Then, at the behavioral level, two mains concepts, among others, drive a “worthwhile changes and stays” process: i) the unsatisfied need function \( f \) (which materialises the motivation to change) and, ii) inertia (the relative resistance to change function \( \Gamma[q(x,y)] \)).

**Step 3.** Habit formation processes consider, i) a repetitive process where an action is repeated again and again in the same recurrent context, ii) a final stage where this action becomes a permanent habit, iii) the convergent process which describes a slow learning habituation process where this action, being repeated again and again in the same recurrent context becomes gradually automatized. The annex arguments this view in some details. At least three vast literatures have examined habit formation: Psychology, Economics and Management. Inexact proximal algorithms, as specific instances of “worthwhile changes and stays” processes, modelize the Psychological approach of habit formation. Management Sciences examines routines in a similar way, as collective recurrent and automatic patterns of interactions between agents within an organization. Then, they are outside the topic of this paper which considers only one agent. Economics sees habits in a very different way, as stocks of past experiences which change the current preference of an agent. In the annex we give a very short survey of these three literatures, giving more details on the Psychological approach which is directly related to our paper, and only some words on the two others.

**Step 4.** Then, it becomes clear that habituation processes are specific instances of worthwhile changes and stays processes. Unsatisfied needs and inertia play a major role. The annex justifies this step.

**Step 5.** In Section 3 we have shown how an inexact proximal algorithm is a specific instance of a “worthwhile
changes and stays” process because: i) minimization of the current proximal payoff, descent conditions and current stopping rules are special cases of worthwhile changes and marginal worthwhile stays, ii) critical points, local and global minimum are specific representations of variational traps, iii) convergence of the proximal algorithm shows how proximal worthwhile changes converge, depending of the shape of the payoff function (which can be convex, lower semicontinuous, or which can satisfy a Kurdyka-Lojasiewicz inequality,...) and the shape of the perturbation term (linear, convex,... with respect to distance or quasi distance).

Let us organize the discussion with respect to resistance to change.

The benchmark case of lower semicontinuous unsatisfied need functions $f$ and strong resistance to change functions $\Gamma[q(x,y)]$ (where costs to be able to change are higher than a quasi distance) have been examined by Soubeyran [1, 2] which considered worthwhile “changes and stays” processes. It has been shown that when a worthwhile changes and stays process converges to a variational trap, this variational formulation offers a model of habit formation which modelizes a permanent habit as the end point of a convergent path of worthwhile changes and temporary habits, where there is no way to do any other worthwhile change, except repetitions.

The case of weak resistance to change was left open. In the context of an exact proximal algorithm, Moreno et al. [5] have examined a specific case of weak resistance to change, namely the quadratic case $\Gamma[q(x,y)] = q(x,y)^2$. However, exact proximal algorithms represent a much more specific case of worthwhile changes, where, each step, the descent condition is optimal. This means that, each step, the process minimizes the proximal payoff $f + \lambda q \Gamma[q(x^k,\cdot)]$ on the whole space $X$. Then, such optimizing worthwhile changes are not step by step economizing behaviors because they require to explore, each step, the whole state space, again and again. The present paper considers the generalized weak resistance to change case in the context of an inexact proximal algorithm instead of an exact one. In both papers the unsatisfied need function $f$ satisfies a Kurdyka-Lojasiewicz inequality.

To summarize, we have compared an inexact generalized proximal algorithm with a “worthwhile changes and stays” process with respect to three aspects: A) as a dynamical system, B) with an end point, C) which converges to
that end point. To end this paper, it remains to compare an inexact generalized proximal algorithm with an habituation process, as it is described in Psychology (see the annex), using the same three criteria to know if a generalized inexact proximal algorithm can modelize adequately an habituation process.

A) **Inexact generalized proximal algorithm and habituation process as dynamical systems.**

An habituation process represents the repetition of an action of a given kind (some activity related to a given goal), in order to satisfy a recurrent unsatisfied need in a stable context. The repetition concerns the action and what becomes more and more the same is “the way of doing it” (the script). This repetition follows a succession of worthwhile changes and stays. An inexact proximal algorithm represents a step by step processes, a succession of moves in order to minimize or decrease some proximal payoff function. Usually, both dynamical processes are unable to reach the goal in one step. Each step, the level of satisfaction of the recurrent need increases, but some unsatisfaction remains.

An habituation process is driven by two balancing forces: a motivation to change function $M[A(x,y)]$ (an habit must serve us), and a resistance to change function $D[C(x,y)]$, (habits are hard to form and hard to break because learning and unlearning are costly). An inexact proximal algorithm is driven by the two terms $f(y)$ and $\Gamma[q(x,y)]$ of its proximal payoff $f(y) + \lambda k \Gamma[q(x,y)]$ where $q(x,y) = C(x,y)$. This balance describes the goal-habit interface.

The rationality of an habituation process is to improve by repetition the way of doing a similar action in the same context. The agent improves with costs to change. He satisfices, doing worthwhile changes, without exploring too much each step (local consideration and exploration; see [1][2] for this important aspect). An inexact proximal algorithm follows, each step, some descent condition and marginal stopping rule, without optimizing each step.

The influence of the past differs from one process to the other. For an habituation process the impact of the past can be very important (the past sequence matters much). For an inexact proximal algorithm it is as if only the last action matters. The influence of the past is minimal (it is as if the agent has a short memory). The influence of the future seems identical in both cases: myopia seems to be the rule. Only the next future action matters. Agent’s behavior driven by habits are not forward looking. For more forward looking worthwhile to change behavior; see [1][2].
B) **Comparing end points.**

Our inexact proximal algorithm converges to a critical point, which is not an end point, unless it can be shown that a critical point is a variational trap (as done in Section 4). A variational trap is worthwhile to reach and not worthwhile to leave. An habituation process ends in a permanent habit which is hard to form and hard to break. It represents the vestige of a past repeated behavior.

C) **Convergence: How the strength of resistance to change impacts habit formation.**

In the context of habit formation, the main point is to know how more or less bounded needs and the strength of resistance to change (weak, strong) favors or not the speed of convergence. This point is very important because we have seen that an habituation process is gradual. It represents a progressive increase in automaticity. The speed of convergence of the inexact proximal algorithm modelizes in a nice way a slow learning process of automatization (how the same way of doing an action emerges gradually from repetitions in the same recurrent context). Actions become more and more similar and converge to a limit action which is a variational trap, a permanent habit, where the agent prefers to stay than to move.

In this behavioral context, our last Theorem 3.2 is a powerful result. It shows the importance of the strength of resistance to change for habit’s formation. Take the leading example of $\Gamma[q(x,y)] = q(x,y)^\alpha$, where $\alpha > 1$, which represents the case of weak resistance to change “in the small” (but strong resistance to change in the large), as opposed to the strong resistance to change “in the small” (but weak resistance to change in the large) case $0 < \alpha \leq 1$; see [1, 2]. Let us make more precise this point. If $U[A] = A^\lambda, \lambda > 0$ and $D[C] = C^\mu, \mu > 0$, the relative resistance to change function has been defined as the relative desutility $U^{-1}[D[C(x,y)]] = \Gamma[q(x,y)] = q(x,y)^\alpha$ of costs to change $C(x,y) = q(x,y)$. For $\alpha > 1$ (the case of this paper), it is weak in the small (and strong in the large), because $q^\alpha < q$ for $0 < q < 1$ and $q^\alpha \geq q$ for $q \geq 1$. Then, $\alpha > 1$ is equivalent to $\mu > \lambda$. A benchmark case is $0 < \lambda < 1 < \mu$. A small exponent $0 < \lambda < 1$ modelizes a concave utility function $U[A] = A^\lambda$. This means that advantages to change satisfy less and less (a satiation effect with decreasing motivation). A large exponent $\mu > 1$ modelizes a convex desutility.
function $D[C] = C^\mu$. This means that costs to change are more and more painful (a fatigue effect). If advantages to change and costs to change have the same small size, $0 < A = C = Z < 1$, the desutility of costs to change $D[Z] = Z^\mu$ is lower than the utility of advantages to change $U[Z] = Z^\lambda$ if,

$$D[Z] = Z^\mu < U[Z] = Z^\lambda \iff Z^{\mu/\lambda} < 1 \iff Z < 1.$$  

This modelizes the case of weak resistance to change in the small. Moreover, when $\Gamma[q(x,y)] = q(x,y)^\alpha$, $\alpha > 1$, then $h(\theta) = (\alpha - 1)(1 - \theta)/\theta > 0$ for $0 < \theta < 1$. In this context, Theorem 3.2 exhibits two polar cases:

ii) weak resistance to change in the small: $h(\theta) \geq 1 \iff \alpha \geq 1/(1 - \theta) > 1$. In this case where $\alpha$ is very high,

$$q(\bar{x},x^k) \leq c_1Q^k = c_1e^{(\log Q)k} \text{ with } \log Q < 0 \text{ from } 0 < Q < 1;$$

iii) still weak, but stronger resistance to change in the small : $0 < h(\theta) < 1 \iff 1 < \alpha < 1/(1 - \theta)$. In this case where $\alpha$ is lower (but still high because $\alpha > 1$),

$$q(\bar{x},x^k) \leq c_2k^{\omega(\theta)} = c_2e^{\omega(\theta)(\log k)} \text{ with } \omega(\theta) = h(\theta)/|h(\theta) - 1| < 0 \text{ from } 0 < h(\theta) < 1.$$

Then, when $k$ goes to infinite the speed of convergence is higher in case ii) than in case iii), because $k > \log k$ for $k$ high enough. This shows the following striking result:

When resistance to change in the small is weak, an even weaker resistance to change in the small favors the speed of convergence, hence the speed of the habituation process. The intuition is that the agent is motivated to repeat more and more in the same way the same action in the same context, because the desutility of costs to small changes is low.

Finally, when $\theta = 1$, habit formation is in finite time, a nice result!

**Loss aversion and resistance to change**

In Economics, the famous concept of “loss aversion” has been defined by the two Nobel Prize, Kahneman and Tversky [9,12] in the case of risk and also for riskless choices (our case). Let us see the connexion with the variational concept of weak resistance to change in the small. Loss aversion means that losses $C > 0$ loom larger than gains $A > 0$ of the same size $C = A = Z > 0$. Using our notation, this is equivalent to say that the loss aversion index
\[ L(Z) = D[C] / U[A] > 1 \] is strictly higher than 1, for all changes \( C = A = Z > 0 \). Other indexes of “loss aversion” have been given (for example \( \tilde{L}(Z) = D'[C] / U'[A] \), see Abdellaoui et al. \[33\] and Körberling and Wakker \[39\]. In our case, where we consider only small changes \( 0 < Z < 1 \), the relative resistance to change index is \( \Gamma[C] = U^{-1} D(C) \). If \( U[A] = A^\lambda, \lambda > 0 \) and \( D[C] = C^\mu, \mu > 0 \), then, \( L[Z] = Z^{\mu - \lambda} (\tilde{L}[Z] = (\mu / \lambda)Z^{\mu - \lambda}) \) and \( \Gamma[Z] = Z^\alpha \), with \( \alpha = \mu / \lambda > 0 \). The present paper focuses attention on weak resistance to small changes, where \( \alpha > 1 \), which appears to be equivalent to “gain attraction for small changes” (small gains loom larger than losses of the same size): \( L[Z] = Z^{\mu - \lambda} < 1 \) for \( 0 < Z < 1 \) and \( \mu - \lambda > 0 \). Then, this paper has examined the speed of “habit formation” in this context of “gain attraction for small changes”. Soubeyran \[1, 2\] has examined the opposite case of loss aversion for small changes which represents also strong resistance to small changes.

### 6 Final Remarks

The main message of this paper is that, using the behavioral context of a recent “Variational rationality approach” \[1,2\], a generalized proximal algorithm can modelize fairly well an habituation process as described in Psychology. This is the case even when resistance to change (inertia) is weak (our paper). This opens the door to a new vision of proximal algorithms. They are not only very nice mathematical tools in Optimization theory, with striking computational aspects. They can also be nice tools to modelize the dynamics of human behaviors.


#### 7.1 Habit as Learned Automatic Behaviors in Psychology

- **The definition of habits**

  In Psychology habits represent “learned sequences of acts that have become automatic responses to specific
cues, and are functional in obtaining certain goals or end states”; see Verplanken and Aarts [40]. An habit is also defined as a more or less fixed way of thinking, willing, feeling and doing, acquired through previous repetition of a physical or mental experience. For Duhigg [41], an habit as an automatized action which follows an automatized three steps pattern: a given trigger which activates the action, a process (or script) that the action follows, and a reward (benefit or gain). Habits concern both physical actions or mental acts (habitual thoughts, habits of mind; see [42]).

- Repetition is a necessary condition for habits to develop, and frequency of past behaviour correlates with habit strength. But it is not repetition that matters. The main aspect of habit is the automaticity of behaviour occurring in stable contexts (“habit as frequency or automaticity”; see Gardner [43]).

- Context stability is an important feature. Habits are triggered (activated) by specific internal cues (internal states, like moods and goals) or external cues (the presence of typical interaction partners or external goals) in the same recurrent situation; see [44].

- Habits represent a form of automaticity. Following [45] the four horsemen of automaticity are: “awareness, intentionality, efficiency, and controllability”.

i) are more or less conscious. They occur, most of the time, outside of awareness, although some of them are conscious. Habitual behavior often goes unnoticed in persons exhibiting it.

ii) are intentional (wanted) in the sense of being goal directed; see [46]. They are goal-directed acts which do not develop randomly, but are formed first and foremost because they serve us. They develop by the systematic experience of rewarding consequences. Hence the rewarding properties of habits make them functional from the perspective of the individual who develops them. In particular when we consider habits that are unwanted from the perspective of an outsider, it is important to realize that, from the individual’s perspective such habits are functional and thus “wanted” in achieving some goal; see [47].

iii) can be difficult to control and to break. Willpower distinguishes a bad habit from an addiction or a mental
disease. If a person still seems to have control over the behavior then it is just a habit.

iv) are mentally efficient. They save on deliberation efforts.

- **Habits are hard to form: a progressive learning process**

  At the origin of an habit, there is the desire to satisfy some recurrent need or goal, conscious or not, and an action which can help to satisfy this goal. Then, an habit is learned by repetition of this action, in the same recurrent context, to satisfy the same recurrent need. It represents an increase in automaticity with number of repetitions up to an asymptote. Habituation is a simple form of learning, in which an agent, after a repeated exposure to a stimulus, stops responding to that stimulus in varied manners. The process by which new behaviours become automatic is habit formation. Habit formation is a slow process, an incremental accrual of information over time in procedural memory. Automaticity forms progressively from a history of behavioral repetition. The initial repetitions cause a large increase in automaticity. After that, each new repetition generates less amount of automaticity; see Laly et al. [48]. Learning depends of the mental accessibility of past behavior (past behavior frequency, habit frequency, response frequency, habit strength). Then, habits express self-identity. They represent a disposition, a skill, a knowledge structure, the ability to do something in the same way, in the same context. Habit formation involves the creation of associations in memory between actions and stable features of the context in which they are performed, and an incremental increase in the link between the context and the action.

- **Habits can be hard to break: weak or strong resistance to change**

  Depending of its strength, it can be hard or not to break an habit. Resistance to change and inertia are core concepts to understand, both, how to form a good habit, and how to break a bad one, because learning and unlearning can be very slow and costly processes. At the level of a group, Lewin (see [49]) is the father of the resistance to change concept. He defines a three stages model of change which clearly shows, at each stage, the
different difficulties to be able to abandon the old position and to imagine the new one (unfreezing), to be able to move from the old position to the new one (transition), and finally to be able to stay in the new position (refreezing). At the organizational level, Rumelt in [50] defines inertia by five criteria: distorted perception, dulled motivation, failed creative response, political deadlocks and action disconnects. In Economic switching costs and adjustments costs have been used to modelize different frictions. In the context of his “Variational rationality” approach, and at the individual level, Soubeyran [1, 2] has defined resistance to change as the desutility of costs to be able to change. Costs to be able to change abound (see Soubeyran, 2009, 2010). Among a huge list, they could be costs to improve motivation, inhibition costs to break a bad habit (costs to stop doing something in a given way in a given context, monitoring costs to resist temptations, learning costs to be able to start doing a new action… ego depletion costs to self regulate his behavior…. To break a bad habit, an agent will have either to change the context or to inhibit the automatic response. Habits can be weak (temporary) or strong (permanent). The strength of an habit depends of its the degree of automaticity (more or less conscious, intentional, difficult to control and mentally efficient). Its strength increases with the number of repetitions and the stability of the context. Resistance to change increases with the degree of automaticity. Old habits are hard to break and new habits are hard to form because the behavioural patterns we repeat are imprinted in our neural pathways. Repetition in the same context increases the strength of an habit and, conversely, repetitions help to form new habits. Then, in the context of the variational rationality approach in [1, 2], permanent habits are variational traps, worthwhile to form and not worthwhile to break. They are the end points of habituation processes, the vestiges of past repeated behaviors.

- **Habit formation**

  There are four theories to explain habit formation; see [51].

  - The “direct-context-cuing” theory (see [52]) consider that habits are learned dispositions to repeat past responses. Contexts activate habitual responses directly, without the mediation of goal states.
- The “attitudes cuing” theory, for example the theory of planned behaviour, see [53], tells us that intentions to perform behaviors can be predicted with high accuracy from attitudes toward the behavior, subjective norms, and perceived behavioral control.

- The “goal-directed behaviors” theory (see [47, 54]) sees habits as a form of goal-directed automatic behavior such that the very activation of the goal to act automatically evokes the habitual response.

- The “habits-goals interface” theory (see [55]) considers that once a habit is formed, perception of contexts triggers the associated response without a mediating goal. However, habits interface with goals. Habits can either benefit or hurt goals. Goals can guide habits by providing the initial outcome-oriented impetus for response repetition. As such, habits often are a vestige of past goal pursuit. But when a habit forces an action, and, at the same time a conscious goal pushes for another action, they are in conflict.

### 7.2 Habits as Stocks of Past Experiences in Economics

In this domain, there is also a huge literature where a current habit is modelized as a stock of past behaviors which determines the present preference of the agent with respect to present consumption. In standard models of addictions (see [56]) and habit formation (see [57, 58]) preferences have the given current numerical representation

\[ U_n = U(c_n, h_n), \]

where the current state \( h_n \) represents a stock of habits, \( c_n \) stands for current consumption, and \( n \) indexes time. The habit persistence hypothesis implies that instantaneous utility does not only depend on current consumption, but also on a stock of habits, \( h_n \). The formulation can be substractive or multiplicative; see [59]. The substractive form of habit persistence is

\[ U(c_n, h_n) = V(c_n - \gamma h_n) \]

where \( 0 \leq \gamma \leq 1 \) represents the strength of habits. The multiplicative form is

\[ U(c_n, h_n) = V(c_n h_n^{-\gamma}). \]

Habit formation is modelized by a state function \( h_{n+1} = \rho(h_n, c_n) \). Most of the time the stock of habits \( h_n \) evolves according to an autoregressive law

\[ h_{n+1} = \lambda h_n + (1 - \lambda)c_n, \]

where \( 0 \leq \lambda < 1 \) represents the degree of persistence of the habit reference level. This defines a linear state function \( \rho(.). \)
7.3 Routines as collective patterns of interactions in Management

At the interaction level of an organization, an enormous literature considers routines as organizational habits in the context of the stability and change dynamics of organizations. The excellent survey is [60] which lists the main points which characterize routines as: patterns of interactions, collective activities, mindlessness vs. effortful accomplishments, processes (ways of doing, scripts), context dependent (embeddedness and specificity), path dependent, and triggered by related actors and external cues. Routines have several effects. They favor coordination, control, truce and stability. They also economize on cognitive resources, store knowledge, and reduce uncertainty. However organizational routines are outside our considerations because our paper deals with one agent.

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