Generating all minimal integral solutions to AND–OR systems of monotone inequalities: Conjunctions are simpler than disjunctions

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Abstract

We consider monotone ∨, ∧-formulae of m atoms, each of which is a monotone inequality of the form \( f_i(x) \geq t_i \) over the integers, where for \( i = 1, \ldots, m \), \( f_i : \mathbb{Z}^n \to \mathbb{R} \) is a given monotone function and \( t_i \) is a given threshold. We show that if the ∨-degree of \( \phi \) is bounded by a constant, then for linear, transversal and polymatroid monotone inequalities all minimal integer vectors satisfying \( \phi \) can be generated in incremental quasi-polynomial time. In contrast, the enumeration problem for the disjunction of \( m \) inequalities is NP-hard when \( m \) is part of the input. We also discuss some applications of the above results in disjunctive programming, data mining, matroid and reliability theory.

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1. Introduction

Consider a system of linear inequalities

\[
\sum_{j=1}^{n} a_{ij} x_j \geq t_i \quad \text{for } i = 1, \ldots, m,
\]  

where \( a_{ij} \) are given non-negative reals, and where we assume that the variables can take only binary values. Due to the non-negativity of the coefficients, if a vector \( x \geq 0 \) satisfies some of these inequalities and \( y \geq x \), then \( y \) satisfies the same inequalities (and possibly some others as well), i.e., the system (1) is monotone. We say that \( x \in \{0, 1\}^n \) is a minimal feasible solution for a subset \( I \subseteq \{1, \ldots, m\} \) of the inequalities (1), if \( x \) satisfies all inequalities belonging to \( I \), and any binary vector \( y \neq x \) such that \( y \leq x \) violates at least one of these inequalities. Lawler et al. [25] considered

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the problems of generating all minimal feasible solutions satisfying

(P1) all \( m \) inequalities of (1), and
(P2) at least one of the \( m \) inequalities of (1).

These or equivalent problems arise in a number of areas, including integer programming, scheduling, and polyhedral combinatorics (see e.g. [5,6,25,31,33]). Note that the number of minimal feasible solutions may not be limited by a polynomial of \( n \) and \( m \). A generation algorithm is said to be incrementally polynomial (or quasi-polynomial,\(^1\) or exponential) if for an arbitrary subset \( \mathcal{X} \) of minimal feasible solutions it can find an additional minimal feasible solution \( x \not\in \mathcal{X} \), or recognize that \( \mathcal{X} \) contains all such solutions, in time polynomially (or quasi-polynomially, or exponentially) limited in \( n, m \) and \( |\mathcal{X}| \). Equivalently, an algorithm is incrementally polynomial (quasi-polynomial, or exponential) if for arbitrary integer \( k \), it can generate \( k \) minimal feasible solutions, or all of them if \( k \) is too large, in time polynomial (quasi-polynomial, exponential) in \( n, m \) and \( k \).

When \( m = 1 \), problems (P1) and (P2) coincide, and incrementally efficient generation of all minimal feasible solutions is possible (see e.g. [25]). For the general case, when \( m > 1 \), only incrementally exponential algorithms were proposed for (P1) and (P2) in [25], and it was conjectured that no incrementally polynomial time algorithm can solve these problems, unless \( P = NP \). Let us note that incremental polynomiality of a generation problem is equivalent with the fact that for all subsets \( \mathcal{X} \) of minimal feasible solutions we can decide in polynomial time if \( \mathcal{X} \) is complete, or not, see e.g. [24]. Thus, a generation problem is called NP-hard if the above decision problem is NP-hard, or in other words, if no algorithm can generate all minimal feasible solutions in time polynomial in \( n, m \), and the number of minimal feasible solutions, unless \( P = NP \). These problems were reconsidered recently in [12], and contrary to the conjecture of [25], (P1) was shown to be tractable in incremental quasi-polynomial time (which makes it very unlikely to be NP-hard), while (P2) was shown to be tractable in incremental polynomial time for fixed \( m \), and NP-hard in the general case.

This motivated us to study more complex monotone systems, and generalize the above results in three directions.

First, we generalize the above enumeration problems to integer variables. More precisely, we consider the inequalities (1) with the variables running over an arbitrary integer box \( \mathcal{C} = \{x \in \mathbb{Z}^n | 0 \leq x \leq c\} \), where \( c = (c_1, \ldots, c_n) \in \mathbb{Z}^n_+ \) is a given integer vector (some components of which may be infinite).

Second, we consider not only conjunctions (like in (P1)), or disjunctions (like in (P2)), but arbitrary monotone expressions of linear inequalities. Specifically, let \( Y_i: \mathcal{C} \to \{0,1\} \) be the characteristic functions corresponding to the inequalities of (1), i.e. \( Y_i(x) = 1 \) iff \( \sum_{j=1}^{m} a_{ij} x_j \geq t_i, i = 1, \ldots, m \). Then we associate to any given monotone \( \lor, \land \)-formula \( \phi \) in \( m \) propositional variables a system \( \Sigma_\phi \) of inequalities for which a vector \( x \in \mathcal{C} \) is feasible iff \( \phi(Y_1(x), Y_2(x), \ldots, Y_m(x)) = 1 \). Since \( \Sigma_\phi \) is a monotone system, the notion of minimal feasible solutions of \( \Sigma_\phi \) is well defined, and we can consider the corresponding enumeration problem:

(P3) Generate all minimal feasible solutions of \( \Sigma_\phi \).

Note that when \( \phi = Y_1 \land Y_2 \land \cdots \land Y_m \), then (P3) is the integer variant of (P1), while for \( \phi = Y_1 \lor Y_2 \lor \cdots \lor Y_m \), problem (P3) is the integer variant of (P2).

Let us further associate to each monotone \( \lor, \land \)-formula \( \phi = \phi(Y_1, \ldots, Y_m) \) a polynomial \( P_\phi \in \mathbb{Z}[y_1, \ldots, y_m] \) defined by replacing logical conjunctions by arithmetic additions, and logical disjunctions by arithmetic multiplications. For instance, if \( \phi = Y_1 \lor (Y_2 \land (Y_3 \lor Y_4)) \), then we have \( P_\phi(y_1, y_2, y_3, y_4) = y_1 (y_2 + (y_3 y_4)) = y_1 y_2 + y_1 y_3 y_4 \). We call \( P_\phi \) the evaluation polynomial of \( \phi \). It turns out that the degree of this polynomial is intimately related to the complexity of the associated enumeration problem (P3).

**Theorem 1.** If the degree of the evaluation polynomial \( P_\phi \) is bounded by a constant, then we can generate all minimal feasible solutions to system \( \Sigma_\phi \) in incremental quasi-polynomial time in terms of \( n \) and \( m \).

Let us note that if the degree of \( P_\phi \) is not bounded, then generating minimal feasible solutions for \( \Sigma_\phi \) is NP-hard already for the Boolean case \( \mathcal{C} = \{0,1\}^n \) of (P2), see [12].

Finally, in a third direction, we extend Theorem 1 to transversal, polymatroid and 2-monotonic inequalities.

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\(^1\) A function \( f(x) \) is quasi-polynomial if \( f(x) = 2^{\text{polylog}(x)} \).
Given a subset of integral vectors $\mathcal{H} \subseteq \mathbb{C}$ and non-negative real weights $w : \mathcal{H} \to \mathbb{R}_+$, we call the function

$$f_{\mathcal{H},w}(x) = \sum \{ w(a) | a \in \mathcal{H}, a \not\geq x \}$$

a (weighted) transversal function over $\mathbb{C}$. Note that for any given $x \in \mathbb{C}$ we can compute the value $f_{\mathcal{H},w}(x)$ in $O(n|\mathcal{H}|)$ time, and that for the Boolean case $\mathbb{C} = \{0, 1\}^n$ we can equivalently define $f_{\mathcal{H},w}(x)$ as the total weight of all hyperedges of the hypergraph $\mathcal{H}$, whose complements intersect the support of $x$.

An integer-valued monotone function $f : \mathbb{C} \to \mathbb{Z}_+$ is called polymatroid if $f(0, \ldots, 0) = 0$ and $f$ is submodular, i.e., $f(x \lor y) + f(x \land y) \leq f(x) + f(y)$ holds for all vectors $x, y \in \mathbb{C}$, where $x \lor y = (\max\{x_j, y_j\})_{j = 1, \ldots, n}$ and $x \land y = (\min\{x_j, y_j\})_{j = 1, \ldots, n}$.

Following the terminology of e.g. [34, 29], a real-valued monotone function $f : \mathbb{C} \mapsto \mathbb{R}$ is said to be 2-monotonic if there exists a permutation $\sigma \in \mathbb{S}_n$ of the coordinate set $[n] \overset{\text{def}}{=} \{1, \ldots, n\}$ such that $f(x + \epsilon^{\sigma(i)} - \epsilon^{\sigma(j)}) \geq f(x)$ whenever $x \in \mathbb{C}, x_\sigma(j) > 0, x_\sigma(i) < c_\sigma(i)$ and $i < j$ (where $\epsilon^i$ denotes the $i$th unit vector). For instance, any non-negative linear function $f(x) = \sum_{j=1}^n a_j x_j$ is 2-monotonic with respect to the permutation $\sigma$ for which $a_\sigma(1) \geq a_\sigma(2) \geq \cdots \geq a_\sigma(n)$.

Let us note that transversals are both monotone and submodular, thus they are also polymatroid, whenever they take only integer values. In what follows we will be dealing with polymatroid and 2-monotonic functions whose value at any point $x \in \mathbb{C}$ can be evaluated in polynomial time. Some applications of monotone systems defined via transversal and polymatroid functions are discussed in Section 2.

Analogously to the case of linear inequalities, given a set of inequalities

$$f_i(x) \geq t_i, \quad i = 1, \ldots, m,$$  

(2) where $f_i(x)$, are monotone functions over $\mathbb{C}$ and $t_i \in \mathbb{R}$ for $i = 1, \ldots, m$, and given a monotone $\lor, \land$-formula $\phi$ in $m$ variables, we can associate to (2) and $\phi$ a monotone system $\Sigma_{\phi}$: A vector $x \in \mathbb{C}$ is called feasible for $\Sigma_{\phi}$ if $\phi(Y_1(x), \ldots, Y_m(x)) = 1$, where $Y_i(x) = 1$ iff $f_i(x) \geq t_i$, $i = 1, \ldots, m$. Let $\mathcal{F}_{\phi}$ denote the set of all minimal feasible solutions to system $\Sigma_{\phi}$.

**Theorem 2.** If (2) involves linear and/or transversal inequalities, i.e., if we have $f_i(x) = a_i^T x$ for some non-negative vectors $a_i$, for $i \in I$, and $f_i = f_{\mathcal{H}^i,w^i}$ for some subsets $\mathcal{H}^i \subseteq \mathbb{C}$ and non-negative weights $w^i : \mathcal{H}^i \mapsto \mathbb{R}_+$, for $i \in [m] \setminus I$, and if the degree of the evaluation polynomial $P_{\phi}$ is bounded, then we can generate $\mathcal{F}_{\phi}$ in incremental quasi-polynomial time in terms of $n, m$ and $\max_{i \in [m] \setminus I} \|a_i\|_\infty + 1)$.

Note that for the Boolean case $\mathbb{C} = \{0, 1\}^n$, linear inequalities are transversal inequalities corresponding to the hypergraph $\mathcal{H}$ consisting of $n$ singletons $\{1\}, \{2\}, \ldots, \{n\}$. In particular, if the degree of $P_{\phi}$ is not bounded, then no efficient (incrementally quasi-polynomial) generation of minimal feasible solutions for $\Sigma_{\phi}$ is possible, unless $P = \text{NP}$ [12, 15]. On a general integer box $\mathbb{C}$ (with bounded $c$), a linear function can be represented as a transversal function $f_{\mathcal{H},w}$, but with $|\mathcal{H}| = \|c\|_1$.

**Theorem 3.** If (2) involves polymatroid inequalities, and if the degree of the evaluation polynomial $P_{\phi}$ is bounded, then we can generate $\mathcal{F}_{\phi}$ in incremental quasi-polynomial time in terms of $n, m$ and $\max_{1 \leq i \leq m} t_i$.

Let us remark that the above theorem provides efficient for $\mathcal{F}_{\phi}$ only if $\max_{1 \leq i \leq m} t_i$ is bounded by a polynomial or quasi-polynomial expression of $n$ and $m$. In fact, generating all minimal feasible solutions to a single polymatroid inequality over $\{0, 1\}^n$ is already NP-hard, if the right-hand side is not bounded [10]. Due to this fact, integrality of polymatroid functions is essential in our analysis.

**Theorem 4.** If (2) involves 2-monotonic inequalities, and if the degree of the evaluation polynomial $P_{\phi}$ is bounded, then we can generate $\mathcal{F}_{\phi}$ in incremental quasi-polynomial time in terms of $n, m$.

Let us add that in case of a single 2-monotonic inequality, all minimal feasible solutions can be generated in incremental polynomial time [8, 18, 30].

The rest of this article is organized as follows. In the next section, we give some applications of the above stated results to enumeration problems arising in different areas. In Section 3, we relate the efficiency of the enumeration
problem for monotone systems to an important parameter, which we call the \textit{duality index}. We first consider the composition of monotone systems on an abstract level, and show that, while the efficiency of enumeration is preserved under taking conjunctions, the same is not true in the case of disjunctions, even if we consider only two systems. We then concentrate on the special cases of polymatroid, transversal and 2-monotonic inequalities, and show that the efficiency of enumeration is preserved under taking the disjunction of a constant number of these. This is achieved by proving a bound on the duality index of the disjunction, in terms of the duality indices of the individual inequalities. This comprises the main technical part of the paper, which will be proved in Section 4.

2. Applications

\textit{Disjunctive programming:} Let $A_1 \in \mathbb{R}_{+}^{r_1 \times n}$, \ldots, $A_m \in \mathbb{R}_{+}^{r_m \times n}$ be non-negative real matrices, and $b^1 \in \mathbb{R}_+^{r_1}$, \ldots, $b^m \in \mathbb{R}_+^{r_m}$ be positive real vectors. Consider the following disjunctive normal form (DNF) of linear monotone inequalities [5]:

$$
\bigvee_{i=1}^{m} (A_i x \geq b_i), \quad x \in \mathbb{R}_+^{n}.
$$

(3)

It follows from Theorem 1 that, if $m$ is bounded by a constant then all minimal integer solutions of (3) can be enumerated in incremental quasi-polynomial time. In contrast, when $m$ is unbounded but $\max\{r_1, \ldots, r_m\} \leq \text{const}$, Theorem 1 implies a quasi-polynomial incremental algorithm for enumerating all maximal infeasible vectors for (3).

\textit{Data mining:} Given a binary database $\mathcal{D} \subseteq \{0, 1\}^n$, and an integer threshold $t$, a subset $X \subseteq [n]$ is said to be $t$-\textit{frequent} if $s(X) = |\{Y \in \mathcal{D} : Y \supseteq X\}| \geq t$ and it is called $t$-\textit{infrequent} if $s(X) < t$. It is easy to see that the function $f(X) = |\mathcal{D}| - s(X)$ is a transversal function with respect to the hypergraph $\mathcal{D}$. Let $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be $m$ binary databases, $t_1, \ldots, t_m$ be real thresholds, and consider the family

$$
\mathcal{F} = \min \{X \subseteq [n] \mid \exists i \in [m] : X = t_i\text{-infrequent with respect to } \mathcal{D}_i\}.
$$

For instance, each database $\mathcal{D}_i$ may represent the set of items purchased in each weekday $i = 1, \ldots, m = 7$, with $\mathcal{F}$ representing the family of minimal collections of items that lie below a specified purchase threshold in at least one of the 7 days of the week. Clearly, $\mathcal{F}$ is the family of minimal true vectors for the disjunction of transversal inequalities, and thus, for constant $m$, the elements of $\mathcal{F}$ can be enumerated in incremental quasi-polynomial time by Theorem 2.

The generation of maximal frequent and minimal infrequent sets arises in the generation of association rules in data mining applications, see e.g. [2,3,22].

\textit{Sparse boxes:} Another notion related to data mining applications is that of sparse boxes. Let $\mathcal{F}$ be a set of points in $\mathbb{R}^n$, and $t \leq |\mathcal{F}|$ be a given integer. A \textit{maximal $t$-box} is a closed $n$-dimensional hyper-rectangle which contains at most $t$ points of $\mathcal{F}$ in its interior, and which is maximal with respect to this property (i.e., cannot be extended in any direction without strictly enclosing more points of $\mathcal{F}$). Typically, the set of points $\mathcal{F}$ represents the set of records in a quantitative database, and maximal sparse boxes correspond to empty or nearly empty regions in the data space [4,16,19,26,27].

It is not difficult to see that the family $\mathcal{F}_{\mathcal{F}, t}$ of maximal sparse boxes, with respect to a given set of $n$-dimensional points $\mathcal{F}$ and a given threshold $t \in \mathbb{Z}_+$, can be represented as the set of minimal feasible vectors of a transversal inequality over a $2n$-dimensional box $B$ [13]. Given $m$ databases $\mathcal{D}_1, \ldots, \mathcal{D}_m$, and thresholds $t_1, \ldots, t_m$, one may be interested in finding all maximal regions in space which are sparse with respect to at least one of the databases, i.e., finding the disjunction $\mathcal{F} = \bigvee_{i=1}^{m} \mathcal{F}_{\mathcal{D}_i, t_i}$. Theorem 2 implies that the family $\mathcal{F}$ can be generated in quasi-polynomial time if the number of databases $m$ is constant. In contrast, mining all maximal boxes that are sparse for all $m$ databases can be done in incremental quasi-polynomial time regardless of whether $m$ is constant or not (see Theorem 2 and [13]). Let us add that only exponential algorithms were previously known in the literature for mining maximal sparse boxes [19].

\textit{Matroid intersections:} Given $m$ matroids $M_1, \ldots, M_m$, defined on the common ground set $V$ by $m$ independence oracles, Lawler et al. [25] considered the problem of enumerating all maximal sets $X \subseteq V$ independent in all the matroids. They gave an exponential-time enumeration algorithm whose running time is $O(|V|^{m+2})$ per each generated maximal independent set. Since a set $X \subseteq V$ is independent in a matroid if and only if $\text{rank}^*(V \setminus X) \geq \text{rank}^*(V)$, where $\text{rank}^*(\cdot)$ is the rank function for the dual matroid (which is a polymatroid function), Theorem 3 implies that the above problem can be solved in incremental quasi-polynomial time regardless of $m$. (Specifically, $k$ maximal
sets independent in \( M_1, \ldots, M_m \) can be generated in \( K^{O(\log K)} \) time and \( \text{poly}(K) \) independence tests, where \( K = \max(k, |V|, m) \). When \( m \) is fixed, Theorem 3 also implies an incremental quasi-polynomial time for generating all maximal sets \( X \) independent in at least one of the matroids \( M_1, \ldots, M_m \). Our next example deals with graphic matroids.

**Reliability:** Let \( R \) be a finite set of vertices, \( R_1, \ldots, R_m \subseteq R \) be \( m \) possibly intersecting subsets of \( R \), and \( E_1, \ldots, E_n \subseteq \binom{R}{2} \) be a collection of \( n \) sets of edges on \( R \). Given a set \( X \subseteq [n] \) and \( i \in [m] \), define \( c_i(X) \) to be the number of connected components of the graph \( (R_i, \bigcup_{j \in X} E_j \cap \binom{R}{2}) \). Then for any integral threshold \( t_i \), the inequality \( f_i(X) = |R_i| - c_i(X) \geq t_i \) is matroidal. In particular, if \( t_i = |R_i| - 1 \) and \( c_i([n]) = 1 \) then the family \( \mathcal{F}_i \) is the set of all minimal collections of the input sets of edges \( E_1, \ldots, E_n \) which interconnect all vertices in \( R_i \). In network reliability applications (see e.g. [17,17,32]), the sets of edges \( E_1, \ldots, E_n \) correspond to relays, each controlled by a single switch which may work or fail, and the sets of vertices \( R_1, \ldots, R_m \) correspond to regions, or sets of nodes in the network, whose connectivity is to be observed. It may be the case that the connectivity of the whole network is measured by the connectivity of these regions, e.g. the network is considered working properly if at least one of the regions \( R_i \) is connected, or more generally if a certain monotone Boolean expression \( \phi \) on the connectivity of these regions is satisfied. It follows from Theorem 3 that if the number of prime implicants of \( \phi \) is bounded by a constant, then all minimal collections of relays maintaining the connectivity of the network, as defined by the Boolean expression \( \phi \), can be enumerated in incremental quasi-polynomial time.

**Statistics:** Let \( (S, 2^S, \mu_1), \ldots, (S, 2^S, \mu_m) \) be \( m \) probability spaces defined on some finite sample space \( S \). Given a set \( \mathcal{X} \subseteq 2^S \) of events, we are interested in finding all minimal collections \( \mathcal{X} \subseteq \mathcal{X} \) of events the probability of the union of which exceeds some threshold \( t \), with respect to at least one of the measures \( \mu_1, \ldots, \mu_m \), i.e.,

\[
\left( \Pr_{\mu_1} \left[ \bigcup_{X \in \mathcal{X}} X \right] \geq t \right) \lor \cdots \lor \left( \Pr_{\mu_m} \left[ \bigcup_{X \in \mathcal{X}} X \right] \geq t \right).
\]

The above condition is an example of the disjunction of transversal functions, and for constant \( m \), the family of minimal such collections can be enumerated in quasi-polynomial time by Theorem 2. This remains true for arbitrary monotone \( \lor, \land \)-conditions of bounded \( \lor \)-degree.

### 3. Our approach and further results

Our approach is to utilize a general enumeration method for minimal elements of a monotone system, the so-called **joint generation**, proposed first in [9,23], and analyzed at greater detail in [14]. For a subset \( \mathcal{X} \subseteq \mathcal{C} \) let us denote by \( \mathcal{I}(\mathcal{X}) \) the set of all maximal vectors not above any vectors of \( \mathcal{X} \), i.e., \( \mathcal{I}(\mathcal{X}) = \{ \mathcal{y} \in \mathcal{C} | \nexists \mathcal{x} \in \mathcal{X} : \mathcal{x} \leq \mathcal{y} \} \). For instance, if \( \mathcal{F}_\phi \) denotes the family of all minimal feasible solutions for the system \( \Sigma_\phi \), as before, then \( \mathcal{I}(\mathcal{F}_\phi) \) is the set of all maximal feasible solutions.

The method jointly generates \( \mathcal{F}_\phi \cup \mathcal{I}(\mathcal{F}_\phi) \), iteratively extending two partial sets \( \mathcal{X} \subseteq \mathcal{F}_\phi \) and \( \mathcal{Y} \subseteq \mathcal{I}(\mathcal{F}_\phi) \), by generating either an element in \( \mathcal{F}_\phi \setminus \mathcal{X} \) or in \( \mathcal{I}(\mathcal{F}_\phi) \setminus \mathcal{Y} \), solving in each step the so-called **dualization problem** (see e.g. [20]). In particular, incrementally extending a given subset \( \mathcal{X} \subseteq \mathcal{F}_\phi \) requires solving at most \( |\mathcal{I}(\mathcal{F}_\phi) \cap \mathcal{I}(\mathcal{X})| \) dualization problems each of size at most \( |\mathcal{X}| + |\mathcal{I}(\mathcal{F}_\phi) \cap \mathcal{I}(\mathcal{X})| \) with \( n \) variables. Let us denote by \( \delta(n, M) \) the complexity of solving a dualization problem in \( n \) variables and of size \( M = |\mathcal{X}| + |\mathcal{Y}| \). Even though no polynomial upper bound on \( \delta(n, M) \) is currently available, it is known that \( \delta(n, M) = \text{poly}(n)M^{O(\log M)} \), see [12,21].

Note that the joint generation method utilizes the given inequalities in (2) only to check the feasibility of a given vector \( x \in \mathcal{C} \), and that any method for generating \( \mathcal{F}_\phi \) that uses only such feasibility tests has to perform at least \( |\mathcal{F}_\phi| + |\mathcal{I}(\mathcal{F}_\phi)| \) tests (and in fact has to generate both \( \mathcal{F}_\phi \) and \( \mathcal{I}(\mathcal{F}_\phi) \), see e.g. [22]). Furthermore, one can easily see that the dualization problem can be stated as the enumeration of minimal feasible solutions of a transversal or a polymatroidal inequality (given \( \mathcal{X} \subseteq \mathcal{C} \), the function \( f : \mathcal{C} \mapsto \{0, 1, \ldots, |\mathcal{X}| \} \), defined by \( f(x) = |\{a \in \mathcal{X} : a \not\leq x\}| \), for \( x \in \mathcal{C} \) is both transversal and polymatroidal, and the minimal feasible solutions of the inequality \( f(x) \geq |\mathcal{X}| \) correspond to \( \mathcal{I}(\mathcal{X}) \).

The above discussion shows that the quantity \( q_{\mathcal{F}_\phi}(k) \) defined by

\[
q_{\mathcal{F}_\phi}(k) = \max_{\mathcal{X} \subseteq \mathcal{F}_\phi, |\mathcal{X}| \leq k} |\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_\phi)|
\]

(4)
is an important parameter intimately related to the generation of the monotone system \( \mathcal{F}_\phi \). We shall call it the duality index of the monotone system \( \mathcal{F}_\phi \). Using this notion, let us then summarize the above in the following statement.

**Proposition 1.** Given a system of monotone inequalities (2), and a monotone Boolean formula \( \phi \), let \( \Sigma_\phi \) denote the monotone system associated to these, and let \( \mathcal{F}_\phi \) denote the set of all minimal feasible solutions to \( \Sigma_\phi \) (as before). Then, for an arbitrary subset \( \mathcal{X} \subseteq \mathcal{F}_\phi \), we can find \( x \in \mathcal{F}_\phi \setminus \mathcal{X} \), or recognize that \( \mathcal{X} = \mathcal{F}_\phi \) in time \( \ell(\delta(n, k + \ell) + T) \) time, where \( k = |\mathcal{X}| \), \( \ell = q_{\mathcal{F}_\phi}(|\mathcal{X}|) \), and \( T \) is the maximum time for a single feasibility test for (2).

In particular, \( \mathcal{F}_\phi \) can be generated in incremental quasi-polynomial time whenever the duality index \( q_{\mathcal{F}_\phi}(k) \) is bounded by a quasi-polynomial. Hence in order to derive Theorems 1, 2, 3, and 4 from Proposition 1, it now suffices to bound the duality index of \( \mathcal{F}_\phi \). In the rest of the paper we focus on the duality index of various monotone systems, and in particular, on obtaining the duality index of complex monotone systems from the duality indices of their component systems.

Given a monotone subset \( \mathcal{A} \subseteq \mathcal{C} \), i.e., for which \( x \in \mathcal{A}, y \geq x \) imply \( y \in \mathcal{A} \), let us denote by \( \min \mathcal{A} \) the set of all minimal vectors of \( \mathcal{A} \). Furthermore, for an arbitrary subset \( \mathcal{A} \subseteq \mathcal{C} \) let us denote by \( \mathcal{A}^+ = \{ x \in \mathcal{C} | x \geq y \) for some \( y \in \mathcal{A} \} \) the smallest monotone subset (ideal) of \( \mathcal{C} \) containing \( \mathcal{A} \). Thus, if \( \mathcal{F} \) is the set of all minimal feasible solutions of a monotone inequality \( f(x) \geq t \) over \( \mathcal{C} \), then \( \mathcal{F}^+ \) is the set of all feasible solutions for the same inequality.

If \( \mathcal{A}^+ \) and \( \mathcal{B}^+ \) are monotone subsets of \( \mathcal{C} \), then their conjunction and disjunction are defined as \( \mathcal{A}^+ \land \mathcal{B}^+ = \mathcal{A}^+ \cap \mathcal{B}^+ \) and \( \mathcal{A}^+ \lor \mathcal{B}^+ = \mathcal{A}^+ \cup \mathcal{B}^+ \). Let us note that the same operations can naturally be extended to their sets of minimal elements, \( \mathcal{A} = \min \mathcal{A}^+ \) and \( \mathcal{B} = \min \mathcal{B}^+ \), by defining \( \mathcal{A} \land \mathcal{B} = \min \{ a \lor b | a \in \mathcal{A}, b \in \mathcal{B} \} \) and \( \mathcal{A} \lor \mathcal{B} = \min \mathcal{A} \lor \mathcal{B} \). In particular, if \( \mathcal{A} \) and \( \mathcal{B} \) are the minimal feasible solutions to two systems of monotone inequalities over \( \mathcal{C} \), then \( \mathcal{A} \land \mathcal{B} \) consists of all minimal vectors satisfying both systems of inequalities, while \( \mathcal{A} \lor \mathcal{B} \) consists of all minimal vectors satisfying at least one of the systems. Using these definitions, we can thus talk about monotone formulae of arbitrary monotone systems. In particular, if \( \phi \) is an arbitrary monotone \( \lor, \land \)-formula in \( m \) propositional variables, \( \Sigma_\phi \) is the monotone system defined by \( \phi \) on the set of inequalities (2), and \( \mathcal{F}_\phi \) denotes the set of all minimal feasible solutions of the inequality \( f_i(x) \geq t_i \), for \( i = 1, \ldots, m \), then \( \mathcal{F}_\phi = \mathcal{F}_1 \land \mathcal{F}_2 \land \cdots \land \mathcal{F}_m \) is the set of all minimal feasible solutions of the system \( \Sigma_\phi \).

Our first result shows that the duality index of conjunctions of monotone systems can effectively be limited in terms of the duality indices of the component systems.

**Theorem 5.** Let \( \mathcal{F}_i \subseteq \mathcal{C} \) be monotone systems for \( i = 1, \ldots, m \), and let \( \phi = \mathcal{Y}_1 \land \mathcal{Y}_2 \land \cdots \land \mathcal{Y}_m \), i.e., \( \mathcal{F}_\phi = \mathcal{F}_1 \land \mathcal{F}_2 \land \cdots \land \mathcal{F}_m \). Then, we have

\[
q_{\mathcal{F}_\phi}(k) \leq P_\phi(q_{\mathcal{F}_1}(k), \ldots, q_{\mathcal{F}_m}(k)) = \sum_{i=1}^{m} q_{\mathcal{F}_i}(k),
\]

where \( P_\phi \) is the evaluation polynomial of \( \phi \).

**Proof.** Clearly, it is enough to prove the statement for \( m = 2 \). Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two arbitrary families of vectors in \( \mathcal{C} \). Then, we claim that, for all subfamilies \( \mathcal{X} \subseteq \mathcal{F}_1 \land \mathcal{F}_2 \) we have

\[
|\mathcal{F}(\mathcal{X}) \cap \mathcal{F}(\mathcal{F}_1 \land \mathcal{F}_2)| \leq (q_{\mathcal{F}_1} + q_{\mathcal{F}_2})(|\mathcal{X}|).
\]

To see the claim, let us consider the sets of vectors

\[
\mathcal{F}_1' = \{ a \in \mathcal{F}_1 | a \lor b \in \mathcal{X} \text{ for some } b \in \mathcal{F}_2 \}
\]

and

\[
\mathcal{F}_2' = \{ b \in \mathcal{F}_2 | a \lor b \in \mathcal{X} \text{ for some } a \in \mathcal{F}_1 \},
\]

let \( z \in \mathcal{F}(\mathcal{X}) \cap \mathcal{F}(\mathcal{F}_1 \land \mathcal{F}_2) \) an arbitrary element, and define

\[
X = \{ a \in \mathcal{F}_1 | a \leq z \} \quad \text{and} \quad Y = \{ b \in \mathcal{F}_2 | b \leq z \}.
\]

Then, we must have either \( X = \emptyset \) or \( Y = \emptyset \), since otherwise we would have elements \( a \in \mathcal{F}_1 \) and \( b \in \mathcal{F}_2 \) for which \( a \lor b \leq z \), contradicting that \( z \in \mathcal{F}(\mathcal{F}_1 \land \mathcal{F}_2) \).
Now, if $X = \emptyset$, then $z \in \mathcal{I}(\mathcal{F}_1) \cap \mathcal{I}(\mathcal{F}_1')$, since it is clear that $z$ is then an independent element with respect to $\mathcal{F}_1$, and furthermore $z \in \mathcal{I}(\mathcal{F})$ implies that for each $i \in [n]$, such that $z_i \neq e_i$, we have an element $a \in \mathcal{F}_1' \subseteq \mathcal{F}_1$ such that $a \leq z + e_i$, where $e_i$ is the $i$th unit vector.

Similarly, if $Y = \emptyset$, then $z \in \mathcal{I}(\mathcal{F}_2) \cap \mathcal{I}(\mathcal{F}_2')$ follows.

Thus,

$$|\mathcal{I}(\mathcal{F}) \cap \mathcal{I}(\mathcal{F}_1 \wedge \mathcal{F}_2)| \leq |\mathcal{I}(\mathcal{F}_1') \cap \mathcal{I}(\mathcal{F}_1)| + |\mathcal{I}(\mathcal{F}_2') \cap \mathcal{I}(\mathcal{F}_2)|$$

$$\leq q_{\mathcal{F}_1}(|\mathcal{F}_1'| + q_{\mathcal{F}_2}(|\mathcal{F}_2'|)$$

$$\leq (q_{\mathcal{F}_1} + q_{\mathcal{F}_2})(\max(|\mathcal{F}_1'|, |\mathcal{F}_2'|))$$

$$\leq (q_{\mathcal{F}_1} + q_{\mathcal{F}_2})(|\mathcal{F}|),$$

where the last two inequalities follow from the monotonicity of $q(\cdot)$. □

The above theorem implies that if (2) involves monotone inequalities the duality indices of which are (quasi)-polynomially bounded, then, so is the duality index of the conjunction system, implying a quasi-polynomially efficient generation of minimal feasible solutions, by Proposition 1.

In fact, an efficient bound on the duality index is known for several types of monotone inequalities. Let us denote temporarily by $\mathcal{A}$ the set of all minimal feasible solutions to a single monotone inequality $f(x) \geq t$. For a monotone linear function $f(x) = \sum_{j=1}^{n} a_j x_j \geq t$, where $a_j \geq 0$ for $j = 1, \ldots, n$, or more generally, for a 2-monotonic function $f: \mathcal{C} \mapsto \mathbb{R}$, we have by [12] that

$$q_{\mathcal{A}}(k) \leq nk. \tag{5}$$

If $f(x) = f_{\mathcal{H},w}(x) \geq t$ is a transversal function, then it was shown in [15] that

$$q_{\mathcal{A}}(k) \leq |\mathcal{H}| k, \tag{6}$$

regardless of the weights. Note that both of these bounds are sharp, up to a constant factor. Finally, if $f$ is a polymatroid function, then it was shown in [10] that

$$q_{\mathcal{A}}(k) \leq \max\{n, k^{\log(\epsilon)/c(2n,k)}\}, \tag{7}$$

where $c(x, \beta)$ is the unique positive root of the equation $2^c(x^{\epsilon} / \log \beta - 1) = 1$ (note that the above bound implies that $q_{\mathcal{A}}(k) \leq 2nk^{\log(\epsilon)}$ and that $c(2n,k) \approx \log \log k$ for large $k$). The exponent of (7) is asymptotically tight [11].

**Corollary 1.** If the conjunction of inequalities (2) consists of 2-monotonic, transversal and/or polymatroid functions (the latter ones with quasi-polynomially limited right-hand sides), then all minimal feasible solutions of (2) can be generated in quasi-polynomially incremental time.

Unfortunately, as the following claim shows, no result analogous to Theorem 5 can hold unconditionally for disjunctions of monotone sets.

**Theorem 6.** For each $\ell \geq 1$ there exist monotone systems $\mathcal{A}, \mathcal{B} \subseteq \{0, 1\}^{4\ell}$, for which $q_{\mathcal{A}}(k) \leq (2\ell + 1)k$ and $q_{\mathcal{B}}(k) \leq (2\ell + 1)k$, while $q_{\mathcal{A} \vee \mathcal{B}}(|\mathcal{A} \vee \mathcal{B}|) = 2^{|\mathcal{A} \vee \mathcal{B}|}$.

To prove Theorem 6, we need a few definitions and a lemma. Given a finite set $V$ and a hypergraph $\mathcal{H} \subseteq 2^V$, let us define the independence-degree of a hyperedge $H \in \mathcal{H}$ by

$$d(H) = |\{I \in \mathcal{I}(\mathcal{H}) : |H \setminus I| = 1\}|,$$

where, recall, $\mathcal{I}(\mathcal{H})$ is the family of maximal independent sets of $\mathcal{H}$ (i.e., inclusion-maximal subsets of $V$ which do not contain any hyperedge of $\mathcal{H}$). For a threshold $r$ let

$$\mathcal{H}_r = \{H \in \mathcal{H} : d(H) \leq r\}.$$
Lemma 1. Let $\mathcal{H} \subseteq 2^V$ be a given hypergraph and $r$ be a given integer threshold. If for all $I \in \mathcal{I}(\mathcal{H})$ there exists a vertex $x \in V \setminus I$ for which $I \cup \{x\}$ contains hyperedges only from $\mathcal{H}_r$, then we have for all positive integers $k$ that

$$q_{\mathcal{H}}(k) \leq r.$$

Proof. Let $\mathcal{B}$ be an arbitrary subhypergraph of $\mathcal{H}$ of cardinality at most $k$. If $I \in \mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})$, then there exists a vertex $x_I \in I$ for which $I \cup \{x_I\}$ contains hyperedges only from $\mathcal{H}_r$, according to the assumption. For each $I \in \mathcal{I}(\mathcal{B})$, the set $I \cup \{x_I\}$ must contain a hyperedge from $\mathcal{B}$, by definition. Let us associate to each $I \in \mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})$ such a hyperedge $X_I \subseteq I \cup \{x_I\}$, $X_I \in \mathcal{B}$. Then, we have $X_I \in \mathcal{X} \cap \mathcal{H}_r$ for all $I \in \mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})$. Thus, we have $d(X_I) \leq r$ for each, i.e., there are at most $r$ independent sets $I' \in \mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})$ for which $X_I = X_{I'}$. Therefore,

$$|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})| \leq r|\mathcal{X} \cap \mathcal{H}_r| \leq r|\mathcal{X}| \leq r$$

follows. \square

Proof of Theorem 6. Let $X = \{1, \ldots, 2\ell\}$, $Y = \{2\ell + 1, \ldots, 4\ell\}$, and $V = X \cup Y$. Furthermore, let

$$\mathcal{M}_a = \{i, i + \ell\} | i = 1, \ldots, \ell\} \quad \text{and} \quad \mathcal{M}_b = \{i + 2\ell, i + 3\ell\} | i = 1, \ldots, \ell\}$$

be perfect matchings on the sets $X$ and $Y$, respectively, and let $\mathcal{F}$ denote the family of all minimal transversals to $\mathcal{M}_a \cup \mathcal{M}_b$. Let us finally define

$$\mathcal{A} = \mathcal{M}_a \cup \{T \cup \{y\} | T \in \mathcal{F}, y \in Y \setminus T\}, \quad \text{and} \quad \mathcal{B} = \mathcal{M}_b \cup \{T \cup \{x\} | T \in \mathcal{F}, x \in X \setminus T\}.$$ 

It is easy to see that every hyperedge of $\mathcal{A} \cup \mathcal{B}$ contains an edge from $\mathcal{M}_a \cup \mathcal{M}_b$, and since this latter family is a subfamily of $\mathcal{A} \cup \mathcal{B}$,

$$\mathcal{A} \setminus \mathcal{B} = \mathcal{M}_a \setminus \mathcal{M}_b$$

follows, immediately implying the last equality in the statement.

To complete the proof, we need to verify that both $\mathcal{A}$ and $\mathcal{B}$ are uniformly dual-bounded, as claimed. Since the roles of $X$ and $Y$ are symmetric in $\mathcal{A}$ and $\mathcal{B}$, it is enough to show this for one of them, say for $\mathcal{A}$.

We shall show the claim using Lemma 1 with $r = 2\ell + 1$. To this end, let us first observe that $\mathcal{A}$ has three types of maximal independent sets, and refer to them, respectively, as red, blue, and yellow, to simplify notations:

$$\mathcal{I}(\mathcal{A}) = \mathcal{F} \cup \{Y \cup (T \cap X) \setminus \{x\} | T \in \mathcal{F}, x \in T \cap X\} \cup\{(T \cap X) \cup (Y \setminus e) | T \in \mathcal{F}, e \in \mathcal{M}_b\}.$$ 

To see this, let us first observe that if $I \in \mathcal{I}(\mathcal{A})$ contains at least one endpoint of each of the edges of $\mathcal{M}_a \cup \mathcal{M}_b$, then it contains exactly one such endpoint from each, and $I \in \mathcal{F}$ (type red). If $I$ does not contain either endpoints of an edge $e \in \mathcal{M}_a$, then it must contain exactly one endpoint from each other $e' \in \mathcal{M}_a$, and must contain $Y$ (type blue). Finally, if $I$ does not contain either endpoints of a set $e \in \mathcal{M}_b$, then it must contain exactly one endpoint of each $e' \in \mathcal{M}_a$, and contains $Y \setminus e$ (type yellow).

Let us next show that

$$\mathcal{A}_{2\ell+1} = \mathcal{A} \setminus \mathcal{M}_a.$$ 

Indeed, $d((i, i + \ell)) \geq |\mathcal{F}| > 2\ell + 1$ for all $\ell \geq 1$, and hence $\mathcal{A}_{2\ell+1} \subseteq \mathcal{A} \setminus \mathcal{M}_a$. On the other hand, for $H = T \cup \{y\}$, $T \in \mathcal{F}$ and $y \in Y \setminus T$, we have $d(H) = 2\ell + 1$. To see this let us denote by $e \in \mathcal{M}_b$ the edge containing $y$. Then, for every $x \in H \cap X$ we have a unique blue set containing $H \setminus \{x\}$; for both endpoints $i \in e$ the set $H \setminus \{i\}$ is a red set; and finally for every $y' \in H \cap Y \setminus e$ there is a unique yellow set containing $H \setminus \{y'\}$.

Finally, we claim that for each $I \in \mathcal{I}(\mathcal{A})$ there exists a point $x_I \notin I$ such that $I \cup \{x_I\}$ contains hyperedges of $\mathcal{A}$ only from $\mathcal{A}_{2\ell+1}$, i.e., does not contain an edge of $\mathcal{M}_a$. For a red $I = T \in \mathcal{I}(\mathcal{A})$ any $x_I \in Y \setminus I$ is such a point (and there are $\ell \geq 1$). For a blue $I = (T \cap X) \setminus \{x\} \cup Y \in \mathcal{I}(\mathcal{A})$ we can set $x_I = x$. Finally, for a yellow set $I = (T \cap X) \cup (Y \setminus e) \in \mathcal{I}(\mathcal{A})$ we can set $x_I$ to be either endpoints of $e \in \mathcal{M}_b$. 

Thus, Lemma 1 can be applied, and hence
\[ |\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{A})| \leq (2^\ell + 1)|\mathcal{X}| \]
follows for all subfamilies \( \mathcal{X} \subseteq \mathcal{A} \). □

Despite this negative result for disjunctions, for the special cases of 2-monotonic, transversal or polymatroid systems
the following better result can be regarded as an analogue of Theorem 5.

**Theorem 7.** Assume that either all functions \( f_i(x), i = 1, \ldots, m \) in (2) are linear and/or transversal, or all of them
are polymatroid, or all of them are 2-monotonic, and let \( \phi = Y_1 \vee Y_2 \vee \cdots \vee Y_m \). Then
\[ q_{\mathcal{F}_\phi}(k) \leq P_{\phi}(q_1(k), \ldots, q_m(k)) = \prod_{i=1}^m q_i(k), \]
where \( P_{\phi} \) is the evaluation polynomial of \( \phi \), and where \( q_i(k) \) are the upper bounds on \( q_{\mathcal{F}_i}(k) \) stated in (5), (6),
and (7).

The statement for 2-monotonic inequalities in Theorem 7 was proved in [12].

We claim next that even though we cannot mix different types of inequalities in Theorem 7, still a result analogous
to Theorem 1 can be derived from Theorems 5 and 7.

**Theorem 8.** If the system of inequalities (2) consists of either \( m \) linear and/or transversal inequalities, \( m \) polymatroid
inequalities, or \( m \) 2-monotonic inequalities, and \( \phi \) is an arbitrary monotone \( \lor, \land \)-formula in \( m \) propositional variables,
then the inequality
\[ q_{\mathcal{F}_\phi}(k) \leq P_{\phi}(q_1(k), q_2(k), \ldots, q_m(k)) \]
holds, where \( P_{\phi} \) is the evaluation polynomial of \( \phi \), and where, as in Theorem 7, the functions \( q_i(k) \) are the upper
bounds on the duality indices of the individual inequalities stated in (5), (6),
and (7).

**Proof.** First, we observe that if \( P_{\phi} \) is the evaluation polynomial for a monotone formula \( \phi \), and \( P'_{\phi} \) is the evaluation
polynomial for the conjunctive normal form (CNF) representation of \( \phi \) then
\[ P'_{\phi}(x_1, \ldots, x_m) \leq P_{\phi}(x_1, \ldots, x_m), \]
for all positive integers \( x_1, \ldots, x_m \). Thus we may assume without loss of generality that the formula \( \phi(Y_1, \ldots, Y_m) = \bigwedge_{j=1}^r \bigvee_{i \in I_j} Y_j \) is given in CNF form, i.e., \( \mathcal{F}_\phi = \mathcal{F}_1 \land \cdots \land \mathcal{F}_r \), where \( \mathcal{F}_1, \ldots, \mathcal{F}_r \) are minimal feasible solutions
satisfying monotone disjunctions of linear/transversal or polymatroid inequalities. From Theorem 7, we conclude that
\[ q_{\mathcal{F}_j}(k) \leq \prod_{i \in I_j} q_i(k) \quad \text{for } j = 1, \ldots, r. \]

From Theorem 5, we conclude that
\[ q_{\mathcal{F}}(k) \leq \sum_{j=1}^r q_{\mathcal{F}_j}(k) \leq \sum_{j=1}^r \prod_{i \in I_j} q_i(k) = P_{\phi}(q_1(k), \ldots, q_m(k)). \] □

Theorems 1, 2 and 3 readily follow from Theorem 8 in view of Proposition 1 and the bounds of (5), (6) and (7).

In the next section we state and prove two main lemmas from which Theorem 7 can be derived, and which may be
of interest on their own.
4. Main lemmas

4.1. Aggregating polymatroid inequalities

Let \( f_i : \mathcal{C} \to \mathbb{Z}_+ \), be a polymatroid function, \( t_i \in \mathbb{Z}_+ \) be a given positive integer threshold, and denote by \( \mathcal{F}_i \subseteq \mathcal{C} \) the set of all minimal feasible solutions to the polymatroid inequality \( f_i(x) \geq t_i, i = 1, \ldots, m \). Let us further define

\[
(f_1 \wedge \cdots \wedge f_m)(x) = \sum_{i=1}^{m} \min \{ f_i(x), t_i \}, \quad \text{and}
\]

\[
(f_1 \vee \cdots \vee f_m)(x) = \prod_{i=1}^{m} t_i - \prod_{i=1}^{m} (t_i - \min \{ f_i(x), t_i \})
\]

for all \( x \in \mathcal{C} \).

**Lemma 2.** Both functions, \( g = f_1 \wedge \cdots \wedge f_m \) and \( h = f_1 \vee \cdots \vee f_m \), are polymatroid. Furthermore, the sets \( \mathcal{F}_1 \wedge \cdots \wedge \mathcal{F}_m \) and \( \mathcal{F}_1 \vee \cdots \vee \mathcal{F}_m \), respectively, consist of all minimal feasible solutions of the polymatroid inequalities

\[
g(x) \geq \sum_{i=1}^{m} t_i \quad \text{and} \quad h(x) \geq \prod_{i=1}^{m} t_i.
\]

**Proof.** We have to show that the monotonically increasing functions \( g(x) \) and \( h(x) \) are polymatroid. It is well known and easy to see that if \( f_i(x) \) is polymatroid, then so is \( f_i^*(x) = \min \{ f_i(x), t_i \} \). Since a sum of polymatroid functions is polymatroid, we conclude that \( g(x) \) is polymatroid. To prove that \( h(x) \) is polymatroid as well, write \( \mathcal{C} \) as the product of \( n \) intervals \( \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \). Then (by a straightforward generalization of Proposition 1.1 in [28]) it suffices to show that for any \( i \in [n] \), for any \( z \in \mathcal{C}_i \), and for any \( x \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_i-\{z\} \times \mathcal{C}_{i+1} \times \cdots \times \mathcal{C}_n \), the difference

\[
\partial(x, i, z) = h(x + e^i) - h(x) = \prod_{j=1}^{m} [t_j - f_j^*(x)] - \prod_{j=1}^{m} [t_j - f_j^*(x + e^i)]
\]

as a function of \( x \) is monotonically decreasing in \( x \) (where \( e^i \) is the \( i \)th unit vector). This readily follows from the representation

\[
\partial(x, i, z) = \sum_{j=1}^{m} \left( \prod_{k=1}^{j-1} [t_k - f_k^*(x + e^i)] \cdot \prod_{k=j+1}^{m} [t_k - f_k^*(x)] \right) [f_j^*(x + e^i) - f_j^*(x)].
\]

Lemma 2 implies that any monotone \( \vee, \wedge \)-formula of polymatroid inequalities can be replaced by an equivalent polymatroid inequality:

**Corollary 2.** Let \( \phi \) be a monotone \( \vee, \wedge \)-formula in \( m \) variables, \( f_i(x) \) be a polymatroid function, \( t_i \in \mathbb{Z}_+ \) be non-negative integral threshold, and let \( \mathcal{F}_i \) denote the set of all minimal feasible solutions of the polymatroid inequality \( f_i(x) \geq t_i, \) for \( i = 1, \ldots, m \). Then, \( \mathcal{F}_{\phi} = \phi(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m) \) is the set of all minimal feasible solutions of the system \( \Sigma_{\phi} \), and also of the single polymatroid inequality

\[
(\phi(f_1, f_2, \ldots, f_m))(x) \geq P_{\phi}(t_1, \ldots, t_m),
\]

where \( P_{\phi} \) is the evaluation polynomial of \( \phi \).

4.2. Disjunction of transversal inequalities

To bound the duality index for the disjunction of general transversal inequalities, it will be enough to get a bound for a special class of transversal inequalities. More specifically, let us associate to a system of non-negative
Lemma 3. If \( a, b, m \in \mathbb{N}, p, q \in \mathbb{N}^m, \) and \( w \in \mathbb{R}_{+}^m \) such that \( w^T p \geq a \) and \( w^T q \leq b \) then

\[
\max_{1 \leq i \leq m} \frac{p_i}{q_i} \geq \frac{w^T p}{w^T q} \geq \frac{a}{b}.
\]

Lemma 4. Let \( f_w \) be a non-negative separable mapping as in (10), and let \( t \in \mathbb{R}_{+} \). Assume further that \( \mathcal{X}, \mathcal{Y} \subseteq \mathcal{C} \) are subsets of vectors for which \( \mathcal{X} \neq \emptyset, (\ell_1, \ldots, \ell_n) \notin \mathcal{X}, \) and which satisfy the following separation constraints:

(i) \( f_w(x) \geq t \) holds for all \( x \in \mathcal{X}; \)
(ii) \( f_w(y) < t \) for all \( y \in \mathcal{Y}. \)

Then, there exists a subfamily \( \mathcal{Y}' \subseteq \mathcal{Y} \) such that

(a) \( x \nless y \lor y' \) for all \( y, y' \in \mathcal{Y}' \) and \( x \in \mathcal{X}, \) and
(b) \( v(x) = |\{i \in [n] : x_i > \ell_i\}| \) denotes the number of non-minimal entries of \( x \in \mathcal{C}, \) then

\[
|\mathcal{Y}'| \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}} v(x)} \geq \frac{|\mathcal{Y}|}{n|\mathcal{X}|}.
\]

Proof. Let \( L = \{(i, z) | i \in [n], z \in \mathcal{C}_i \setminus \{\ell_i\}\}, \) and let us prove the claim by induction on the size of \( L. \) Clearly, if \( |L| \leq 1, \) then \( |\mathcal{X}| = 1 \) and \( |\mathcal{Y}| \leq 1, \) and thus the claim holds trivially.

For \( (i, z) \in L \) let us define

\( \mathcal{X}(i, z) = \{x \in \mathcal{X} | x_i < z\} \) and \( \mathcal{Y}(i, z) = \{y \in \mathcal{Y} | y_i < z\}, \)

and let \( U = \{(i, z) | \mathcal{X}(i, z) = \emptyset\}. \)

Let us observe first that if \( x \leq y \lor y' \) for some \( x \in \mathcal{X}, \) and \( y, y' \in \mathcal{Y}(i, z) \) for some \( (i, z) \in L, \) then we must have \( x \in \mathcal{X}(i, z). \) This observation implies first of all that (a) holds for all families \( \mathcal{Y}(i, z) \subseteq \mathcal{Y} \) with \( (i, z) \in U. \)

Let us also observe that the sets \( \mathcal{X}(i, z) \subseteq \mathcal{C} \) and \( \mathcal{Y}(i, z) \subseteq \mathcal{C} \) satisfy the conditions (i) and (ii) for every \( (i, z) \in L \setminus U \) with a smaller \( i \)th chain \( \mathcal{C}' = \mathcal{C}_i \setminus \{z' \mid z' \geq z\}, \) i.e., for which the associated set \( L' = L \setminus \{(i, z') | z' \in \mathcal{C}_i, z' \geq z\} \) satisfies \( |L'| < |L|. \) Thus, by our inductive hypothesis, there exists a subset \( \mathcal{Y}(i, z)' \subseteq \mathcal{Y}(i, z) \) which satisfies (a) by our first observation, and for which we have

\[
|\mathcal{Y}(i, z)'| \geq \frac{|\mathcal{Y}(i, z)|}{\sum_{x \in \mathcal{X}(i, z)} v(x)}.
\]

Thus, we can prove the claim by induction, if we can show that

\[
\max_{(i, z) \in U} \left( \frac{|\mathcal{Y}(i, z)'|}{|\mathcal{Y}(i, z)|} \right), \max_{(i, z) \in L \setminus U} \left( \frac{|\mathcal{Y}(i, z)|}{\sum_{x \in \mathcal{X}(i, z)} v(x)} \right) \right) \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}} v(x)}.
\]

If we have \( |\mathcal{Y}(i, z)'| \geq |\mathcal{Y}|/\sum_{x \in \mathcal{X}} v(x) \) for some \( (i, z) \in U, \) then we are done, with \( \mathcal{Y}' = \mathcal{Y}(i, z). \)

Let us assume next that \( U \neq \emptyset \) and that

\[
|\mathcal{Y}(i, z)| < \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}} v(x)} \quad \text{for all} \quad (i, z) \in U.
\]
Clearly, if $z, z' \in \mathcal{C}_i$, and $z \geq z'$, then $\mathcal{Y}(i, z) \supseteq \mathcal{Y}(i, z')$. Thus, by defining $\ell'_i = \max\{z | (i, z) \in U\}$ for $i \in [n]$, setting $\ell' = (\ell'_1, \ldots, \ell'_n)$, and introducing

$$\tau = \nu(\ell') = |\{i \in [n] | \ell'_i > \ell_i\}| \quad \text{and} \quad \mathcal{Y} = \bigcup_{(i, z) \in U} \mathcal{Y}(i, z),$$

we have

$$|\mathcal{Y}| \geq \left(1 - \frac{\tau}{\sum_{x \in \mathcal{Y}} \nu(x)}\right) |\mathcal{Y}|, \quad (11)$$

Defining further $\mathcal{C}_i = \{z' \in \mathcal{C}_i | z' \geq \ell'_i\}$ for $i \in [n]$, and setting $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$, we can see that the sets $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{C}$ satisfy (i) and (ii) with an associated set $L = L \setminus U$, i.e., for which $|L| < |L|$ in this case (we assumed $U \neq \emptyset$). Thus, by our inductive hypothesis there exists a subset $\mathcal{Y}' \subseteq \mathcal{Y}$ for which (a) holds and for which we have

$$|\mathcal{Y}'| \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{Y}'} \nu(x)}, \quad (12)$$

where $\nu(x) = |\{i \in [n] | x_i > \ell'_i\}|$ (note that $\ell'_i = \min \mathcal{C}_i$).

Let us also note that

$$\sum_{x \in \mathcal{X}} \nu(x) \leq \sum_{x \in \mathcal{Y}} \nu(x) - \tau. \quad (13)$$

This is because $\mathcal{X}(i, z) \neq \emptyset$ for all $z > \ell'_i$, by the definition of $\ell'_i$, and thus for all $i \in [n]$ there exists a vector $x^i \in \mathcal{X}$ for which $x_i^i = \ell'_i$, and therefore $\nu(x^i) = \nu(x^i) - 1$ for all indices for which $\ell'_i > \ell_i$.

From inequalities (11), (12) and (13) we can conclude that

$$|\mathcal{Y}'| \geq \frac{1 - \frac{\tau}{\sum_{x \in \mathcal{Y}} \nu(x)}}{\sum_{x \in \mathcal{Y}'} \nu(x)} \sum_{x \in \mathcal{Y}} \nu(x) - \tau \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{Y}} \nu(x)},$$

implying that the set $\mathcal{Y}'$ satisfies both (a) and (b).

Let us finally assume that $U = \emptyset$, and prove

$$\max_{(i, z) \in L} \frac{|\mathcal{Y}(i, z)|}{\sum_{x \in \mathcal{X}(i, z)} \nu(x)} \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{Y}} \nu(x)} \quad (14)$$

by Lemma 3, using the weights $w(i, z)/(W - t)$ for $(i, z) \in L$, where $W = \sum_{(i, z) \in L} w(i, z)$. Observe that due to the strict inequality in (ii), we may assume that $W > t$.

Let us note first that

$$\sum_{(i, z) \in L} \frac{w(i, z)}{W - t} |\mathcal{Y}(i, z)| = \sum_{y \in \mathcal{Y}} \sum_{(i, z) \in L} \frac{w(i, z)}{W - t} = \sum_{y \in \mathcal{Y}} \frac{W - f_w(y)}{W - t} |\mathcal{Y}|, \quad (15)$$

is implied by the inequalities $W - f_w(y) > W - t$ for all $y \in \mathcal{Y}$, which follow by (ii).

Let us note next that

$$\sum_{(i, z) \in L} \frac{w(i, z)}{W - t} \sum_{x \in \mathcal{X}(i, z)} \nu(x) = \sum_{x \in \mathcal{X}} \sum_{(i, z) \in L} \frac{w(i, z)}{W - t} \nu(x)$$

$$= \sum_{x \in \mathcal{X}} \frac{W - f_w(x)}{W - t} \nu(x) \leq \sum_{x \in \mathcal{X}} \nu(x) \quad (16)$$

is implied by the inequalities $W - f_w(x) \leq W - t$ for all $x \in \mathcal{X}$, which follow by (i).

Thus, (15) and (16) imply (14) by Lemma 3, and thus conclude our proof. \(\square\)
Corollary 3. Assume that $f_{w^i}$ is a non-negative separable mapping as in (10), and that $t_i \in \mathbb{R}_+$, for $i = 1, \ldots, m$.
Assume further that $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{C}$ are nonempty collections of vectors satisfying the following separation condition:

(i) For all $x \in \mathcal{X}$ we have $f_{w^i}(x) \geq t_i$ for at least one of the indices $i \in \{1, \ldots, m\}$.
(ii) For all $y \in \mathcal{Y}$ we have $f_{w^i}(y) < t_i$ for all indices $i \in \{1, \ldots, m\}$.

Then, there exists a subset $\mathcal{Y}' \subseteq \mathcal{Y}$ such that $x \not\in \mathcal{Y}'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}'$, and

$$|\mathcal{Y}'| \geq \frac{|\mathcal{Y}|}{(n|\mathcal{X}|)^m}.$$  

Proof. We shall show the claim by induction on $m$. If $m = 1$, then the statement follows from Lemma 4, and the fact that $v(x) \leq n$ for all $x \in \mathcal{C}$ (note that $0 \not\in \mathcal{X}$ since otherwise $\mathcal{Y}$ would be empty). In the general case, let us consider a partition of $\mathcal{X}$,

$$\mathcal{X} = \bigcup_{i \in \{1, \ldots, m\}} \mathcal{X}^i$$

such that $f_{w^i}(x) \geq t_i$ for all $x \in \mathcal{X}^i$ and for all $i \in \{1, \ldots, m\}$. Assume without loss of generality that $\mathcal{X}^i \neq \emptyset$ for $i = 1, \ldots, j$, for some $j \leq m$. Then for the sets $\mathcal{Y}$ and $\mathcal{X}^j$ together with the separable mapping $f_{w^i}$ and threshold $t_i$, the conditions of Lemma 4 hold. Thus there exists a subfamily $\mathcal{Y}' \subseteq \mathcal{Y}$ for which $x \not\in \mathcal{Y}'$ for all $y, y' \in \mathcal{Y}'$ and $x \in \mathcal{X}^j$ holds and for which

$$|\mathcal{Y}'| \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}^j} v(x)}.$$  

Let us next consider the families $\mathcal{Y}', \mathcal{X}^1 \cup \cdots \cup \mathcal{X}^{j-1}$, together with $f_{w^i}$ and $t_i$ for $i \in \{j - 1\}$. Then conditions (i) and (ii) hold for this input, and thus we can apply our inductive hypothesis, and conclude that there exists a subfamily $\mathcal{Y}' \subseteq \mathcal{Y}$ such that $x \not\in \mathcal{Y}'$ for all $y, y' \in \mathcal{Y}'$ and $x \in \mathcal{X}^1 \cup \cdots \cup \mathcal{X}^{j-1}$ hold, and for which

$$|\mathcal{Y}'| \geq \frac{|\mathcal{Y}|}{\prod_{i=1}^{j-1} (\sum_{x \in \mathcal{X}^i} v(x))}.$$  

Putting these together completes the proof of the claim. \qed

Let us remark that the bound in Corollary 3 cannot be improved by more than a factor of $O(m^{2m})$, i.e., it is tight within a constant factor whenever $m$ is constant. Let us also observe from the proof of Corollary 3 that this bound, and consequently all corresponding bounds in our previous theorems can be improved by a factor of $m^m$. Consequently, in our claims the evaluation polynomial $P_\phi$ associated to a Boolean expression $\phi$ could be replaced by $Q_\phi$ obtained from $\phi$ by replacing conjunctions by arithmetic addition, and disjunctions by the arithmetic $\diamond$ operation defined by $a_1 \diamond a_2 \diamond \cdots \diamond a_r = a_1 a_2 \cdots a_r / r^r$. A similar remark also applies to the bound on the duality index for the disjunction of 2-monotonic inequalities, stated in Theorem 7.

4.3. Proof of Theorem 7

We finally derive Theorem 7 from Lemmas 2 and 4. As mentioned earlier, the statement for 2-monotonic inequalities was proved in [12]. So we need only to prove the other two cases.

Let $\mathcal{X} \subseteq \mathcal{F}_\phi$ be an arbitrary subfamily of minimal feasible solutions of the system of size $k$, and let $\mathcal{Y} = \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_\phi)$. Clearly, if the functions $f_1, \ldots, f_m$ are polymatroid, then the statement follows from Corollary 2 and the bound (7). Assume next that the functions $f_1, \ldots, f_m$ are transversal. In this case we apply Lemma 4 as follows. Let $\mathcal{X}^1 \cup \cdots \cup \mathcal{X}^m = \mathcal{X}$ be a partition of $\mathcal{X}$ such that $f_i(x) \geq t_i$ for $x \in \mathcal{X}^i$ and $i \in \{1, \ldots, m\}$. Let $I \subseteq \{1, \ldots, m\}$ be the set of indices $i$ for which the function $f_i$ is linear. For $i \in I$, if the function $f_i(x) = \sum_{j=1}^m a_{ij} x_j$ for $x \in \mathcal{C}$, where $a_{ij} \geq 0$ for all $j = 1, \ldots, n$, we define the non-negative weight $w^i(j, z) = a_{ij}$ for $j \in \{1, \ldots, n\}$ and $z \in \mathcal{C}_j$. Then clearly, for all $x \in \mathcal{C}$,
such that $x \leq y \lor y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$. If $\bigcup_{i \in \mathcal{I}} \mathcal{X}^i = \emptyset$, we let $\mathcal{Y} \defeq \emptyset$. For $i \in [m]\setminus I$, if the function $f_i$ is transversal with respect to the family of vectors $\mathcal{H}_i \subseteq \mathcal{C}$, and corresponding weight $w^i : \mathcal{H}_i \mapsto \mathbb{R}_+$, then we define a non-negative weight function $w^i(a, z)$ over $a \in \mathcal{H}_i$ and $z \in \{0, 1\}$ as follows: $w^i(a, z) = 0$ if $z = 0$, and $w^i(a, z) = w^i(a)$ if $z = 1$. For $x \in \mathcal{C}$, define the vector $\eta^i(x) \in \{0, 1\}^{\mathcal{H}_i}$, whose $a$-component, for $a \in \mathcal{H}_i$, is given by $\eta^i_a(x) = 1$ if and only if $a \not\in x$. Note that for $x \in \mathcal{C}$, we have $f_i(x) \geq t_i$ if and only if $f_{w^i}(\eta^i(x)) \geq t_i$, for $x \in \mathcal{C}$. Thus, if $\mathcal{X}^i \neq \emptyset$, we can apply Lemma 4 to the families $\eta^i(\mathcal{X}) \defeq \{\eta^i(x) | x \in \mathcal{X}^i\}$ and $\eta^i(\mathcal{Y}) \defeq \{\eta^i(y) | y \in \mathcal{Y}\}$ to conclude that there exists a subset $\mathcal{Y}' \subseteq \mathcal{Y}$ of size $|\mathcal{Y}'| \geq |\mathcal{Y}|/\prod_{i \in \mathcal{I}} q_i(k)$, such that $\eta^i(x) \notin \eta^i(y \lor y')$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}'$. Continuing this way, for all $i \in [m]\setminus I$, we get finally a subfamily $\mathcal{Y}' \subseteq \mathcal{Y}$ of size at least $|\mathcal{Y}'|/\prod_{i \in \mathcal{I}} q_i(k)$, such that $x \notin y \lor y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}'$. Note, however, that $\mathcal{Y} \subseteq \mathcal{F}(\mathcal{X})$ implies that, for any distinct $y, y' \in \mathcal{Y}$, there exists an $x \in \mathcal{X}$, such that $y \lor y' \geq x$. Consequently, there cannot be more than one element in $\mathcal{Y}'$, i.e., $|\mathcal{Y}'| \leq 1$, and the theorem follows. □

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References