Gramian-based Reachability Metrics for Bilinear Networks

Yingbo Zhao, Jorge Cortés

Department of Mechanical and Aerospace Engineering
UC San Diego

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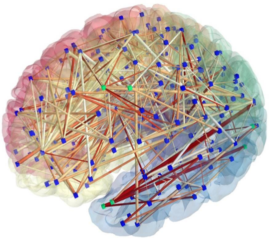
Bilinear networks provide good model for natural and artificial processes



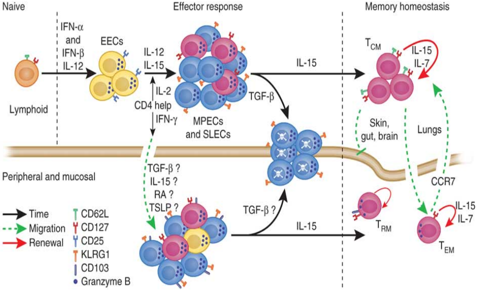


Figure: Left: brain network (courtesy: ftdtalk), Right: the generation of cell population (courtesy: Nature Immunology).

- Biological systems: brain network, population generation processes, physiological regulation processes (regulation of CO_2 in the respiratory system).

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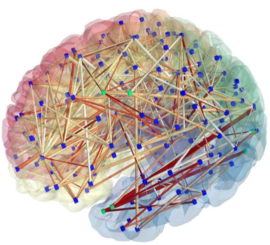

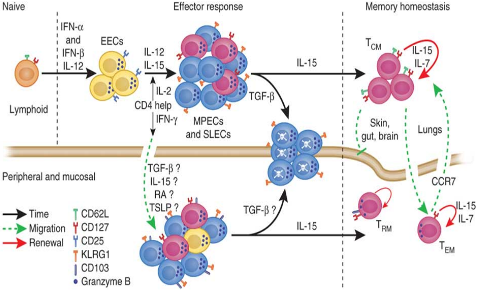

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- Physical systems: thermal exchange, chemical reactions.

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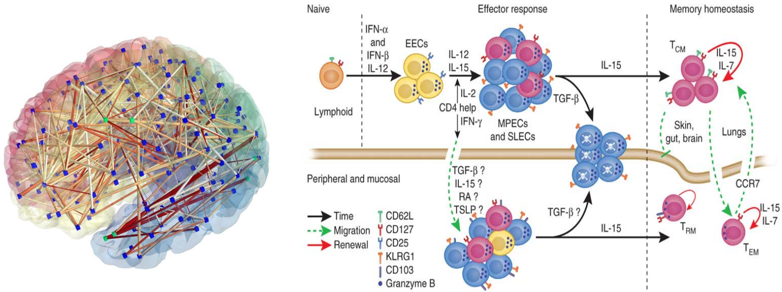

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- Biological systems: brain network, population generation processes, physiological regulation processes (regulation of CO_2 in the respiratory system).
- Physical systems: thermal exchange, chemical reactions.
- Economic systems: control of capital by varying the rate of interest.

State space description of bilinear networks

$$(A, F, B): \quad x(k+1) = Ax(k) + \sum_{i=1}^{m} (F_i x(k) + B_i) u_i(k).$$

Controller i can affect the states of some nodes (reflected by B_i) as well as the interconnections among neighboring nodes (reflected by F_i).

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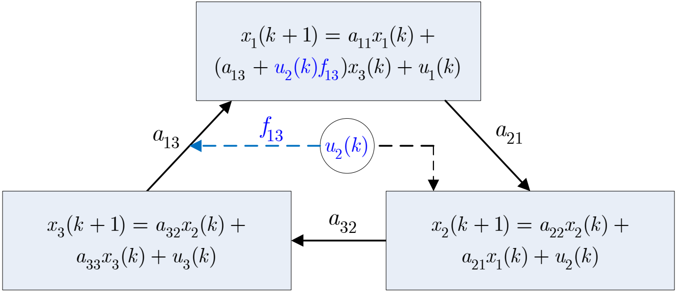

Figure: A bilinear ring network of 3 nodes: the bilinear term corresponds to interconnection modulation.

A motivating problem

To achieve certain performance (e.g., reachability) under certain constraints (e.g., sparsity or input energy), should one control a node state directly or modulate (strengthen, weaken, or create) an interconnection?

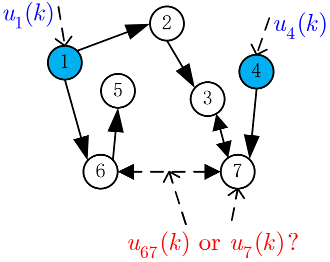

Figure: Modulating an interconnection v.s. controlling another node.

Reachability/controllability of bilinear systems (A, F, B) has been studied mostly as a binary property

(Limited) literature review on controllability of bilinear systems

- W. Boothby and E. Wilson (SIAM JCO, 1979): determination of transitivity.
- D. Koditschek and K. Narendra (IEEE TAC 1985): controllability of planar continuous-time bilinear systems.
- U. Piechottka and P. Frank (Automatica 1992): uncontrollability test for 3-d continuous-time systems.
- M. Evans and D. Murthy (IEEE TAC 1977), T. Goka, T. Tarn, and J. Zaborszky (Automatica 1973): controllability of discrete-time homogeneous-in-the-state bilinear system using rank-1 controllers.
- L. Tie and K. Cai (IEEE TAC 2010): near controllability of discrete-time systems.
- ...

Controllability of linear networks $(A, \mathbf{0}, B)$ has been studied both qualitatively and quantitatively

- F. Pasqualetti, S. Zampieri, and F. Bullo (IEEE TCNS 2014): "Controllability metrics, limitations and algorithms for complex networks."
- T. Summers and J. Lygeros (World Congress 2014): "Optimal sensor and actuator placement in complex dynamical networks."
- Y. Liu, J. Slotine, and A. Barabási (Nature 2011): "Controllability of complex networks."
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Reachability of bilinear systems (A, F, B) has been studied extensively as a binary property. However, the degree of reachability remains open.

Outline

- Reachability Gramian for bilinear systems.
- Gramian-based reachability metrics for bilinear systems.
- Actuator selection algorithm with guaranteed reachability performance.
- Addition of bilinear inputs to linear networks.

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Reachability metrics in terms of control energy

Minimum energy control of bilinear systems

$$\begin{aligned} \min_{\{u\}^{K-1}} \quad & \sum_{k=0}^{K-1} u^T(k)u(k) \\ \text{s.t.} \quad & \forall k = 0, \dots, K-1, \\ & x(k+1) = Ax(k) + \sum_{i=1}^{m} (F_i x(k) + B_i) u_i(k) \\ & x(0) = \mathbf{0}_n, \quad x(K) = x_f. \end{aligned}$$

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- The necessary optimality conditions for the solution $\{u^*\}^{K-1}$ lead to a nonlinear two-point boundary-value problem without an analytical solution.

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- The necessary optimality conditions for the solution $\{u^*\}^{K-1}$ lead to a nonlinear two-point boundary-value problem without an analytical solution.
- When $F_i = \mathbf{0}$, the bilinear network becomes linear and

$$u^*(k) = B^T (A^T)^{K-k-1} \mathcal{W}_{1,K}^{-1} x_f, \quad \sum_{k=0}^{K-1} (u^*(k))^T u^*(k) = x_f^T \mathcal{W}_{1,K}^{-1} x_f,$$

where $\mathcal{W}_{1,K} \triangleq \sum_{k=0}^{K-1} A^k BB^T (A^T)^k$ is the K-step controllability Gramian.

Gramian-based reachability metrics for linear networks

$$
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\min_{\{u\}^{K-1}} \quad & \textstyle\sum_{k=0}^{K-1} u^T(k)u(k) \\
\text{s.t.} \quad & \forall k = 0, \ldots, K-1, \\
& x(k+1) = Ax(k) + \textstyle\sum_{i=1}^{m} B_i u_i(k) \\
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\end{aligned}
$$

- Gramian-based lower bound on the minimum input energy:

$$
\sum_{k=0}^{K-1} (u^*(k))^T u^*(k) = x_f^T \mathcal{W}_{1,K}^{-1} x_f > x_f^T \mathcal{W}_1^{-1} x_f,
$$

where $\mathcal{W}_1 = \lim_{K\to\infty} \mathcal{W}_{1,K} = \sum_{k=0}^{\infty} A^k BB^T(A^T)^k$.

Gramian-based reachability metrics for linear networks

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- Gramian-based quantitative reachability metrics:

  $\lambda_{min}(\mathcal{W}_1)$: worst case minimum input energy,

  $tr(\mathcal{W}_1)$: average minimum control energy over $\{x \in \mathbb{R}^n | \|x\| = 1\}$,

  $\det(\mathcal{W}_1)$: volume of ellipsoid reachable using unit-energy control inputs.

Reachability Gramian for bilinear systems

Definition (Reachability Gramian for bilinear systems)

The reachability Gramian for a stable discrete-time bilinear system (A, F, B) is

$$\mathcal{W} = \sum_{i=1}^{\infty} \mathcal{W}_i,$$

where

$$\mathcal{W}_i = \sum_{k_1,\ldots,k_i=0}^{\infty} \mathcal{P}_i(\{k\}_1^i)\mathcal{P}_i^T(\{k\}_1^i),$$

$$\mathcal{P}_1(\{k\}_1^1) = A^k B \in \mathbb{R}^{n\times m},$$

$$\mathcal{P}_i(\{k\}_1^i) = A^{k_i} F(I_m \otimes \mathcal{P}_{i-1}(\{k\}_1^{i-1})) \in \mathbb{R}^{n\times m^i}, \; i \geq 2.$$

Properties of the Reachability Gramian $\mathcal{W} = \sum_{i=1}^{\infty} \mathcal{W}_i$

- $\mathcal{W}_i = \sum_{k_i=0}^{\infty} A^{k_i} \left(\sum_{j=1}^{m} F_j \mathcal{W}_{i-1} F_j^T\right) (A^{k_i})^T, \quad \mathcal{W}_1 = \sum_{k_1=0}^{\infty} A^{k_1} B B^T (A^{k_1})^T.$

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- The reachability Gramian $\mathcal{W}$ satisfies the generalized Lyapunov equation

$$A\mathcal{W}A^T - \mathcal{W} + \sum_{j=1}^{m} F_j \mathcal{W} F_j^T + BB^T = \mathbf{0}_{n\times n}.$$

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- A unique positive semi-definite solution $\mathcal{W}$ exists if and only if

$$\rho(A \otimes A + \sum_{j=1}^{m} F_j \otimes F_j) < 1.$$

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- The subspace $Im(\mathcal{W})$ is invariant under the bilinear dynamics $(A, F, B) \rightarrow$ any target state x_f that is reachable from the origin belongs to $Im(\mathcal{W})$.

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Gramian-based reachability metrics

Theorem (The reachability Gramian is a metric for reachability)

For the bilinear control system (A, F, B), *For* $K \in \mathbb{Z}_{\geq 1}$, *if*

$$\|u(k)\|_\infty \leq 2^{-1}\Big(\sum_{i,j=1}^{m} \|F_j^T \Psi F_i\|\Big)^{-1}\beta,$$

for all $k = 0, 1, \cdots, K-1$, *then*

$$\sum_{k=0}^{K-1} u^T(k)u(k) \geq x^T(K)\mathcal{W}^{-1}x(K),$$

where

$$\beta \triangleq -\sum_{j=1}^{m}\|A^T\Psi F_j + F_j^T\Psi A\| + \Big(\big(\sum_{j=1}^{m}\|A^T\Psi F_j + F_j^T\Psi A\|\big)^2 - 4\sum_{i,j=1}^{m}\|F_j^T\Psi F_i\| \cdot \lambda_{\max}(A^T\Psi A - \mathcal{W}^{-1})\Big)^{1/2},$$

$$\Psi \triangleq \mathcal{W}^{-1} - \mathcal{W}^{-1}B(B^T\mathcal{W}^{-1}B - I_m)^{-1}B^T\mathcal{W}^{-1}.$$

Gramian-based (local) reachability metrics

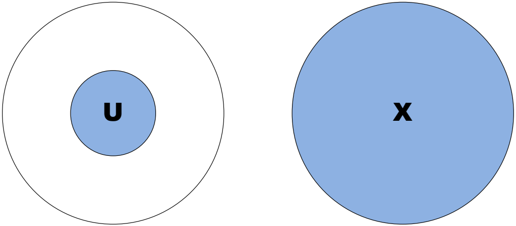


Figure: Input and state regions where the Gramian-based reachability metrics hold.

$$\|u(k)\|_\infty \leq 2^{-1}\Big(\sum_{i,j=1}^{m}\|F_j^T \Psi F_i\|\Big)^{-1}\beta$$

$$\Rightarrow \sum_{k=0}^{K-1} u^T(k)u(k) \geq x^T(K)\mathcal{W}^{-1}x(K)$$

Gramian-based (local) reachability metrics

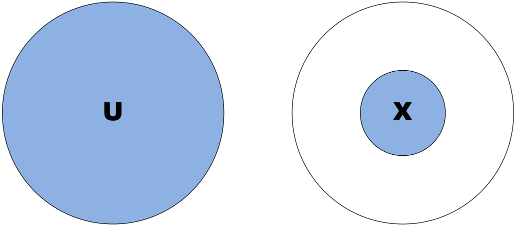
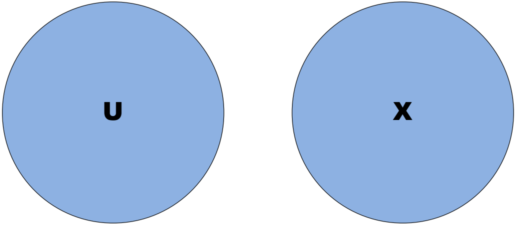


Figure: Input and state regions where $\sum_{k=0}^{K-1} u^T(k)u(k) \geq x^T(K)\mathcal{W}^{-1}x(K)$ hold.

- W. Gray and J. Mesko (IFAC 1998): "Energy functions and algebraic gramians for bilinear systems."
- P. Benner, T. Breiten, and T. Damm (IJC 2011): "Generalised tangential interpolation for model reduction of discrete-time mimo bilinear systems."

There is no non-trivial global reachability metrics



For general bilinear systems,

$$\inf \frac{\sum_{k=0}^{K-1} u^T(k)u(k)}{\|x(K)\|^2} = 0.$$

Gramian-based reachability metrics

The relation $\sum_{k=0}^{K-1} u^T(k)u(k) \geq x^T(K)\mathcal{W}^{-1}x(K)$ implies the following reachability metrics:

- $\lambda_{\min}(\mathcal{W})$: worst-case minimum reachability energy
- $tr(\mathcal{W})$: average minimum reachability energy over the unit hypersphere in state space
- $\det(\mathcal{W})$: the volume of the ellipsoid containing the reachable states using inputs with no more than unit energy

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Actuator selection

The actuator selection problem is to solve

$$\max_{S \subseteq V} f(\mathcal{W}(S)),$$

where $V = \{1, \cdots, M\}$, $S = \{s_1, \cdots, s_m\}$, f can be tr, $\lambda_{\min}$ or det.

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Theorem (Increasing returns property of the mapping from S to $\mathcal{W}(S)$)

For any $S_1 \subseteq S_2 \subseteq V$ and $s \in V \backslash S_2$,

$$\mathcal{W}(S_2 \cup \{s\}) - \mathcal{W}(S_2) \geq \mathcal{W}(S_1 \cup \{s\}) - \mathcal{W}(S_1).$$

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Maximization of supermodular function $(tr(\cdot))$ under cardinality constraint is NP hard in general.

Lower bound on reachability metrics

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Theorem (Lower bound on reachability metrics)

Let $f : \mathbb{R}^{n \times n} \to \mathbb{R}_{\geq 0}$ be either tr, $\lambda_{\min}$ or det. *Then*

$$f(\mathcal{W}(S)) \geq \sum_{s \in S} f(\mathcal{W}(s)),$$

for any set S of m actuators.

The greedy algorithm based on $f(\mathcal{W}(S)) \geq \sum_{s \in S} f(\mathcal{W}(s))$

In this example, baseline system parameters (A, F_0, B_0) are taken from T. Hinamoto and S. Maekawa (IEEE Transactions on Circuits and Systems, 1984) with 3 actuator candidates. For more details, please see http://arxiv.org/pdf/1509.02877.pdf.

S	$tr(W(S))$	$\lambda_{\min}(W(S))$	$\det(W(S))$	S	$tr(W(S))$	$\lambda_{\min}(W(S))$	$\det(W(S))$
$\{0\}$	14.42	0.027	0.242	$\{0,2\}$	19.91	0.07	3.32
$\{1\}$	5.03	0.023	0.025	$\{0,3\}$	18.69	0.05	1.13
$\{2\}$	4.04	3×10^{-5}	9×10^{-7}	$\{0,1,2\}$	26.50	0.137	46.15
$\{3\}$	3.03	1.6×10^{-6}	4×10^{-11}	$\{0,1,3\}$	25.28	0.125	28.68
$\{0,1\}$	20.98	0.09	11.704	$\{0,2,3\}$	24.19	0.103	8.34

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- $\sum_{s \in S} tr(\mathcal{W}(s))$ is a good estimate of $tr(\mathcal{W}(S))$.
- For $\lambda_{\min}(W(S))$ and $\det(W(S))$, the sum of individual contribution is far from the combinatorial contribution.

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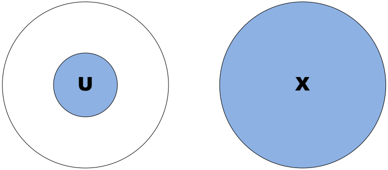
$\{1\}$	5.03	0.023	0.025	$\{0,3\}$	18.69	0.05	1.13
$\{2\}$	4.04	3×10^{-5}	9×10^{-7}	$\{0,1,2\}$	26.50	0.137	46.15
$\{3\}$	3.03	1.6×10^{-6}	4×10^{-11}	$\{0,1,3\}$	25.28	0.125	28.68
$\{0,1\}$	20.98	0.09	11.704	$\{0,2,3\}$	24.19	0.103	8.34

- $\sum_{s \in S} tr(\mathcal{W}(s))$ is a good estimate of $tr(\mathcal{W}(S))$.
- For $\lambda_{\min}(W(S))$ and $\det(W(S))$, the sum of individual contribution is far from the combinatorial contribution.
- Actuators with a large individual contribution provide a large combinatorial contribution.

So far, we have shown for bilinear networks

$$(A, F, B): \quad x(k+1) = Ax(k) + \sum_{i=1}^{m} (F_i x(k) + B_i) u_i(k).$$

- Input and state regions where $\sum_{k=0}^{K-1} u^T(k)u(k) \geq x^T(K)\mathcal{W}^{-1}x(K)$ hold.



- Actuator selection $\max_{S \subseteq V} f(\mathcal{W}(S))$ through $f(\mathcal{W}(S)) \geq \sum_{s \in S} f(\mathcal{W}(s))$ for $f = tr(\cdot), \lambda_{\min}(\cdot), \det(\cdot)$.

Back to the motivating problem

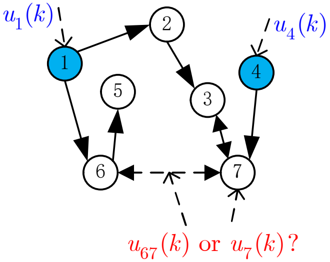
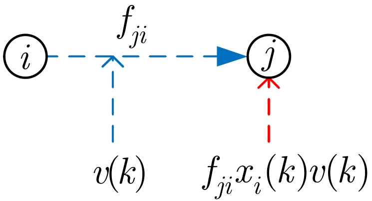


Figure: Modulating an interconnection v.s. controlling another node.

Controlling a node or an interconnection?



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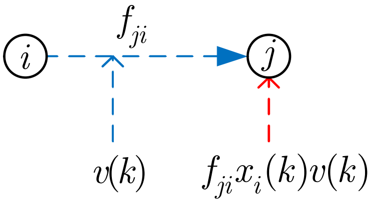

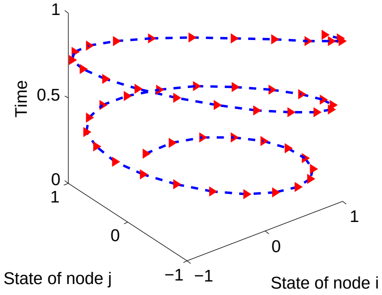

- No energy constraint → node > edge.

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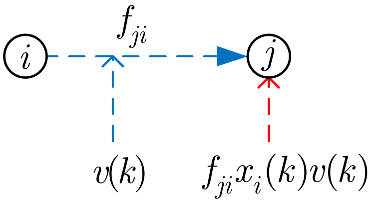

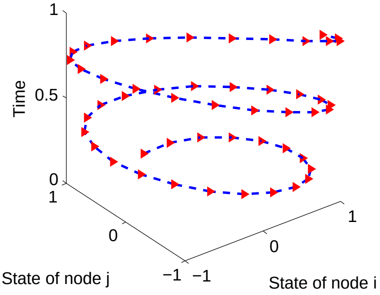

- No energy constraint $\rightarrow$ node $>$ edge.
- Otherwise, the answer depends on specific problem setup.

Difficult-to-control networks

Definition (Difficult-to-control networks)

A class of networks is said to be difficult to control (DTC) if, for a fixed number of control inputs, the normalized worst-case minimum reachability energy grows unbounded with the scale of the network, i.e.,

$$\lim_{n\to\infty} \sup_{x_f\in\mathbb{R}^n} \inf_{\{u\}^\infty : u(k)\in\mathbb{R}^m} \frac{\|\{u\}^\infty\|^2}{\|x_f\|^2} \to \infty.$$

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- For linear networks $(A(n), \mathbf{0}_{n\times nm}, B(n))$,

$$\sup_{x_f\in\mathbb{R}^n : \|x_f\|^2=1} \inf_{\{u\}^\infty : u(k)\in\mathbb{R}^m} \|\{u\}^\infty\|^2 = \lambda_{\min}^{-1}(\mathcal{W}_1(n)).$$

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- A typical class of DTC linear networks is stable and symmetric networks for which $\lambda_{\min}^{-1}(\mathcal{W}_1(n))$ increases exponentially with rate $\frac{n}{m}$ for any choice of $B(n)\in\mathbb{R}^{n\times m}$ whose columns are canonical vectors in $\mathbb{R}^n$ (Pasqualetti *et al.* IEEE TCNS 2014).

Will interconnection modulation change a DTC network?

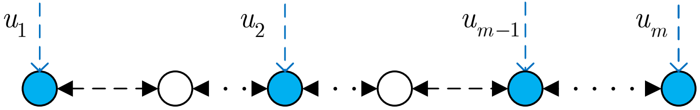

Figure: Linear symmetric line networks with finite controlled nodes are DTC.

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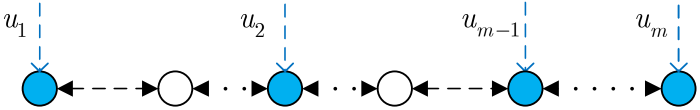

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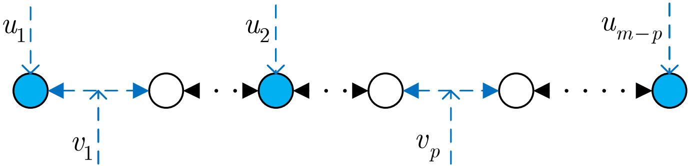

Figure: Linear symmetric line networks with finite controlled nodes AND finite controlled interconnections.

DTC linear symmetric networks remain DTC after certain types of interconnection modulation

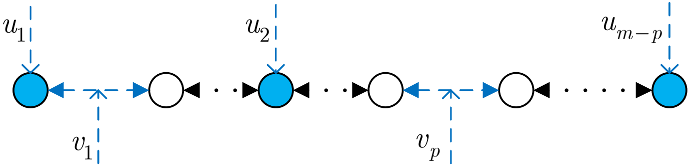

Figure: Linear symmetric line networks with finite actuators are DTC.

Theorem (DTC linear symmetric networks under bilinear control)

Consider a class of DTC linear symmetric networks $(A(n), \mathbf{0}_{n\times nm}, B(n))$*. The class of bilinear networks* $(A(n), F(n), B(n))$ *is also DTC if the number of nonzero entries in the matrix* $F(n) \in \mathbb{R}^{n\times nm}$ *and* $\|F(n)\|_{\max} \triangleq \max_{i,j} |F_{ij}(n)|$ *are uniformly bounded with respect to n.*

DTC linear symmetric networks remain DTC after certain types of interconnection modulation

Theorem (Linear symmetric networks with self-loop modulation)

Consider the class of bilinear networks given by

$$x(k+1) = (A + \alpha v(k) I_n) x(k) + \sum_{j=1}^{m} B_j u_j(k),$$

with $A = A^T$, $|tr(\alpha I_n)| \leq \mu$ *and* $\rho(A) < \sqrt{1 - T_m^{-1}}$, *where* $T_m \triangleq \lceil \frac{n}{m} \rceil - 1$. *Then the reachability Gramian of the network satisfies, for any* $n > m^{-1}\mu^2$,

$$\lambda_{\min}(\mathcal{W}) \leq \frac{(1 - T_m \alpha^2)^{-1}}{1 - \rho^2(A) - T_m^{-1}} \rho^{2T_m}(A).$$

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The term $\frac{(1 - T_m \alpha^2)^{-1}}{1 - \rho^2(A) - T_m^{-1}}$ decreases in n and $\lim_{n \to \infty} \frac{(1 - T_m \alpha^2)^{-1}}{1 - \rho^2(A) - T_m^{-1}} = (1 - \rho^2(A))^{-1}$.

Conclusion and future work

$$(A, F, B): \quad x(k+1) = Ax(k) + \sum_{i=1}^{m}(F_i x(k) + B_i)u_i(k).$$

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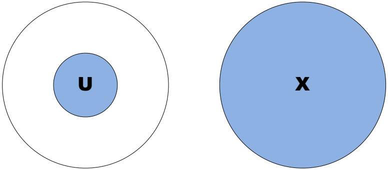

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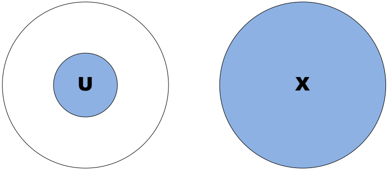

- Actuator selection $\max_{S \subseteq V} \; f(\mathcal{W}(S))$ through $f(\mathcal{W}(S)) \geq \sum_{s \in S} f(\mathcal{W}(s))$ for $f = tr(\cdot), \lambda_{\min}(\cdot), \det(\cdot)$.

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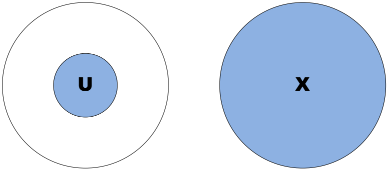

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  Future work 1: optimal selection of actuators.

Conclusion and future work

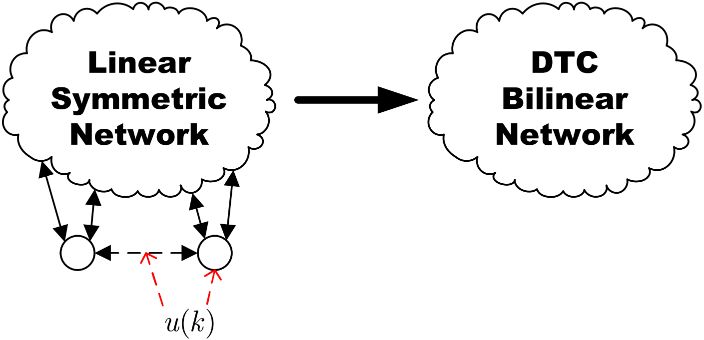
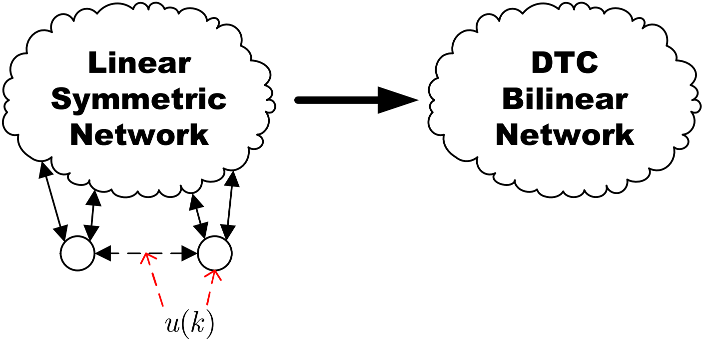


Figure: DTC linear symmetric networks remain DTC after addition of bilinear control.

Conclusion and future work



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Future work 2: whether bilinear control can do quantitatively better.

Thank You