Portfolio Choice with Illiquid Assets*

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October, 2012

Abstract

We investigate the effect of illiquidity on portfolio choice. We model illiquidity as the restriction that the illiquid asset can only be traded at infrequent, stochastic intervals. We find that illiquidity leads to increased and state-dependent risk aversion; a large reduction in the allocation to both liquid and illiquid risky assets and an increased preference for cash; and substantial welfare costs. The features of illiquidity that drive these results are the uncertainty regarding the length of the illiquidity period and the need to finance consumption through liquid assets, not the simple inability to trade. We extend the model to allow for state-dependent probabilities of trade, distinguishing between ‘normal’ times and a ‘liquidity crisis’. We show that the possibility of a liquidity crisis leads to limited arbitrage in normal times and we derive the risk premium associated with the onset of a liquidity crisis.

JEL Classification: G11, G12

Keywords: Asset Allocation, Liquidity, Alternative Assets, Liquidity Crises

*We thank Andrea Eisefeldt, Will Goetzmann, Katya Kartashova, Leonid Kogan, Francis Longstaff, Jun Liu, Chris Mayer, Liang Peng, Eduardo Schwartz, and Dimitri Vayanos, and seminar participants at the Bank of Canada, USC, the USC-UCLA-UCI Finance Day, and the Q-group meetings for comments and helpful discussions. We thank Sarah Clark for providing data on illiquid assets for calibration.
1 Introduction

Many assets are illiquid and cannot be immediately traded by investors when desired. Illiquidity is the result of difficulty in finding a counterparty with which to trade. This difficulty arises because appropriate counterparties often need to have specialized abilities and capital which are in limited supply, as in the case of structured credit products, small equity and bond issues, or large real estate and infrastructure projects. Other illiquid assets, such as private equity and venture capital limited partner investments, have stochastic exit and re-investment timing due to the uncertainty regarding the exit from the underlying investments. Illiquidity may also arise as financial intermediaries receive negative shocks and withdraw from market making. In this paper, we investigate the effects of this illiquidity risk on asset allocation.

We extend the Merton portfolio choice framework to include two risky assets – a liquid and an illiquid security – along with a liquid risk-free asset. The illiquid asset can only be traded contingent on the arrival of a trading opportunity. A key feature of our paper is that these trading opportunities arrive stochastically, modeled as an i.i.d. Poisson process. We can interpret these random trading times as the outcome of an unmodeled search process to find an appropriate counterparty following Diamond (1982) or as the random exit time when capital is returned from a private equity or venture capital investment. Our notion of illiquidity captures the inability to trade the illiquid asset for an uncertain period of time. This uncertainty implies that the investor is exposed to risk that cannot be hedged.

Illiquidity affects the portfolio choice problem in two important ways. First, the investor’s immediate obligations – consumption or payout – can only be financed through liquid wealth, implying that liquid and illiquid wealth are imperfect substitutes. If the investor’s liquid wealth drops to zero, these obligations cannot be met until after the next rebalancing opportunity. As a result, the investor is willing to reduce her allocation to both the liquid and illiquid risky assets in order to minimize the probability that a state with zero liquid wealth – as opposed to zero total wealth – is reached. This concern corresponds to real-world situations where investors or investment funds are insolvent, not because their assets under management
have hit zero, but because they cannot fund their immediate obligations. Illiquidity risk substantially reduces the optimal allocation to the illiquid asset compared to the case when all assets can be continuously rebalanced. A standard calibration indicates that if the expected time between liquidity events is once a year, the investor should cut her investment in the illiquid asset by 33% relative to an otherwise identical but fully liquid asset.

Second, the presence of illiquid assets that cannot be rebalanced until a liquidity event induces endogenous time-varying risk aversion. The share of illiquid securities in the investor’s portfolio can deviate from the optimal position and this ratio becomes a state variable in the investor’s asset allocation problem. Intuitively, the investor’s ability to fund intermediate consumption depends on her liquid wealth, and thus her effective level of risk aversion endogenously increases in the fraction of wealth held in illiquid securities. The inability to trade the illiquid asset also implies that an investor should be prepared for large, skewed changes in the relative value of illiquid to liquid holdings in her portfolio. Illiquid wealth grows on average faster than liquid wealth, even if the liquidity premium is zero, because investors consume out of liquid wealth. As a result, the investor rebalances to a mixture with a lower allocation illiquid assets than her long-run average target when a liquidity event arrives.

We extend the baseline model in three ways to explore the determinants of the costs of illiquidity. First, illiquidity is costly because the investor’s portfolio deviates from the optimal allocation and because there is a risk that intermediate consumption needs will not be funded. To disentangle these two components of the cost, we compare welfare and portfolio policies between our setting and a version where the investor has no intermediate consumption needs and cares only about the value of her total portfolio at a future uncertain date. We find the inability to fund intermediate consumption needs is substantially more important than deviations from optimal portfolio allocation. Absent the motive to smooth intermediate consumption, the effect of illiquidity on portfolio policies and welfare is minimal.

Second, we explore whether the cost of illiquidity is due the inability to trade for a certain period or instead to uncertainty in the lengths of the non-trading periods. We compare welfare
and portfolio policies between our setting and a version of our model with a deterministic rebalancing interval, similar to the models of Longstaff (2001, 2009) and Schwartz and Tebaldi (2010). We find that the deterministic version leads to qualitatively similar but quantitatively weaker results. Both the allocation to the risky assets and investor welfare are significantly lower in the stochastic case. Furthermore, varying the length of the illiquidity period has a major impact on welfare and optimal policies in the stochastic case, but only a minor effect in the deterministic case. Hence, our results suggest that the uncertainty associated with the next opportunity to trade is a major component of the cost of illiquidity.

Having established that the main effects are driven by uncertainty in trading times and the inability to smooth intermediate consumption, we turn to the manner in which consumption is smoothed. Agents dislike illiquidity because it disrupts the ability to smooth consumption across time – governed by the elasticity of intertemporal substitution parameter (EIS) – and across states – controlled by the investor’s risk aversion parameter. Using the recursive preference specification of Duffie and Epstein (1992), we find that the welfare cost of illiquidity is highest for agents that unwilling to substitute across time (low EIS) but are willing to substitute across states (low risk aversion). We find that the amount of investment in the illiquid asset is primarily dictated by risk aversion. Then, since a very risk averse agent is unlikely to invest in the illiquid risky asset anyway, she faces lower welfare costs of illiquidity. In contrast, a low EIS investor dislikes states of the world where illiquid wealth, and hence current consumption, is low relative to total wealth and future consumption. Since the EIS has a quantitatively minor impact on the allocation to the illiquid asset, lowering the EIS while holding risk aversion constant increases the welfare cost to the investor.

We extend the model to study random episodes of illiquidity, by considering two distinct regimes. The first regime represents ‘normal’ times, where all assets are fully liquid. The second regime represents a dry up of liquidity – a liquidity crisis – where now one of the two assets becomes illiquid and can only be traded at infrequent intervals.\footnote{For instance, during the financial crisis over 2008-2009, many markets that are liquid during normal times}
model is thus applicable to a wide selection of assets that normally liquid, but are subject to occasional market freezes, which Brunnermeier (2009), Gorton (2010), Tirole (2011), and others have highlighted as a stylized fact of the 2008-2009 financial crisis.

The possibility of a liquidity crisis leads to limited arbitrage in normal times. The investor is averse to entering ‘arbitrage’ trades, defined as trading in two perfectly correlated securities with different Sharpe ratios. The hidden cost of these arbitrage trades is that, even though both securities are currently fully liquid, the investor may be unable to continuously trade one of them in the event of a crisis. This potential inability to rebalance introduces risk into the trade, especially since the investor is able to finance consumption only from one leg of the arbitrage trade during a crisis. As a result, the investor will underinvest in apparent arbitrage opportunities, even when realizing the arbitrage involves no short positions.

We use the extended model to quantify the risk premium associated with a systematic liquidity crisis. This ‘liquidity crisis risk premium’ refers to the risk premium of an Arrow-Debreu security that pays off at the onset of a liquidity crisis. This notion of risk premium is distinct from the ‘liquidity premium’, defined as the price discount of an illiquid security. We find that, for typical parameter values, the investor is be willing to pay an annual premium of 0.5% to 2% over the actuarial probability, in order to receive liquid funds at the onset of a crisis.

Our analysis falls into a large literature dealing with portfolio choice with frictions. Our work is related to portfolio choice models where investors cannot trade continuously (see e.g. Kahl, Liu and Longstaff, 2003; Koren and Szeidl, 2003; Dai, Li, Liu, and Wang, 2010; Longstaff, 2009; de Roon, Guo and ter Horst, 2009, and Schwartz and Tebaldi, 2010). In contrast to these papers, the length of the illiquidity period in our setting is stochastic rather completely froze. Examples include the commercial paper market (Anderson and Gascon, 2009), the repo market (Gorton and Metrick, 2009), residential and commercial mortgage-backed securities (Gorton, 2009; Dwyer and Tkac, 2009; Acharya and Schnabl, 2010), and structured credit (Brunnermeier, 2009). This is not simply a question of a seller reducing prices to a level where a buyer is willing to step in. As Tirole (2011) and Krishnamurthy, Nagel, and Orlov (2011) comment, there were no bids, at any price, representing “buyers’ strikes” in certain markets where whole classes of investors simply exited markets.
than deterministic.\textsuperscript{2} This unhedgeable illiquidity risk is a primary determinant of the cost of illiquidity. In our numerical solution, we find that the welfare cost of illiquidity is substantially greater if the length of the illiquidity period is random versus known in advance.

Our specification of random opportunities to trade puts our work falls squarely in the long tradition of search models started by Diamond (1982), where agents need to wait until the arrival of a Poisson event to transact. A number of authors have used this search technology to consider the impact of illiquidity (search) frictions in various over-the-counter markets. Duffie, Garleanu, and Pedersen (2005, 2007) consider only risk-neutral and CARA utility cases and restrict asset holdings at two levels. In Vayanos and Weill (2008), agents can only go long or short one unit of the risky asset. Garleanu (2009) and Lagos and Rocheteau (2009) allow for unrestricted portfolio choice, but Garleanu considers only CARA utility and Lagos and Rocheteau focus on showing the existence of equilibrium with search frictions. We find that the wealth effects associated with CRRA preferences are an important component of the cost of illiquidity risk.

Our work is related to models with transaction costs (e.g. Constantinides, 1986; Vayanos, 1998; Lo, Mamaysky and Wang, 2004). This literature views illiquidity as an explicit transaction cost which investors pay when rebalancing. Our work is similar in the sense that in the presence of fixed transaction costs the investor is unwilling to rebalance continuously. However, in our setting the investor is unable to trade, even at a cost. In the transaction costs model, the shadow cost of illiquidity is bounded by the level of transaction costs. In contrast, in our paper the shadow cost of illiquidity is significant because it is unbounded; liquidity cannot be generated, e.g. a counter-party found, simply by paying a cost. Longstaff’s (2001) approach where investors can trade continuously, but with only bounded variation, is close in

\textsuperscript{2}To the best of our knowledge, the only other work that features random opportunities to trade is Rogers and Zand (2002), who solve a model with random trading opportunities and no liquid risky asset using asymptotic expansions near the Merton benchmark $1/\lambda \to 0$. However, Rogers and Zand (2002) include no formal proof that these expansions are valid and the behavior of the model as $1/\lambda \to 0$ can be very different from the Merton benchmark. In particular, even as $1/\lambda \to 0$, the investor is still trading on a set of measure zero, hence would never take a short position in either liquid or illiquid wealth. In contrast to Rogers and Zand (2002), we solve the ODEs characterizing the investors’ problem numerically.
spirit to the literature on transactions costs, since the illiquid asset is partially marketable at all times.\footnote{Since our setup is an incomplete market, our work is also related to the literature on unhedgeable human capital risk (see e.g., Heaton and Lucas, 1996; Koo, 1998; Santos and Veronesi, 2006). In this literature, part of the investor’s total wealth cannot be traded, which introduces a motive to hedge using the set of tradeable securities. Our investor also hedges illiquidity risk by changing the allocation to liquid assets – even when the correlation between liquid and illiquid asset returns is zero. An important distinction is that our illiquid asset is infrequently traded, unlike human capital which is never traded.}

Last, our paper is related to the “endowment model” of asset allocation for institutional long-term investors made popular by David Swensen’s work, *Pioneering Portfolio Management*, in 2000. Swensen’s thesis is that highly illiquid markets, such as private equity and venture capital, have large potential payoffs to research and management skill, which are not competed away because most managers have short horizons. Leaving aside whether there are superior risk-adjusted returns in alternative investments, the endowment model does not consider the illiquidity of these investments. In addition to economically characterizing the impact of illiquidity risk on portfolio choice, our certainty equivalent calculations are quantitatively useful for investors to take into account the effect of illiquidity on risk-return trade-offs.

## 2 Illiquidity in Asset Markets

Here, we motivate and quantify our notion of illiquidity based on a number of stylized facts about asset markets. In particular, the long times between trades, low turnover, and the difficulty of finding counterparties in over-the-counter markets imply investors need to wait for an indeterminate period before rebalancing their portfolio between liquid and illiquid investments. Connecting our model to the data, the typical holding period for an illiquid asset and the turnover in institutions’ illiquid asset portfolios provide guidance for calibrating the average time between trading opportunities in our model.

As we see in Table 1, outside ‘plain vanilla’ fixed income and public equities, many assets markets are characterized by pronounced illiquidity where there can be long intervals between trades. Even within fixed income and public equities, there are sub-asset classes that are
illiquid. While the public equity market has a turnover well over 100%, corporate bonds have a turnover around 25-35%. The average municipal bond trades only twice per year and the entire market has a turnover of less than 10% per year. Transactions times for many over-the-counter equities, such as those traded on the pinksheet or NASDAQ BB markets, are often longer than a week with a turnover of approximately 35%. In real estate markets, the typical holding period is 4-5 years for residential housing and 8-11 years for institutional real estate. Institutional infrastructure horizons are typically 50 years or longer, and there can be periods of 40-70 years between sales for investments in art. Last, typical holding periods for venture capital and private equity portfolios are 3 to 10 years.

These asset markets are large and often rival the size of the public equity market. For instance, the market capitalization of NYSE and NASDAQ is approximately $17 trillion. The estimated size of the residential real estate market is $16 trillion and the estimated size of the (direct) institutional real estate market is $9 trillion. Further, the share of illiquid assets in many investors’ portfolios is very large. Kaplan and Violante (2011) show that individuals hold the majority of their net wealth in illiquid assets, with 91% and 81% of households’

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4The time between transactions may not be a true measure of liquidity for an investor desiring to trade, but investors in many of these markets face long waits to find appropriate counterparties after the decision has been made to sell or buy assets, particularly for assets that have very idiosyncratic features in private equity, real estate, infrastructure, and fine art markets. Levitt and Syverson (2008) report, for example, a typical time to sale between 110-135 days after initial listing of a house. The standard deviation of the time to sale is even larger than the mean and Levitt and Syverson note that some houses never sell.

5The notion of a stochastic liquidity event is valid even when a fixed horizon is stated for a given investment vehicle, like a 10-year horizon for private equity. Even though the investment horizon is nominally fixed, in many cases partnerships return investor’s money prior to the partnership’s formal 10-year end. Further, these times are random. For example, in private equity, the median investment duration is 4 years with 16% returned before 2 years and 26% returned after 6 years. (see, e.g. Lopez-de-Silanes, Phalippou, and Gottschalg (2010)). The turnover from trade of private equity investments on the secondary market is much lower. While data on private equity portfolio turnover is not typically reported. Kensington, a Canadian private equity fund, reports a 2% turnover in 2008. Alpinvest, a large private equity fund-of-funds reports flows that imply a turnover of approximately 6%. This compares with typical turnover of over 70% for mutual funds (see Wermers, 2000).

6NYSE and NASDAQ market capitalizations are approximately $12 trillion and $5 trillion as of July 2012 from nyxdata.com and nasdaqtrader.com. The estimated size of the U.S. residential real estate market is at December 2011 and is estimated by Keely et al. (2012), down from a peak of $23 trillion in 2006. The estimate of the U.S. institutional real estate market is by Florance et al. (2010), with the institutional real estate market losing $4 trillion from 2006 to 2010. The direct real estate market dwarfs the traded REIT market, with the FTSE NAREIT All Equity REITs Index having a total market capitalization of approximately $500 billion at the end of June 2012.
net portfolios tied up in illiquid positions, mostly housing, taking median and mean values, respectively. High net worth individuals in the U.S. allocate 10% of their portfolios to “treasure” assets like fine art, jewelry, and the share of treasure assets rises to almost 20% in other countries. Further, the share of illiquid assets in institutions’ portfolios has dramatically increased over the last 20 years.

Our framework is especially relevant to the institutional setting. The largest endowments hold significantly more illiquid assets in their portfolios, with endowments over $1 billion holding 60% in alternatives. For example, in 2008, Harvard University held close to two-thirds of its portfolio in illiquid assets. Its endowment shrank from $37 billion to $26 billion from June 2008 to June 2009 due to the financial crisis. Since the endowment contributed over one-third of all revenues to the university (for many schools like Arts and Sciences and Radcliffe the reliance on endowment distributions was even higher at 52% and 83%, respectively), Harvard University experienced severe liquidity problems. Many illiquid assets could not be sold to meet cash needs. Harvard University was a very high profile example of an institution that experienced the effects of illiquidity risk.

3 Baseline Model

In this section we describe the setup of our baseline model.

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8Pension funds increased their holdings in illiquid (“other”) asset classes from 5% in 1995 to close to 20% in 2010, as reported in the “Global Pension Asset Study 2011” by Towers Watson. Data from the National Association of College and University Business Officers (NACUBO) show that the (dollar-weighted) average share of illiquid “alternatives” in university endowment portfolios rose from 25% in 2002 to 52% in 2010.

9Harvard University did not hold a liability-matching portfolio in the sense of Merton (1993) and it also needed cash to meet substantial collateral calls on (negative) swap positions. (See Munk, N., “Rich Harvard, Poor Harvard,” Vanity Fair, August 2009 and Ang, A., “Liquidating Harvard,” Columbia Business School case.) The model we present in Section 3 is the traditional Merton (1969, 1971) formulation that does not have liabilities. Taking into account liabilities only makes the effects of illiquidity risk on asset allocation more severe, as many illiquid assets often do not generate cashflows prior to termination that match the liabilities of typical institutions. Harvard University solved its liquidity problems primarily by stopping new capital projects and issuing debt. In the process, it more than doubled its leverage ratio from 9% to 20%. Note that our model also permits shorting the risk-free asset.
3.1 Information

The information structure obeys standard technical assumptions. Specifically, there exists a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ supporting the vector of two independent Brownian motions $Z_t = (Z^1_t, Z^2_t)$ and an independent Poisson process $(N_t)$. $\mathcal{P}$ is the corresponding measure and $\mathcal{F}$ is a right-continuous increasing filtration generated by $Z \times N$.

3.2 Assets

There are three assets in the economy: a risk-free bond $B$, a liquid risky asset $S$, and an illiquid risky asset $P$. The riskless bond $B$ appreciates at a constant rate $r$:

$$dB_t = r B_t \, dt$$  \hspace{1cm} (1)

The second asset $S$ is a liquid risky asset whose price follows a geometric Brownian motion with drift $\mu$ and volatility $\sigma$:

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dZ^1_t.$$  \hspace{1cm} (2)

The first two assets are liquid and holdings can be rebalanced continuously.

The third asset $P$ is an illiquid risky asset, for which the price process evolves according to a geometric Brownian motion with drift $\nu$ and volatility $\psi$:

$$\frac{dP_t}{P_t} = \nu \, dt + \psi \rho \, dZ^1_t + \psi \sqrt{1 - \rho^2} \, dZ^2_t,$$  \hspace{1cm} (3)

where $\rho$ captures the correlation between the returns on the two risky assets.

The illiquid asset $P$ differs from the first two assets $B$ and $S$ in one important ways. In particular, the illiquid asset $P$ can only be traded at stochastic times $\tau$, which coincide with the arrival of a Poisson process with intensity $\lambda$. Thus, the expected time between rebalancing
events is $1/\lambda$.\textsuperscript{10} When a trading opportunity arrives, the investor is able to costlessly rebalance her holdings of the illiquid asset up to any amount.\textsuperscript{11} Here, $P_t$ reflects the fundamental value of the illiquid asset, which varies randomly irrespective of whether trading the asset is possible.

In addition, the illiquid asset $P$ cannot be pledged as collateral. Investors can issue non-state contingent debt by taking a short position in the riskless bond $B$; however, they cannot issue risky debt using the illiquid asset as collateral. If investors were allowed to do so, they could convert the illiquid asset into liquid wealth and thus implicitly circumvent the illiquidity friction.\textsuperscript{12} Our assumption is motivated by the difficulty of finding a counterparty who is willing to lend cash using illiquid assets as collateral. For instance, Krishnamurthy, Nagel, and Orlov (2011) find evidence suggesting that money market mutual funds, which are the main providers of repo financing, were unwilling to accept private asset-backed securities as collateral between the third quarter of 2008 and the third quarter of 2009. Even when illiquid assets like real estate, private equity, and even art can be used as collateral, an investor cannot borrow an amount equal to the whole value of the illiquid asset position and thus our decomposition into liquid and illiquid assets is still valid.

Finally, we assume the standard discount rate restriction as in the Merton one-asset model

$$\beta > (1 - \gamma) r + \frac{1 - \gamma}{2\gamma} \left( \frac{\mu - r}{\sigma} \right)^2.$$ \textsuperscript{(4)}

\textsuperscript{10}Our specification of illiquidity risk is that an asset becomes more illiquid as $\lambda \to 0$. In the extreme case where $\lambda = 0$, the illiquid asset can never be traded. When $\lambda \to \infty$, the model is similar to the standard Merton model where trading is continuous.

\textsuperscript{11}Transaction costs exacerbate the effects of illiquidity in our model. Adding transactions costs to the model results in an interval of non-rebalancing when the liquidity event arrives, as shown by Constantinides (1986), Liu (2004), and others.

\textsuperscript{12}Alternatively, we could re-interpret $P$ as the fraction of illiquid wealth that cannot be collateralized. In the case of real estate, we could interpret the illiquid asset $P$ as the fraction of the value of the property that cannot be used as collateral against a mortgage or a home equity line. This interpretation assumes that the amount that the asset can be collateralized does not vary over time and that the constraint is always binding. We could extend the model to allow the investor to endogenously choose the amount of collateralized borrowing every period, up to a limit. This model is equivalent to a hybrid model of infrequent trading and transaction costs, with similar qualitative effects.
and that the illiquid asset has at least as high a Sharpe ratio as the liquid asset

\[
\frac{\nu - r}{\psi} \geq \frac{\mu - r}{\sigma}.
\] (5)

### 3.3 Investor

The investor has CRRA utility over sequences of consumption, \(C_t\), given by:

\[
\mathbb{E} \left[ \int_0^{\infty} e^{-\beta t} U(C_t) dt \right],
\] (6)

where \(\beta\) is the subjective discount factor and \(U(C)\) is given by

\[
U(C) = \begin{cases} 
C^{1-\gamma} & \text{if } \gamma > 1 \\
\frac{1}{1-\gamma} \ln(C) & \text{if } \gamma = 1.
\end{cases}
\] (7)

We focus on the case \(\gamma > 1\) and present the results for log utility, \(\gamma = 1\), in the online appendix.

We take an infinite horizon for the investor because any effects of illiquidity and illiquidity risk are magnified with finite horizons. For example, if opportunities to trade arise every 10 years, on average, then an investor with a one-year horizon views the illiquid asset as a very unattractive asset. Thus, the portfolio weights, effects on consumption policies, and certainty equivalent compensations for bearing illiquidity risk should all be viewed as conservative lower bounds for finite-horizon investors.

The agent’s wealth is comprised of two components, liquid and illiquid wealth. The first includes the amount invested in the liquid risky asset and the risk-free asset. The second, which equals the amount invested in the illiquid asset, cannot be immediately consumed or converted into liquid wealth. The joint evolution of the investor’s liquid, \(W_t\), and illiquid
wealth, \(X_t\), is given by:

\[
\frac{dW_t}{W_t} = (r + (\mu - r)\theta_t - c_t) dt + \theta_t\sigma dZ^1_t - \frac{dI_t}{W_t} \\
\frac{dX_t}{X_t} = \nu dt + \psi\rho dZ^1_t + \psi\sqrt{1 - \rho^2} dZ^2_t + \frac{dI_t}{X_t}.
\]

(8)

(9)

The agent invests a fraction \(\theta\) of her liquid wealth into the liquid risky asset, while the remainder \((1 - \theta)\) is invested in the bond. The agent consumes out of liquid wealth, so liquid wealth decreases at rate \(c_t = C_t/W_t\). When a trading opportunity arrives, the agent can transfer an amount \(dI_t\) from her liquid wealth to the illiquid asset.

Following Dybvig and Huang (1988) and Cox and Huang (1989), we restrict the set of admissible trading strategies, \(\theta\), to those that satisfy the standard integrability conditions.

Our first result is that trading risk eliminates any willingness by the investor either to short the illiquid asset or to net borrow in liquid wealth to fund long purchases of the illiquid asset.

**Proposition 1** Any optimal policies will have \(W > 0\) and \(X \geq 0\) a.s.

Thus, without loss of generality, we restrict our attention to solutions with \(W_t > 0\) and \(X_t \geq 0\).

### 4 Solution to the Baseline Problem

Because markets are not dynamically complete, we use dynamic programming techniques to solve the investor’s problem. First, we establish some basic properties of the solution. Then, we compute the investor’s value function and optimal portfolio and consumption policies.

#### 4.1 Value Function

The agent’s value function is equal to the discounted present value of her utility flow,

\[
F(W_t, X_t) = \max_{\{\theta, I, c\}} E_t \left[ \int_t^\infty e^{-\beta(s-t)} U(C_s) ds \right].
\]

(10)
Our first step is to establish bounds on (10). The trader’s value function must be bounded below by the problem in which the illiquid asset does not exist, and the value function must be bounded above by the problem in which the entire portfolio can be continuously rebalanced. We refer to these as the Merton (1969, 1971) one-stock and two-stock problems, respectively. Hence, there exist constants $K_{M1}$ and $K_{M2}$ such that

$$K_{M1} W^{1-\gamma} \leq F(W, X) \leq K_{M2} (W + X)^{1-\gamma} \leq 0. \quad (11)$$

Since the Merton one-asset value function exists (4), our value function is bounded between the one-asset solution and the two-asset solution.

The utility function is homothetic and the return processes have constant moments, hence it must be the case that $F$ is homogeneous of degree $1 - \gamma$

$$F(W, X) = (W + X)^{1-\gamma} H(\xi), \quad \text{where} \quad \xi = \frac{X}{X+W}. \quad (12)$$

The investor’s value function can be therefore represented as a power function of total wealth times a function $H(\xi)$ of her portfolio composition.

The next step involves characterizing the value function at the instant when the agent can rebalance between her liquid and illiquid wealth. When the Poisson process hits and the agent rebalances her portfolio, the value function may discretely jump.\(^\text{13}\) Denote the new, higher, value function after rebalancing occurs as $F^*$, so that the total amount of the jump is $F^* - F$. At the Poisson arrival, the agent is free to make changes to her entire portfolio, and thus we

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\(^{13}\)The change in total wealth when a liquidity event occurs means there are some similarities with models with jump component in prices, as in Liu, Longstaff and Pan (2003). However, a key difference between our setting and the jump-diffusion setting of Liu, Longstaff and Pan (2003) is that in our model of illiquidity, risk aversion is time varying and depends on the share of illiquid assets in the portfolio $\xi$. Since portfolios drift away from optimal diversification, our model features variation in investment and consumption policies even when returns are i.i.d.
have that

\[ F^*(W_t, X_t) = \max_{I \in [-X_t, W_t]} F(W_t - I, X_t + I). \] (13)

Since \( F^* \) must also be homogeneous of degree \( 1 - \gamma \), there exists a function \( H^* \) such that

\[ F^* = (W + X)^{1-\gamma} H^* \left( \frac{X}{X+W} \right). \]

In addition, since rebalancing the illiquid asset is costless when possible, \( H^* \) is a constant function. The homogeneity of the value function implies that when a trading opportunity arrives, the investor rebalances her portfolio so that the fraction of illiquid to total wealth equals \( \xi^* = \arg \max H(\xi) \), and \( H^* = H(\xi^*) \).

The investor’s portfolio and consumption problem can be defined as

**Problem 1 (Baseline)** The investor performs the maximization in (10), subject to the two intertemporal budget constraints (8) and (9), with re-balancing \( (dI_t \neq 0) \) only when the Poisson process \( N^\lambda_t \) jumps.

The following proposition characterizes the solution to the investor’s problem

**Proposition 2 (Baseline)** The solution to Problem 1 is characterized the function \( H(\xi) \) and constants \( H^* \) and \( \xi^* \) that satisfy

\[
0 = \max_{c, \theta} \left[ \frac{1}{1-\gamma} c^{1-\gamma} - \beta H(\xi) + \lambda (H^* - H(\xi)) + H(\xi) A(\xi, c, \theta) + \frac{\partial H(\xi)}{\partial \xi} B(\xi, c, \theta) \right. \\
\left. + \frac{1}{2} \frac{\partial^2 H(\xi)}{\partial \xi^2} C(\xi, c, \theta) \right].
\] (14)

where the functions \( A, B, \) and \( C \) are given by

\[
A(\xi, c, \theta) \equiv (1 - \gamma) \left( r + (1 - \xi)(\mu - r)\theta - c \right) + \xi (\nu - r) \\
- \frac{1}{2} \gamma \left( \xi^2 \psi^2 + (1 - \xi)^2 \sigma^2 \theta^2 + 2 \xi (1 - \xi) \psi \rho \theta \right) \] (15)

\[
B(\xi, c, \theta) \equiv \xi (1 - \xi)(\nu - (r + (\mu - r)\theta - c)) + \gamma \psi \rho \sigma (2 \xi - 1) + \gamma \theta^2 \sigma^2 (1 - \xi) + \gamma \psi^2 \xi \] (16)

\[
C(\xi, c, \theta) \equiv \xi^2 (1 - \xi)^2 \left( \theta^2 \sigma^2 + \psi^2 - 2 \psi \theta \rho \sigma \right) \] (17)
and

\[ H^* = \max_\xi H(\xi) \]  \hspace{1cm} (18)

\[ \xi^* = \arg \max_\xi H(\xi) \]  \hspace{1cm} (19)

When a trading opportunity occurs at time \( \tau \), the trader selects \( I_\tau \) so that \( \frac{X_\tau}{X_\tau + W_\tau} = \xi^* \).

**Lemma 3** \( H(\xi) \) is concave.

The investor’s value function is comprised of two parts. The first part \( (W + X)^{1-\gamma} \) captures the effect of total wealth on the continuation utility. The second component \( H(\xi) \) captures the effect of wealth composition between liquid and illiquid wealth. The function \( H \) is concave and maximized at \( H^* \); deviations from this allocation reduce welfare for two reasons. First, there is the standard effect from lack of optimal diversification. Second, there is an asymmetric effect arising from the fact that consumption is funded by liquid wealth only. In the section below, we explore the second effect in more detail.

### 4.2 Imperfect Substitutability of Liquid and Illiquid Wealth

Here, we discuss some properties of the solution to provide intuition for the results. We will begin by emphasizing the way our model changes the basic Merton continuous trading intuitions and conclude by describing how those changes depend on the consumption smoothing properties of CRRA utility, as opposed to the CARA and risk-neutral utilities mostly used in prior work.

The solution to our problem differs from the solution to the Merton setup because liquid and illiquid wealth are imperfect substitutes. This non-substitutability is particularly acute when the investor’s portfolio is comprised mostly of illiquid assets. To understand the implication of this imperfect substitutability, we first examine the behavior of the solution at the limits \( X \to \infty (\xi \approx 1) \) and \( X \to 0 (\xi \approx 0) \); as we show later, the behavior of the solution at
the extremes sheds light into the effect of illiquid holdings on the investor’s optimal policies in the interior values of $\xi$.

**Lemma 4** At the boundaries, the value function satisfies

$$\lim_{X \to \infty} F(W, X) = K_\infty W^{1-\gamma}, \quad \lim_{W \to \infty} F(W, X) = 0$$  \tag{20}$$

and

$$\lim_{X \to 0} F(W, X) = K_0, \quad \lim_{W \to 0} F(W, X) = -\infty$$  \tag{21}$$

where the constants $K_0 < K_\infty < K_{M2} < 0$ solve

$$0 = -\beta + \gamma ((1-\gamma)K_0)^{-\frac{1}{\gamma}} + (1-\gamma)r + \frac{1}{2}(1-\gamma)\frac{(\mu - r)^2}{\gamma \sigma^2} + \lambda \left(\frac{H^*}{K_0} - 1\right)$$  \tag{22}$$

$$K_\infty = \frac{1}{1-\gamma} \left[\frac{1}{\gamma} \left(\beta + \lambda + (\gamma - 1)r + \frac{1}{2}(\gamma - 1)\gamma \left(\frac{\mu - r}{\gamma \sigma}\right)^2\right)\right]^{-\gamma}. \tag{23}$$

Lemma 4 demonstrates the imperfect substitutability in two ways. First, the investor cannot achieve bliss even with an unboundedly large endowment of illiquid wealth, since $\lim_{X \to \infty} F(X, W) < 0$. In contrast to liquid wealth, illiquid wealth cannot be immediately transformed into consumption. The investor needs to wait for an opportunity to trade the illiquid security, and this random delay bounds her welfare away from bliss. Second, the investor can reach states with negative infinite utility, if her liquid wealth is low enough, in addition to total wealth. This last result follows from the fact that only liquid wealth can be transformed into consumption immediately.

Next, to obtain intuition about the cost of illiquidity, consider the introduction of a fictitious market, that allows the investor to exchange 1 unit for illiquid wealth for $q$ units of liquid wealth. Between normal rebalancing dates, the investor would be indifferent in participating
in this fictitious market as long as

\[ q = \frac{F_X}{F_W}. \tag{24} \]

When the investor has the opportunity to rebalance, then \( q = 1 \). Between rebalancing dates, the relative price \( q \) differs from one, depending on whether the investor has too much, or too little illiquid wealth \( X \) relative to her desired allocation. The following lemma characterizes the behavior of the relative values of the investor’s portfolio as the value of illiquid holdings becomes large:

**Lemma 5** When illiquid holdings are large, the ratio of the value of illiquid to liquid holdings tends to zero

\[ \lim_{X \to \infty} \frac{qX}{W} = 0. \tag{25} \]

Lemma 5 shows that the relative value of illiquid to liquid wealth becomes arbitrarily small as the investor’s allocation to illiquid wealth increases. Specifically, as the investor’s illiquid holdings become large \( X \to \infty \), the price the investor attaches to her illiquid wealth \( q \) tends to zero sufficiently fast, so that the relative value of her entire illiquid portfolio (25) tends to zero.\(^{14}\) A direct consequence of Lemma 5 is that, as illiquid holdings become large, the investor’s marginal utility of consumption is mainly affected by changes in liquid, rather than illiquid wealth:

**Lemma 6** The elasticity of substitution of the marginal utility of consumption between liquid and illiquid wealth tends to zero

\[ \lim_{X \to \infty} \frac{X \frac{\partial}{\partial X} U’(C)}{W \frac{\partial}{\partial W} U’(C)} = 0. \tag{26} \]

\(^{14}\)This behavior is in contrast with the standard Merton problem in which the investor can freely rebalance. In this case, the relative price is constant, implying that \( \lim_{X \to \infty} \frac{F_XX}{FWW} = \infty \).

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Lemma 6 has strong implications for portfolio allocation. In particular, even if changes in illiquid and liquid wealth are correlated, in the limit the investor treats them as separate gambles. Hence, the investor is reluctant to choose her liquid portfolio allocation to offset the value of her illiquid position. As we show in Section 4.4 below, this imperfect substitutability implies that hedging demands will be zero when illiquid wealth becomes large.

The results of this section illustrate that, as a direct consequence of the inability to trade continuously, the investor treats her liquid and illiquid holdings as imperfect substitutes. In the limit where her illiquid wealth becomes large, the elasticity of substitution between them tends to zero. To gain further intuition about why liquid and illiquid wealth are imperfect substitutes, consider the following approximation to the value function

\[
F(X_t, W_t) = E_t \left[ \int_t^\tau e^{-\beta(s-t)}U(C_s)ds + e^{-\beta(\tau-t)}F^*(W_\tau, X_\tau) \right] 
\approx K_\infty W_t^{1-\gamma} + (K_0 - K_\infty) (W_t + X_t)^{1-\gamma} 
\]

(27)

The investor’s continuation value can be decomposed into two parts: i) the utility she derives from consumption until the next rebalancing date \(\tau\); and ii) her continuation value \(F^*(W_\tau, X_\tau)\) thereafter. These two parts correspond approximately to the two components of equation (27).\(^{15}\) The first component in equation (27) corresponds to the part of the value function capturing the utility of consumption until the next trading day. This part depends only on liquid wealth, \(W\), because the investor can only instantaneously consume out of her liquid holdings. The second term in equation (27) corresponds to the investor’s continuation value immediately after the next trading time. At that instant, the investor can freely convert her illiquid holdings into liquid assets and vice versa.

Equation (27) sheds light into the non-substitution results in Lemmas 4, 5, and 6. In

\(^{15}\)In general, the weights of the exact solution on these two components are not constant but depend on \(\xi\). The approximation is exact for \(X = 0\) and \(X = \infty\) and reasonably accurate for intermediate values using our parameters. This approximation generates a mean squared relative error – weighted by the invariant distribution of \(X/(X + W)\) – of less that 1%. While this is a good approximation for the level of the value function, it cannot necessarily be used to generate good approximations of the optimal policies.
particular, illiquid wealth affects the level of the value function only through the continuation value $F^*$ at the next trading time $t = \tau$. In contrast, liquid wealth can fund consumption both before and after $\tau$. Hence, liquid and illiquid wealth are not perfect substitutes. When the illiquid endowment is large $X \gg W$, this non-substitutability is particularly acute, since variation in liquid wealth becomes unimportant for long-run consumption, and the value function becomes separable in $X$ and $W$. In this case, liquid wealth $W$ is only used to fund immediate consumption, while illiquid wealth is used to fund future consumption. Since consumption preferences are time separable, so is the value function. As a consequence, when $X$ is large, the hedging demand disappears and the correlation between the liquid and illiquid asset returns does not matter for portfolio allocation.

The approximation (27) also makes it clear why the agent cannot achieve bliss through an increasing allocation of the illiquid asset, as we see in Equation 20. The first term in equation (27) bounds the value function away from zero for large values of $X$: the illiquid asset cannot be used to fund immediate consumption and illiquid wealth is inaccessible until after the first trading time. In contrast, the value function is not bounded away from zero for large values of $W$ because liquid wealth can be used for consumption today.

The approximation demonstrates how illiquidity creates additional high-marginal-utility states. In contrast to the standard Merton model, the investor’s marginal value of wealth in our model is high in two types of states: states where total wealth is low and states where liquid wealth is low. Even if the investor has substantial total wealth, if her liquid wealth is low, she cannot fund immediate consumption, leading to high marginal utility. As a result, the investor is concerned with smoothing not only her total wealth $W + X$, but also her liquid wealth $W$. These concerns lead to underinvestment in the illiquid and the liquid risky assets.

### 4.3 Parameter Values

We select our parameter values so that the liquid asset can be interpreted as an investment in the aggregate stock market. We set the parameters of the liquid equity return to be $\mu = 0.12$, 
\( \sigma = 0.15 \), and set the risk-free rate to be \( r = 0.04 \). Table 2 shows that this set of parameters closely matches the performance of the S&P500 before the financial crisis. The mean of the S&P500 including 2008-2010 falls to 0.10 and slightly more volatile, at 0.18, but our calibrated values are still close to these estimated values. We work mostly with the risk aversion case \( \gamma = 6 \), which for an investor allocating money between only the S&P500 and the risk-free asset paying \( r = 0.04 \) produces an equity holding close to a classic 60% equity, 40% risk-free bond portfolio used by many institutional investors.

Table 2 shows that the reported returns on a composite illiquid investment in private equity, buyout, and venture capital has similar characteristics to equity. For example, over the full sample (1981-2010), the mean log return on the illiquid investment is 0.11 with a volatility of 0.17. This is close to the S&P500 mean and volatility of 0.10 and 0.18, respectively, over that period. Table 2 shows that the returns on liquid and illiquid investments are even closer in terms of means and volatilities before the financial crisis. This suggests setting the parameters of the illiquid asset, \( \nu \) and \( \psi \), to be the same as the parameters on the S&P500.

For most of our analysis, we take a conservative approach and set \( \nu = 0.12 \) and \( \psi = 0.15 \) to be the same mean and volatility, respectively, as the liquid asset. This has the advantage of isolating the effects of illiquidity rather than obtaining results due to the higher Sharpe ratios of the illiquid assets. Further, even for individual funds this assumption is not unrealistic, at least for some illiquid asset classes.\(^{16}\) These parameters also mean that our illiquid asset can be interpreted as any composite investment with the same sharpe ratio as public equities, for example a composite fixed income investment. Further, to isolate the effect of illiquidity, in the baseline case we assume that the two risky assets are uncorrelated, \( \rho = 0 \); we explore the effect of correlation by varying \( \rho \) between 0 and 1.

We take a baseline case of \( \lambda = 1 \), or that the average waiting time to rebalance the illiquid asset is one year. As mentioned before, individual private equity, buyout, and venture capital funds can have average investment durations of approximately 4 years, which corresponds to

\(^{16}\)Driessen, Lin and Phalippou (2008) and Gottschalg and Phalippou (2009), for example, estimate private equity fund alphas, with respect to equity market indexes, close to zero.
\( \lambda = 1/4 \). As Section 2 shows, \( \lambda = 1/10 \) is an appropriate horizon for a single large real estate investment by institutions. Since \( \lambda \) is an important parameter, we take special care to show the portfolio and consumption implications for a broad range of \( \lambda \). Fortunately, the economics behind the solution are immune to the particular parameter values chosen.

### 4.4 Optimal Portfolio Policies

In this section we characterize the investor’s optimal asset allocation and consumption policies. Even though the investment opportunity set is constant, the optimal policies vary over time as a function of the amount of illiquid assets held in the investor’s portfolio.

**Participation**

Before characterizing the optimal allocation, we first show the sufficient conditions for the investor to have a non-zero holding of the illiquid asset:

**Corollary 7** *An investor prefers holding a small amount of the illiquid asset to holding a zero position if and only if*

\[
\frac{\nu - r}{\psi} \geq \rho \frac{\mu - r}{\sigma}.
\]

The condition for participation is identical to the Merton two-asset case and depends only on the mean-variance properties of the two securities. Somewhat surprisingly, the degree of illiquidity \( \lambda \) does not affect the decision to invest a small amount in the illiquid asset because of the infinite horizon of the agent: a trading opportunity will eventually arrive where the illiquid asset can be converted to liquid wealth and eventual consumption. However, even though the conditions for participation are the same as the standard Merton case, the optimal holdings of the illiquid and liquid assets are quite different, as we show below.
Illiquid Asset Holdings

Illiquidity induces underinvestment in the illiquid asset. In Table 3, we present the investor’s optimal rebalancing point $\xi^*$ along with the long-run average level illiquid portfolio holdings $E[\xi]$ for different values of $\lambda$. For comparison, and in an abuse of notation, we report the consumption and portfolio policies for an investor able to continuously trade one ($\lambda = 0$) and two ($\lambda = \infty$) risky assets. The optimal holding of illiquid assets at $\lambda = 1$ when rebalancing is possible is 0.37, which is lower than the optimal two-asset Merton holding at 0.60.

In addition to underinvestment in the illiquid asset, the inability to trade means that the investor’s portfolio can deviate from optimal diversification for a long time. Figure 1 plots the stationary distribution of an investor’s holding of the illiquid asset, $\xi$. For most of the time – the 20% to 80% range – the share of wealth allocated in illiquid securities is 0.36 to 0.45, while the 1% to 99% range is 0.30 to 0.65. Furthermore, the distribution in Figure 1 is positively skewed, since illiquid wealth grows faster on average than liquid wealth since only the latter is used to fund consumption. As a result of this skewness, the investor chooses a rebalancing point lower than the mean of the steady-state distribution of portfolio holdings ($\xi^* < E[\xi_t]$).

The degree of skewness is increasing in the illiquidity of the investment. When $\lambda = 1$, the mean holding is 0.41, compared to a rebalancing value of 0.37, while the distribution of portfolio holdings has a standard deviation of 6.3% and normalized skewness coefficient of 1.9. The average distance of illiquid portfolio holdings from the target $\xi^*$ increases with the degree of illiquidity $1/\lambda$; in the case when the investor can trade once every four years on average ($\lambda = 4$), the standard deviation of the investor’s illiquid holdings increases to 12% and the skewness increases to 2.3.

Liquid Asset Holdings

In addition to underinvestment in the illiquid asset, illiquidity affects the investment in the liquid asset. Taking the first order condition from the investor’s value function, the allocation
to the liquid risk asset as a fraction of the investor’s liquid holdings is equal to

$$
\theta_t = \frac{\mu - r}{\sigma^2} \left( -\frac{F_W}{F_{WW} W_t} \right) + \rho \frac{\psi}{\sigma} \left( -\frac{F_{WX} X_t}{F_{WW} W_t} \right).
$$

(29)

The investor’s allocation to the liquid asset as a function of her total wealth is equal to $\theta(1-\xi)$.

There are two aspects of the optimal policy that merit attention. First, even in the case where the liquid and illiquid asset returns are uncorrelated, $\rho = 0$, the allocation to the liquid asset differs from the Merton benchmark due to time-varying effective risk aversion. In Panel a of Figure 2, we compare the curvature of the investor’s value function with respect to liquid wealth $-F_{WW} W/F_W$ (black line) to that of a Merton investor (grey line). We see that for low values of allocation to illiquid assets, the two behave in a similar fashion: as the share of liquid wealth $W$ declines in the investor’s total wealth $W + X$, so does the investor’s aversion to gambles in $W$. However, when the investor’s liquid wealth becomes sufficiently low, the two lines diverge, since liquid wealth is no longer viewed as a substitute for illiquid wealth. The investor in our problem becomes much more averse to taking gambles in terms of liquid wealth than a Merton investor. Hence, her effective risk aversion increases.

Second, in the case where the liquid and illiquid asset are correlated, $\rho \neq 0$, an additional element that influences the demand for the liquid asset is the desire to hedge changes in the value of the illiquid wealth. The strength of this motive depends on the strength of the correlation, $\rho$, and the elasticity of substitution between liquid and illiquid wealth $-F_{WX} X/F_{WW} W$.

In our setting, this effect is dampened, since liquid and illiquid wealth are imperfect substitutes. In Panel b of Figure 2 we plot the second component for the demand for the liquid risky asset $-F_{WX} X/F_{WW} W$ (black line) and contrast it to the term corresponding to a Merton investor, which reduces to $-X/W$ (grey line), for the case where $\rho = 0.6$. Again, we see that for low values of $X$ relative to total wealth, the two lines are very similar, whereas they diverge dramatically as $X$ increases relative to $W$. In our case, the term $-F_{WX} X/F_{WW} W$ converges to zero rather than minus infinity in the Merton case. This striking behavior is a direct con-
sequence of Lemma 6: the hedging motive disappears when illiquid securities comprise the majority of the agent’s portfolio, since illiquid assets are not a substitute for liquid wealth. In this case, the investor chooses the allocation in liquid assets to smooth consumption rather than hedging fluctuations in her illiquid portfolio.

Figure 3 plots the agent’s optimal allocation to the liquid risky asset as a function her current allocation in illiquid assets $\xi$. In general, the investor under-allocates to the liquid risky asset relative to the Merton benchmark. The agent partially compensates for the risk of being unable to trade the illiquid asset for a long period of time by underinvesting in the liquid risky asset. In Table 3 we summarize the average long-run holdings in the liquid risky asset for different degrees of illiquidity $1/\lambda$. Illiquidity negatively impacts the allocation to the liquid risky asset, but less so than the illiquid asset. In the case where $\lambda = 1$, the investor reduces her allocation from 60% to 56% in the liquid risky asset, compared to a reduction from 60% to 37% for the illiquid security, relative to the Merton benchmark.

**Effect of Correlation on Optimal Asset Holdings**

Last, we study the effect of correlation on portfolio policies. We focus on the interesting case when the two securities have different Sharpe ratios; hence, for this comparison we set the expected return of the illiquid asset to $\nu = 0.2$.

In Figure 5 we compare the optimal allocation to the liquid and illiquid asset at the rebalancing moment with the Merton policies, as a function of the correlation coefficient. We see that the effect of correlation is muted relative to the Merton benchmark. In the case where both assets are fully liquid, varying $\rho$ from zero to one leads to large swings in portfolio allocations. As $\rho$ approaches one, the investor takes large offsetting positions in the two assets. In contrast, in our setting the investor never shorts the liquid risky asset, even when the correlation approaches one.$^{17}$

These results further demonstrate that from the investor’s perspective, the liquid and

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$^{17}$In contrast to the Merton problem, here the investor’s problem is well defined even in the case where $|\rho| = 1$ and the illiquid security has a higher Sharpe ratio than the liquid risky asset.
illiquid asset are imperfect substitutes, since only the former can be used to fund short- 
term consumption. This imperfect substitutability dampens the desire to use the liquid risky 
asset to hedge price changes in the illiquid asset. As we show below, deviation of optimal 
diversification is a second-order concern for the investor.

4.5 Consumption

The inability to trade the illiquid asset for long periods of time also affects the agent’s optimal 
consumption policy. The investor’s optimal consumption choice satisfies

\[
U'(C) = F_W(W, X). 
\]

(30)

In contrast to the Merton setting, the investor equates the marginal utility of consumption 
with the marginal value of liquid, as opposed to total wealth, since consumption is funded by 
liquid assets in the short run. Using the form for the value function (12), and denoting by 
c the ratio of consumption to liquid wealth, the investor’s consumption to total wealth ratio 
equals

\[
\frac{C}{W + X} = c(1 - \xi) = \left((1 - \gamma)H(\xi) - H'(\xi)\xi\right)^{\frac{1}{\gamma}}. 
\]

(31)

Figure 4 plots the agent’s optimal consumption to total wealth ratio as a function of the 
illiquid asset’s share \(\xi\) of the agent’s wealth. As we see, the investor always consumes a lower 
fraction of her total wealth than the two-asset Merton benchmark. In contrast to the Merton 
problem, the consumption to wealth ratio is time-varying. In particular, the marginal value 
of liquid wealth varies with the current allocation to the illiquid security, hence, so does the 
investor’s optimal consumption policy.

The investor’s consumption policy sheds further light into the behavior of the marginal 
value of liquid wealth \(F_W\). When the share of illiquid assets in the portfolio is small, the 
share of total wealth consumed is insensitive to portfolio composition \(\xi\). In this region, the
investor smooths lifetime consumption by consuming a higher fraction of liquid wealth today as \( \xi \) increases. In this case, liquid and illiquid wealth are substitutes \( F_{W,X}/F_{WW} \approx -1 \), since the investor’s valuation of liquid wealth falls as illiquid wealth increases. In contrast, as the share of illiquid assets \( \xi \) increases towards one, her marginal value of liquid wealth increases, leading to a lower consumption to total wealth ratio. Consistent with Lemma 6, when illiquid wealth is large, the investor’s marginal utility of consumption becomes insensitive to illiquid asset holdings \( X \).

In Table 3 we compute the long-run average consumption rate \([c(1-\xi)]\) for different values of \( \lambda \). As we vary the expected time until the next trading opportunity \( 1/\lambda \) from 0.1 to 10 years, the fraction of total wealth consumed per year declines from 8.6% to 5.9%. Interestingly, when the length of the illiquidity period is sufficiently long and uncertain \( (\lambda \leq 1/4) \), the investor consumes a lower fraction of her total wealth than an investor who is unable to trade the second asset at all.

4.6 The Welfare Cost of Illiquidity

Here, we quantify the cost of illiquidity on the investor’s welfare. In particular, we compute the fraction of initial wealth \( \alpha \) the investor would be willing to give up at the instant of rebalancing, in order to be fully able to trade the illiquid asset

\[
K_{M2} ((W_t + X_t)(1 - \alpha))^{1-\gamma} = (W_t + X_t)^{1-\gamma}H(\xi^*). \tag{32}
\]

The left hand side of equation (32) is the value function of a Merton investor able to invest in two risky securities. Here, we should note that since \( H(\xi^*) \geq H(\xi_t) \), the investor would be willing to pay at least a fraction \( \alpha \) at any point between rebalancing dates; thus \( \alpha \) is a conservative estimate of the cost of illiquidity.

In Table 3 we compute the welfare cost of illiquidity for different values of \( \lambda \). As we see, the welfare cost of illiquidity can be substantial. Even when the investor can trade on average
once a week ($\lambda = 50$), she would be willing to forego 2.8% of her wealth in order to make the second asset fully liquid. For higher degrees of illiquidity, the cost increases substantially; an investor trading an asset with an average of 10 years between trades ($\lambda = 0.1$), such as institutional real estate, would be willing to give up 22% of her total wealth in order to be able to trade the illiquid asset continuously.

5 Determinants of the Cost of Illiquidity

Here, we explore the key determinants of the cost of illiquidity. First we separate the cost due to suboptimal diversification from the inability to fund consumption and show that smoothing consumption is much more important than suboptimal diversification. Second, we disentangle the effect of illiquidity from illiquidity risk by comparing our setup to a model with deterministic periods of illiquidity. In doing so, we show that the risk from uncertain trading times drives our magnitudes. Finally, we explore the impact of preference parameters, disentangling the effect of the coefficient of risk aversion from the elasticity of intertemporal substitution. We show that the cost of illiquidity is highest for investors who are willing to substitute across states but not across time.

5.1 The Effect of Consumption Smoothing

A substantial determinant of the cost of illiquidity is that consumption is financed through liquid wealth. To quantify the magnitude of this effect and compare it to the cost due to deviations from optimal allocation, we consider an investor without intermediate consumption needs but with a finite and uncertain horizon. The investor consumes her entire wealth $W + X$ at some future stochastic data $\tau$

**Problem 2 (No Intermediate Consumption)** The investor maximizes

$$F_{nc}(W_t, X_t) = \max_{\{0, 1\}} \mathbb{E}_t [U(C_\tau)],$$  \hspace{1cm} (33)
where $\tau$ is a stochastic retirement time that is exponentially distributed according to a Poisson process with arrival rate $\delta$ subject to the budget constraints given by

$$\frac{dW_t}{W_t} = (r + (\mu - r) \theta_t) \, dt + \theta_t \sigma dZ^1_t - \frac{dI_t}{W_t}, \quad (34)$$

equation (9), and $C_t = W_t + X_t$. Re-balancing ($dI_t \neq 0$) occurs only when the Poisson process $N^\lambda_t$ jumps.

The following proposition characterizes the solution

**Proposition 8 (No Intermediate Consumption)** The solution to Problem 2 is characterized the function $H_{nc}(\xi)$ and constants $H^*_nc$ and $\xi^*_nc$ that satisfy

$$0 = \max_{\theta} \left[ \delta \left( \frac{1}{1 - \gamma} - H_{nc}(\xi) \right) + \lambda (H^*_nc - H_{nc}(\xi)) + H_{nc}(\xi) A(\xi, 0, \theta) \
\quad + \frac{\partial H_{nc}(\xi)}{\partial \xi} B(\xi, 0, \theta) + \frac{1}{2} \frac{\partial^2 H_{nc}(\xi)}{\partial \xi^2} C(\xi, 0, \theta) \right]. \quad (35)$$

where the functions $A$, $B$ and $C$ are defined in (15)-(17) and $H^*_nc$ and $\xi^*_nc$ are defined as in (18). When a trading opportunity occurs at time $\tau$, the trader selects $I_\tau$ so that $\frac{X_\tau}{X_\tau + W_\tau} = \xi^*_nc$.

As we see in proposition 8, the solution to the problem without intermediate consumption is very close to our baseline model, up to a difference in the discount rate. To facilitate comparison with our baseline model, we consider the case where the investor’s effective rate of impatience is equal to our baseline calibration, $\delta = \beta$.

In Table 4 we compute optimal policies and welfare costs across different levels of illiquidity $1/\lambda$. In this setting where the cost of illiquidity is only due to deviation from optimal diversification, the effects of illiquidity is small. As we compare Table 4 to the results from the baseline problem (Table 3), we conclude that the primary determinant of the welfare cost of illiquidity is the desire to smooth consumption across states, confirming our intuition in Section 4.2.
5.2 Stochastic Versus Deterministic Trading Opportunities

The discussion in Section 4.2 illustrates that the risk of liquid wealth – and hence intermediate consumption – dropping to zero is a substantial cost of investing in illiquid assets. Our model features an illiquid period whose length is uncertain. To disentangle the effect of the length of the illiquid period from the uncertainty over its duration, we consider the case where the agent is allowed to rebalance her portfolio at fixed intervals, spaced $T$ periods apart. The investor’s problem is

**Problem 3 (Deterministic Rebalancing)** The investor maximizes

$$F_T(W_t, X_t) = \max_{\{\theta, I, c\}} E_t \left[ \int_t^{\infty} U(C_s)ds \right],$$

subject to the budget constraints (8) and (9), with re-balancing ($dI_t \neq 0$) only at $\tau = 0, T, 2T, ...$

As in Problem 1, the value function is homothetic and so

$$F_T(W, X) = (W + X)^{1-\gamma} H_T(\xi), \quad \text{where} \quad \xi = \frac{X}{X + W}.$$

The following proposition characterizes the solution:

**Proposition 9 (Deterministic Rebalancing)** The function $H_T(t, \xi)$ satisfies

$$0 = \max_{c, \theta} \left\{ \frac{1}{1 - \gamma} e^{1-\gamma} - \beta H_T(t, \xi) + \frac{\partial H_T(t, \xi)}{\partial t} + H_T(t, \xi) A(\xi, c, \theta) + \frac{\partial H_T(t, \xi)}{\partial \xi} B(\xi, c, \theta) + \frac{1}{2} \frac{\partial^2 H_T(t, \xi)}{\partial \xi^2} C(\xi, c, \theta) \right\}.$$

where the functions $A$, $B$ and $C$ are defined in (15)-(17).

At the repeated trading times $t = \tau = 0, T, 2T, ...$, the investor will optimally rebalance her
portfolio so that

\[ H_T(\tau, \xi) = \max_\xi \lim_{\epsilon \downarrow 0} H_T(\tau + \epsilon, \xi) \]  
(37)

\[ H^*_T = \max_\xi H_T(\tau, \xi) \]  
(38)

\[ \xi^*_T = \arg \max_\xi H_T(\tau, \xi) \]  
(39)

Then \( \tau = 0, T, 2T, \ldots \), a trading opportunity occurs and the trader selects \( I_\tau \) so that

\[ \frac{X_\tau}{X_\tau + W_\tau} = \xi^*_T. \]

We compare the solution to problem 3 to the baseline case in Figure 6. In the right panel, we plot the value function \( H_T(t, \xi) \) for different times \( t = 0 \ldots T \). We see that the function \( H(t, \xi) \) is concave in \( \xi \). The investor has an target allocation for illiquid securities, and deviations from this allocation reduce welfare. In contrast to the case where the trading time is stochastic, the concavity of \( H \) decreases as the investor approaches the next trading time; at \( t = \tau - \epsilon \), the function \( H \) is almost flat. At this point, the investor anticipates that she will be able to rebalance very soon, so her liquid and illiquid holdings become closer to perfect substitutes.

Table 5 computes the optimal policies and the welfare cost of illiquidity for different lengths of the illiquidity period. In contrast to the case with stochastic trading opportunities, varying the length of the deterministic illiquidity period has only a small effect on optimal policies. For example, when the time until the next trade is known in advance, varying the expected time until the next rebalancing time from \( 1/50 \) to 10 years leads to a drop in the fraction of total wealth consumed per year from 8.9% to 8.4%. Similarly, the effects on welfare are small and are relative insensitive to the length of the illiquidity period; the welfare cost of illiquidity varies between 1.1% (1/50 years) to 2.8% (10 years).

As we compare Table 5 to Table 3, we see that the uncertainty regarding the opportunity to trade is a major component of the cost of illiquidity. In contrast to the case with stochastic trading opportunities, when the trading intervals are known in advance, illiquidity has a level
effect on welfare that is relatively insensitive to the length of the period. To understand these findings, note that the investor’s main concern is to avoid states of the world where her liquid wealth – and therefore her consumption – drops to zero before the next opportunity to trade. If the investor can trade at deterministic intervals, this state can be avoided with probability one by investing an appropriate amount into the riskless asset and consuming a constant fraction.\textsuperscript{18} Thus, the investor can choose a policy such that her only risk is from changes in consumption across trading intervals. In these stochastic case, the distribution of waiting times until the next trade is unbounded, hence the only way the investor can guarantee a minimum level of consumption in all states is to invest all her wealth into the risk-free asset.

5.3 The Effect of Preference Parameters

The discussion in the previous section illustrates that the curvature of the value function is an important determinant of the cost of illiquidity. In particular, the investor fears the probability that she reaches states with very low liquid wealth for two reasons. First, states with low liquid wealth are states with low consumption, and the investor likes to smooth consumption across states; the coefficient of risk aversion captures the magnitude of this preference. Second, if the agent reaches states where liquid wealth is low relative to her total wealth, she faces a steeply increasing consumption profile. The agent dislikes these states because she wants to have smooth consumption paths over time, and the elasticity of intertemporal substitution governs this preference.

Here, we disentangle whether the cost of illiquidity is mainly driven by the desire to smooth consumption across states versus across time. A feature of time-separable preferences is that these two effects are linked. To investigate these two motives separately, we consider the case where the agent has recursive preferences, as in Duffie and Epstein (1992). Specifically, the

\textsuperscript{18}For instance, if the illiquidity period lasts for an interval of $T$, the investor can guarantee a consumption flow of $c^*$ by investing an amount $c^*T$ into the riskless asset.
agent maximizes a utility index \( J \), which is defined recursively according to

\[
J_t = E_t \int_t^{\infty} f(C_s, J_s) \, ds, \quad (40)
\]

where

\[
f(C, J) \equiv \frac{\beta}{1 - \zeta} \left( \frac{C^{1-\zeta}}{(1 - \gamma) J^{\frac{1}{1-\gamma}}} - (1 - \gamma) J \right). \quad (41)
\]

Equation (41) represents the continuous-time equivalent of Epstein and Zin (1989), where \( \beta \) is the subjective discount rate, \( \gamma \) is the coefficient of risk aversion and \( \zeta \) is the inverse of the elasticity of intertemporal substitution. The case of power utility corresponds to \( \gamma = \zeta \).

**Problem 4 (Epstein-Zin)**  
*The investor maximizes*

\[
F_{ez}(W_t, X_t) = \max_{\{\theta, I, c\}} E_t \left[ \int_t^{\infty} f(C_s, J_s) \, ds \right], \quad (42)
\]

subject to the budget constraints (8) and (9), with re-balancing \((dI_t \neq 0)\) only when the Poisson process \(N^\lambda_t\) jumps.

As in Problem 1, the value function is homothetic and so

\[
F_{ez}(W, X) = (W + X)^{1-\gamma} H_{ez}(\xi), \quad \text{where} \quad \xi = \frac{X}{X + W}. \quad (43)
\]

The following proposition characterizes the solution:

**Proposition 10 (Epstein-Zin)**  
The solution to Problem 4 is characterized by the function \( H_{ez}(\xi) \) and constants \( H_{ez}^* \) and \( \xi_{ez}^* \) such that

\[
0 = \max_{c, \theta} \left\{ \beta \left( c^{1-\zeta} \left( (1 - \gamma) H_{ez}(\xi) \right)^{\frac{1}{1-\gamma}} - (1 - \gamma) H_{ez}(\xi) \right) + \lambda (H_{ez}^* - H_{ez}(\xi)) 
+ H_{ez}(\xi) A(\xi, c, \theta) + \frac{\partial H_{ez}(\xi)}{\partial \xi} B(\xi, c, \theta) + \frac{\partial^2 H_{ez}(\xi)}{\partial \xi^2} C(\xi, c, \theta) \right\}. \quad (44)
\]
\[ H_{ez}^* = \max_{\xi} H_{ez}(\xi) \]  
\[ \xi_{ez}^* = \arg \max_{\xi} H_{ez}(\xi) \]

where the functions \( A, B \) and \( C \) are defined in (15)-(17). When a trading opportunity occurs at time \( \tau \), the trader selects \( I_\tau \) so that \( \frac{X_\tau}{X_\tau+W_\tau} = \xi_{ez}^* \).

To quantify the impact of risk aversion and the elasticity substitution on portfolio policies, in Table 6 we compute optimal policies and welfare costs for different preference parameters and levels of illiquidity \( 1/\lambda \). In panel A, we vary the coefficient of risk aversion \( \gamma \) and the likelihood of trading \( \lambda \), holding the EIS fixed. As risk aversion increases, the investor’s policies \( \xi^*, c \) and \( \theta \) converge frictionless benchmark. For instance, in the case where \( \lambda = 10 \), and investor with a risk aversion of \( \gamma = 3 \) would allocate 80% of her total wealth to the illiquid asset, compared to 118% for a Merton investor. In contrast, if the investor had a risk aversion of \( \gamma = 15 \), she would allocate only 22% of her total wealth to the illiquid asset, compared to 24% for a Merton investor. An investor with high risk aversion chooses to invest less in risky assets, hence for her, the cost of illiquidity is small. Indeed, as we see in Panel Aii, the welfare cost of illiquidity decreases with risk aversion.

In panel B, we vary the elasticity of intertemporal substitution \( 1/\zeta \) and the likelihood of trading \( \lambda \), holding risk aversion fixed. We find that the elasticity of intertemporal substitution has a quantitatively small impact on portfolio policies. As we see in Panels Bi and Biv, varying the EIS from 1.5 to 1/6 has essentially no impact on the allocation to the liquid asset, and a small impact on the allocation to the illiquid asset. However, as we see in Panel Biii, varying the EIS has an impact on the investor’s optimal consumption policy. When the investor’s desire to smooth consumption across states increases (\( EIS = 1/6 \)), her optimal fraction of wealth consumed declines faster with \( \lambda \) than when her elasticity of substitution is high. Hence, we find that low EIS magnifies the welfare cost of illiquidity to the investor, as we see in Panel
In summary, we find that the welfare cost of illiquidity increases with the desire to smooth consumption across time ($\zeta$ increases) and decreases with the desire to smooth consumption across states ($\gamma$ increases). These findings are driven by the fact that a high risk averse investor would not choose to invest much in the illiquid asset anyway. However, holding risk aversion constant, lowering the EIS increases the welfare cost to the investor, since it leads to non-smooth consumption paths across time.

6 Liquidity Crises

Our baseline model is applicable to a wide variety of illiquid securities. However, sometimes liquidity dries up even in markets that are otherwise fully liquid. For instance, Krishnamurthy, Nagel, and Orlov (2011) document that in the market for money market funds, a usually liquid market, there were instances of “buyers’ strikes” during the recent financial crisis, where investors were unwilling to trade at any price. Anderson and Gascon (2009) note that the commercial paper market froze not only in the 2008-2009 financial crisis, but it also froze in 1970 when Penn Central railroad collapsed. In both cases, the Federal Reserve stepped in to help restore liquidity. Other examples include the previously mentioned repo market (Gorton and Metrick, 2009); residential and commercial mortgage-backed securities (Gorton, 2009; Dwyer and Tkac, 2009; Acharya and Schnabl, 2010); auction-rate securities (Ang and Bollen, 2010); structured credit (Brunnermeier, 2009); and the auction rate security market, which became illiquid in 2008 and is still frozen in 2012.

Here, we extend the model in Section 3 to allow for infrequent liquidity crises. These liquidity crises are temporary, and they adversely affect the liquidity of otherwise liquid secu-

19As noted by the SEC, “Report on the Municipal Securities Market,” July 31, 2012: in 2008, the auction rate securities (ARS) market totaled approximately $200 billion; in February 2008, the market froze because there were no bidders in the primary auctions, where floating interest rates are set. As there was no secondary market, thousands of customers were unable to sell their ARS holdings. In 2011, there were no new issues of ARS. Other intermediated fund vehicles also became more illiquid during this time: hedge funds, for example, imposed ‘gates’ provisions that did not allow for investors to withdraw capital (see Ang and Bollen, 2010).
rities. We consider two applications of the extended model. First, we show that the possibility of a deterioration in market liquidity leads to limited arbitrage, even in normal times. Second, using the investor’s value function, we derive the cost of liquidity insurance – the risk premium of an asset that pays off on the event of a liquidity crisis.

### 6.1 A Model With Time-Varying Liquidity Risk

The level of market liquidity varies depending on two states $S_t \in \{I, L\}$. In state $L$, corresponding to ‘normal’ times, both assets are perfectly liquid as in the Merton benchmark. In state $I$ – the ‘crisis’ state, the second risky asset $P$ is illiquid, as in the model in Section 3.\(^{20}\)

The state of market liquidity $S_t$ follows a continuous-time Markov process with transition probability matrix between time $t$ and $t + dt$ given by

\[
P = \begin{pmatrix}
1 - \chi^L dt & \chi^L dt \\
\chi^I dt & 1 - \chi^I dt
\end{pmatrix}.
\]

Hence, the frequency and average duration of a liquidity crisis are $\chi_I$ and $1/\chi_L$ respectively.

The investor’s problem is

**Problem 5 (Liquidity Crises)** *The investor maximizes (10) subject to the budget constraints (8) and (9). The state of the economy ($S_t \in \{I, L\}$) evolves as in (47). If $S_t = L$, trade in both assets is continuous; if $S_t = I$, the investor can re-balance ($dI_t \neq 0$) only when the Poisson process $N_t$ jumps.*

As in Problem 1, the value function is homothetic and so

\[
F_{LC}(W, X) = (W + X)^{1-\gamma} H_{LC}(\xi, S), \quad \text{where} \quad \xi = \frac{X}{X + W}.
\]

In this case, the investor’s value function depends not only on her wealth composition $\xi$, but

\[^{20}\]Extending the model to allow for multiple illiquidity regimes, or for state-contingent asset return moments, is straightforward.
also on the condition of market liquidity.

Our first result is that even in normal times, \( S = L \), the investor will not short the potentially illiquid asset or have a short position in liquid wealth:

**Proposition 11** Any optimal policies will have \( W > 0 \) and \( X \geq 0 \) a.s. for both \( S = I \) and \( S = L \).

The proposition shows that the possibility of a liquidity crisis affects portfolio policies in normal times. The transition from liquid to illiquid is a surprise event and occurs without the opportunity to re-balance. Consequently, the portfolio restrictions from the illiquid state (see Proposition 1) are also valid in the liquid state.

The following proposition characterizes the solution to investor’s problem

**Proposition 12 (Liquidity Crises)** The solution to the two state problem is characterized by an optimal consumption policy \( \{c_I^*, c_L^*\} \) and portfolio policies \( \{\theta_I^*, \theta_L^*\} \) and \( \{\xi_I^*, \xi_L^*\} \) in the illiquid and liquid state respectively, as well as constants \( H_I^* \) and \( H_L^* \) and the function \( H_I(\xi) \). The investor’s value function is

\[
H_{LC}(\xi, S) = \begin{cases} 
H_I(\xi), & S = I \\
H_L^*, & S = L
\end{cases},
\]

where

1. The function \( H_I(\xi) \) satisfies the Hamilton-Jacobi-Bellman equation

\[
0 = \max_{c,\theta} \left\{ \frac{1}{1 - \gamma} e^{1-\gamma} - \beta H_I(\xi) + \lambda (H_I^* - H_I(\xi)) + \chi^L (H_L^* - H_I(\xi)) + H_I(\xi)(1 - \gamma)A(\xi, c, \theta) + \frac{\partial H_I(\xi)}{\partial \xi} B(\xi, c, \theta) + \frac{1}{2} \frac{\partial^2 H_I(\xi)}{\partial \xi^2} C(\xi, c, \theta) \right\},
\]

where the functions \( A, B \) and \( C \) are defined in (15)-(17).
2. The constants $H_L^*$ and $H_I^*$ solve

$$0 = \max_{c, \xi} \left\{ \frac{1}{1 - \gamma} c^{1-\gamma} - \beta H_L^* + \chi (H_I(\xi) - H_L^*) + A \left( \xi, \frac{c}{1 - \xi}, \frac{\theta}{1 - \xi} \right) H_L^* \right\}$$

(50)

and

$$H_I^* = \max_{\xi} H_I(\xi)$$

(51)

3. The policies $\{c_I^*, \theta_I^*\}$ and $\{c_L^*, \theta_L^*, \xi_L^*\}$ maximize (49) and (50) respectively. The policy $\xi_I^*$ maximizes (51).

In normal times $S = L$, the investor can freely rebalance between both risky assets and thus the function $H_L(\xi)$ is a constant. During a crisis, $S = I$, the investor’s problem is similar to the problem analyzed in Section 3, hence her value function $H_I$ depends on the ratio of illiquid to total wealth $\xi$. When a liquidity crisis occurs, the investor is constrained to hold her current allocation to the now illiquid security until the next opportunity to trade. Hence, the investor’s optimal portfolio holdings in the liquid state are affected by the possibility of a liquidity crisis.

In Figure 7, we compare portfolio policies across regimes for different values of $\chi_L$, $\chi_I$ and $\lambda$. As we see, the investor reduces her allocation in illiquid asset not only during a crisis, but also during normal times. Similarly, the investor reduces her consumption rate in both regimes. Both of these effects are increasing in the frequency, $\chi_I$, average duration $1/\chi_L$ and severity $1/\lambda$ of liquidity crisis. In addition, the investor holds fewer liquid risky assets, but only during a crisis; her portfolio allocation in liquid risky assets is unaffected in normal times, as long as the two assets are uncorrelated.

In Panel A of Figure 8, we compute the welfare cost a liquidity crisis, defined as the fraction of total wealth the investor would pay \textit{ex-ante} to eliminate the possibility of a crisis. Even though liquidity crises are temporary, they still lead to substantial welfare costs. For instance,
the investor would be willing to forgo 2% of her wealth to eliminate the possibility of a once
in a decade liquidity crisis ($\chi_I = 0.1$), with average duration of two years ($\chi_L = 0.5$), during
which the investor can rebalance on average once a year ($\lambda = 1$).

6.2 Limits to Arbitrage

The results in the previous section illustrate that the possibility of a liquidity crisis leads
to underinvestment in assets that are currently fully liquid but whose liquidity can dry up
during a crisis. Here, we show that the same mechanism leads to underinvestment in arbitrage
opportunities, defined as two perfectly correlated assets with different Sharpe ratios:

**Corollary 13** Limits to Arbitrage: in the case where $|\rho| = 1$ and $\frac{\nu - r}{\psi} \neq \frac{\mu - r}{\sigma}$ the investor’s
portfolio policies $\theta^*_L$ and $\xi^*_L$ are finite and satisfy

$$\frac{\nu - r}{\psi} - \rho \frac{\mu - r}{\sigma} = -\frac{\chi^I}{\psi} \frac{H_I'(\xi^*_L)}{H^*_L(1 - \gamma)}$$

(52)

and

$$\theta^*_L = \frac{\mu - r}{\gamma \sigma^2} - \frac{\psi}{\sigma} \xi^*_L$$

(53)

Faced with a two perfectly correlated securities with different Sharpe ratios, in the absence
of any trading friction, the investor should construct a zero-investment portfolio that has a
positive payoff, and take an infinite position in this arbitrage. In our setting, the the investor
is reluctant to do so – even though both securities are fully liquid and taking advantage of
this arbitrage need not involve shorting risky assets.

The reason for this underinvestment is that, on the event of a crisis, the investor is no
longer free to rebalance her position in order to keep the arbitrage locally riskless. As we
see in equation (52), that the investor will increase her holdings of the illiquid asset until the
marginal welfare loss in the illiquid state ($H_I'(\xi^*_L)$) times the probability of that state occurring
is proportional to the difference in the Sharpe ratios between the liquid and illiquid assets.

6.3 The Cost of Insurance Against Liquidity Crisis

Liquidity crisis are systematic events that affect investors’ marginal utility, hence they command a risk premium. Here we use the extended model to quantify the magnitude of this illiquidity risk premium.

We introduce a fictitious derivative security in zero net supply that allows the investor to hedge the deterioration in market liquidity. By purchasing liquidity protection, the investor pays an annual premium equal to $\hat{\chi}_I$ in order to receive a cash payment of $1$ dollar in the event of a liquidity crisis. The following lemma computes the cost of liquidity insurance that would induce zero demand for the derivative security

**Lemma 14** The annual premium for liquidity protection is equal to

$$\hat{\chi}_I = \chi_I \frac{F_W(W, X, I)}{F_W(W, X, L)} \bigg|_{\xi = \xi^*_L}$$

(54)

Lemma 14 shows that the cost of liquidity insurance is equal to the probability of a liquidity crisis, times the increase in the marginal value of liquid wealth on the event of a crisis. Since liquid wealth becomes more valuable during a liquidity crisis than normal times, the investor is willing to pay a higher rate that the objective probability $\chi_I$ to obtain some protection against a liquidity crisis.

In Panel $B$ of Figure 8 we compare the risk premium $\hat{\chi}_I - \chi_I$ of a liquidity crisis across different values of $\chi_I$, $\chi_L$ and $\lambda$. We see that the investor is willing to pay a substantial premium over the subjective probability $\chi_I$ in order to obtain liquidity during a crisis. For example, the investor would be willing to pay an excess premium of 80 bps per year to obtain liquidity on the event of a once in a decade liquidity crisis ($\chi_I = 0.1$), with average duration of two years ($\chi_L = 0.5$), during which the investor can rebalance on average once a year ($\lambda = 1$).

Comparing Panels $A$ and $B$, we see that the liquidity risk premium and the liquidity
premium are related by distinct concepts. The former refers to the risk premium of an asset that appreciates during a state of high marginal value of liquid wealth – the liquidity crisis. Examples of such assets are on-the-run US Treasury bonds, that tend to appreciate during a liquidity crisis; these assets typically command higher prices than comparable assets (e.g. off-the-run bonds) and thus typically have lower expected returns. In contrast, the liquidity premium is directly related to the welfare cost of illiquidity and refers to the price discount of an asset that is illiquid, regardless of its performance during a crisis.

7 Conclusion

We study the effect of illiquidity on portfolio choice by extending the Merton framework to allow for infrequent and stochastic trading opportunities. We find that illiquidity has substantial effects on portfolio policies and welfare. These effects are mainly driven by the fact that i) consumption is financed through liquid assets; and ii) the arrival of these trading opportunities is stochastic rather than deterministic. In particular, the investor cares more about the uncertainty rather than the expected length of the illiquidity periods, since the former increases the likelihood that her intermediate consumption will drop to zero.

We further explore the risk associated with illiquidity by allowing for time variation in the level of market liquidity. Consistent with the behavior of many asset markets, often liquidity ‘dries up’, leading to market freezes. We show that the possibility of an evaporation in liquidity leads to limited arbitrage even in normal times. Using the extended model, we quantify the risk premium associated with a liquidity crisis. Since the marginal value of liquid wealth increases at the onset of a liquidity crisis, investors are willing to pay a substantial premium over the actuarial probability of a crisis to obtain liquidity in the event of a market freeze.
A Proofs

A.1 Proof of Proposition 1

Consumption is out of liquid wealth only and the illiquid asset cannot be pledged, so \( W_t \leq 0 \) implies zero consumption before the next trading day, leaving the objective function (6) at \(-\infty\). For \(|\rho| < 1\), \( X_t < 0 \) implies that under any admissible investment and consumption policy, there is a positive probability that at the next trading time \( W_t + X_t \leq 0 \), violating limited liability, implying zero consumption, and leaving the objective function (6) at \(-\infty\). For \( \rho = 1\), \( X_t < 0 \) is ruled out by assuming that the illiquid asset has a weakly higher Sharpe ratio than the liquid asset (5). For \( \rho = -1\) the investor would invest positive amounts in the illiquid asset \( X_t \) and the liquid risky asset.

A.2 Proof of Lemma 3

Define \( Q = X + W \) to be total wealth, and let \( \{Q_0, X^1_0\} \) and \( \{Q_0, X^3_0\} \) be two initial values with the associated optimal policies \( \{C^1, \pi^1\} \) and \( \{C^3, \pi^3\} \) where \( \pi = \theta W \). For \( \kappa \in (0, 1) \), we consider a middle initial value \( \{Q_0, X^2_0 = \kappa X^1_0 + (1 - \kappa) X^3_0\} \) and the associated (possibly optimal) policies \( \{C^2 = \kappa C^1 + (1 - \kappa) C^3, \pi^2 = \kappa \pi^1 + (1 - \kappa) \pi^3\} \), which are feasible because of the linearity of the budget constraint. From (8) and (9), we have

\[
dQ_t = [rQ_t + (\mu - r)\pi_s + (\nu - r)X_t - C_t]dt + [\pi_s\sigma + \psi\rho X_t]dZ^1_t + \psi X_t \sqrt{1 - \rho^2} dZ^2_t
\]

(55)

for any time \( t \). Thus, from the construction of our initial values and optimal policies, we have \( Q^2_t = \kappa Q^1_t + (1 - \kappa) Q^3_t \). Next, consider the objective function

\[
E \left[ \int_0^\infty e^{-\beta t} U(C_t) \right] = E \left[ \int_0^\tau e^{-\beta t} U(C_t)dt + e^{-\beta \tau} Q_{1-\gamma}^* H^* \right]
\]

(56)

Because \( U(C) \) is increasing and concave, we have \( U(C^2_t) > \kappa U(C^1_t) + (1 - \kappa) U(C^3_t) \). From Jensen’s inequality and \( H^* < 0 \), we have \( Q^{21-\gamma}_\tau H^* > \kappa Q^{11-\gamma}_\tau H^* + (1 - \kappa) Q^{31-\gamma}_\tau H^* \). Thus, \( E^2 \left[ \int_0^\infty e^{-\beta t} U(C_t) \right] > \kappa E^1 \left[ \int_0^\infty e^{-\beta t} U(C_t) \right] + (1 - \kappa) E^3 \left[ \int_0^\infty e^{-\beta t} U(C_t) \right] \), and so the value function is concave in \( X \) for fixed \( Q \). This is sufficient to show that the value function is concave in \( \xi \) for fixed \( Q \), so \( H \) is concave.

A.3 Proof of Proposition 11

The arguments in the proof of Proposition 1 are sufficient to show that for \(|\rho| < 1\), the objective function is at \(-\infty\) if either \( W \leq 0 \) or \( X < 0 \) for \( S = I \). For \( S = L \), we observe that the state will shift to \( S = I \) without the possibility of re-balancing; as a result, if either \( W \leq 0 \) or \( X < 0 \), the objective function in the liquid state is also equal to \(-\infty\). For \( \rho = 1\), \( X < 0 \) is ruled out by (4).
A.4 Proof of Proposition 13

In the liquid state, the investor’s optimal portfolio policies $\theta_L^*$ and $\xi_L^*$ satisfy the first order conditions:

$$
0 = H_L^*(1 - \gamma)(\mu - r) - H_L^*(1 - \gamma)\gamma \theta_L^* \sigma^2 - H_L^*(1 - \gamma) \rho \psi \xi_L^*
$$

$$
0 = \chi^I H_I^*(\xi_L^*) + H_L^*(1 - \gamma)(\nu - r) - H_L^*(1 - \gamma) \psi^2 \xi_L^* - H_L^*(1 - \gamma) \gamma \rho \psi \theta_L^* \sigma
$$

Setting $\rho = 1$, dividing the first equation by $\sigma H_L^*(1 - \gamma)$ and the second by $\psi H_L^*(1 - \gamma)$, and then subtracting the first equation from the second leads to (52) and (53). Setting, $\rho = -1$, dividing the first equation by $\sigma H_L^*(1 - \gamma)$ and the second by $\psi H_L^*(1 - \gamma)$, and then adding the first equation from the second leads to the analogous expressions.

A.5 Proof of Lemma 14

Consider a derivative security $Z$ that pays a fixed rate of return $\kappa$ when the aggregate state $S$ switches from $L$ to $I$. Denoting by $dN_t^I$ the Poisson count process that denotes the arrival of a liquidity crisis, the price of this security evolves according to

$$
\frac{dZ_t}{Z_t} = (r + \mu_Z - \kappa \chi^I) dt + \kappa dN_t^I.
$$

The investor is indifferent between participating in the market for security $Z$ and her current portfolio policy as long as the excess return $\mu_Z$ is equal to

$$
\mu_Z dt = -\text{cov} \left( \frac{dZ_t}{Z_t}, \frac{dF_W}{F_W} \right)
$$

$$
= \kappa \chi^I \frac{H_L^*(1 - \gamma) - H_I^*(\xi_L^*)(1 - \gamma) + \xi_L^* H_I'(\xi_L^*)}{H_L^*(1 - \gamma)} dt
$$

There exists a fictitious probability measure $Q$ under which the security $Z$ has an expected excess return equal to zero and the probability of a liquidity crisis is equal to

$$
\hat{\chi}_t = \chi^I \frac{F_W(W, X, I)}{F_W(W, X, L)} = \chi^I \frac{H_I^*(\xi_L^*)(1 - \gamma) - \xi_L^* H_I'(\xi_L^*)}{H_L^*(1 - \gamma)}.
$$

Under that measure, the present value of the expected payments $p$ has to equal the expected payoff in the event of a liquidity crisis

$$
E_t^Q \int_t^\tau e^{-r(s-t)} p ds = E_t^Q \left[ e^{-r(\tau-t)} \right]
$$

$$
\Rightarrow \int_t^\infty e^{-(r+\hat{\chi}_t)(s-t)} p ds = \int_t^\infty e^{-(r+\hat{\chi}_t)(\tau-t)} \hat{\chi}_t d\tau
$$

$$
\Rightarrow p = \hat{\chi}_t
$$
References


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<td>4-5 years, but ranges from months to decades</td>
<td>4-6%</td>
<td>For median duration in residences see Hansen (1998) and Case and Shiller (1989), with Miller, Peng, and Sklarz’s (2011) comments on the range. Turnover numbers are computed by Dieleman, Clark and Deurloo (2000).</td>
</tr>
<tr>
<td>Institutional Infrastructure</td>
<td>50-60 years for initial commitment, some as long as 99 years</td>
<td>Negligible</td>
<td>Beeferman (2008), Bitsch, Buchner and Kaserer (2010)</td>
</tr>
<tr>
<td>Art</td>
<td>40-70 years</td>
<td>&lt; 15%</td>
<td>For holding periods see Goetzmann (1993) and Kaplan (1997). Turnover can be inferred from the size of the art market estimated by Skaterschikov (2006) and estimated annual art sales.</td>
</tr>
</tbody>
</table>
### Table 2: Liquid and Illiquid Asset Returns

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Stdev</td>
</tr>
<tr>
<td>Equity</td>
<td>0.103</td>
<td>0.182</td>
</tr>
<tr>
<td>Illiquid Assets</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Private Equity</td>
<td>0.103</td>
<td>0.229</td>
</tr>
<tr>
<td>Buyout</td>
<td>0.092</td>
<td>0.134</td>
</tr>
<tr>
<td>Venture Capital</td>
<td>0.133</td>
<td>0.278</td>
</tr>
<tr>
<td>Illiquid Investment</td>
<td>0.109</td>
<td>0.165</td>
</tr>
</tbody>
</table>

The table reports summary statistics on excess returns on liquid and illiquid assets. Liquid equity returns are total returns on the S&P500. Data on private equity, buyout, and venture capital funds are obtained from Venture Economics and Cambridge Associates. We construct annual horizon log returns at the quarterly frequency. We compute log excess returns using the difference between log returns on the asset and year-on-year rollover returns on one-month T-bills expressed as a continuously compounded rate. The column “Corr” reports the correlation of excess returns with equity. The illiquid investment is a portfolio invested with equal weights in private equity, buyout, and venture capital and is rebalanced quarterly.

### Table 3: Welfare and Optimal Policies (Baseline)

<table>
<thead>
<tr>
<th>Average Turnover</th>
<th>Optimal</th>
<th>Illiquidity</th>
<th>Average policies</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(T) = 1/\lambda$</td>
<td>Rebalance ($\xi^*$)</td>
<td>Welfare cost</td>
<td>$E[\xi]$</td>
</tr>
<tr>
<td>0</td>
<td>0.593</td>
<td>-</td>
<td>0.593</td>
</tr>
<tr>
<td>1/50 years</td>
<td>0.505</td>
<td>0.028</td>
<td>0.521</td>
</tr>
<tr>
<td>1/10 years</td>
<td>0.493</td>
<td>0.029</td>
<td>0.511</td>
</tr>
<tr>
<td>1/4</td>
<td>0.475</td>
<td>0.037</td>
<td>0.485</td>
</tr>
<tr>
<td>1/2</td>
<td>0.442</td>
<td>0.045</td>
<td>0.461</td>
</tr>
<tr>
<td>1</td>
<td>0.373</td>
<td>0.067</td>
<td>0.409</td>
</tr>
<tr>
<td>2</td>
<td>0.251</td>
<td>0.103</td>
<td>0.299</td>
</tr>
<tr>
<td>4</td>
<td>0.132</td>
<td>0.165</td>
<td>0.212</td>
</tr>
<tr>
<td>10</td>
<td>0.048</td>
<td>0.222</td>
<td>0.214</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.593</td>
<td>-</td>
<td>0.593</td>
</tr>
</tbody>
</table>

The table is computed using the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$. 

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### Table 4: The Model Without Intermediate Consumption

<table>
<thead>
<tr>
<th>Turnover</th>
<th>Optimal</th>
<th>Illiquidity</th>
<th>Average policies</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[T] = 1/\lambda$</td>
<td>Rebalance ($\xi^*$)</td>
<td>Welfare cost</td>
<td>$E[\xi]$</td>
</tr>
<tr>
<td>0</td>
<td>0.593</td>
<td>-</td>
<td>0.593</td>
</tr>
<tr>
<td>1/50 years</td>
<td>0.557</td>
<td>0.002</td>
<td>0.555</td>
</tr>
<tr>
<td>1/10 years</td>
<td>0.547</td>
<td>0.002</td>
<td>0.546</td>
</tr>
<tr>
<td>1/4</td>
<td>0.544</td>
<td>0.002</td>
<td>0.543</td>
</tr>
<tr>
<td>1/2</td>
<td>0.535</td>
<td>0.002</td>
<td>0.534</td>
</tr>
<tr>
<td>1</td>
<td>0.527</td>
<td>0.003</td>
<td>0.526</td>
</tr>
<tr>
<td>2</td>
<td>0.523</td>
<td>0.003</td>
<td>0.523</td>
</tr>
<tr>
<td>4</td>
<td>0.520</td>
<td>0.003</td>
<td>0.518</td>
</tr>
<tr>
<td>10</td>
<td>0.518</td>
<td>0.004</td>
<td>0.516</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.593</td>
<td>-</td>
<td>0.593</td>
</tr>
</tbody>
</table>

The table is computed using the following parameter values: $\gamma = 6$, $\mu = \nu = 0.12$, $r = 0.04$, $\sigma = \psi = 0.15$, and $\rho = 0$.

### Table 5: The Model With Deterministic Liquidity

<table>
<thead>
<tr>
<th>Turnover</th>
<th>Optimal</th>
<th>Illiquidity</th>
<th>Average policies</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>Rebalance ($\xi^*$)</td>
<td>Welfare cost</td>
<td>$E[\xi]$</td>
</tr>
<tr>
<td>0</td>
<td>0.593</td>
<td>-</td>
<td>0.593</td>
</tr>
<tr>
<td>1/50 years</td>
<td>0.575</td>
<td>0.011</td>
<td>0.579</td>
</tr>
<tr>
<td>1/10 years</td>
<td>0.555</td>
<td>0.013</td>
<td>0.559</td>
</tr>
<tr>
<td>1/4</td>
<td>0.532</td>
<td>0.016</td>
<td>0.542</td>
</tr>
<tr>
<td>1/2</td>
<td>0.516</td>
<td>0.019</td>
<td>0.522</td>
</tr>
<tr>
<td>1</td>
<td>0.494</td>
<td>0.022</td>
<td>0.512</td>
</tr>
<tr>
<td>2</td>
<td>0.488</td>
<td>0.024</td>
<td>0.501</td>
</tr>
<tr>
<td>4</td>
<td>0.455</td>
<td>0.025</td>
<td>0.481</td>
</tr>
<tr>
<td>10</td>
<td>0.448</td>
<td>0.028</td>
<td>0.464</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.593</td>
<td>-</td>
<td>0.593</td>
</tr>
</tbody>
</table>

The table is computed using the following parameter values: $\gamma = 6$, $\mu = \nu = 0.12$, $r = 0.04$, $\sigma = \psi = 0.15$, and $\rho = 0$. |
Table 6: The Effect of Preference Parameters on Optimal Policies and Welfare

<table>
<thead>
<tr>
<th>λ</th>
<th>i. Optimal Rebalance</th>
<th>ii. Welfare cost</th>
<th>iii. $E(c(1-\xi))$</th>
<th>iv. $E(\theta(1-\xi))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>1.185</td>
<td>0.593</td>
<td>0.356</td>
<td>0.237</td>
</tr>
<tr>
<td>10</td>
<td>0.799</td>
<td>0.493</td>
<td>0.336</td>
<td>0.220</td>
</tr>
<tr>
<td>4</td>
<td>0.666</td>
<td>0.475</td>
<td>0.291</td>
<td>0.201</td>
</tr>
<tr>
<td>2</td>
<td>0.530</td>
<td>0.442</td>
<td>0.285</td>
<td>0.199</td>
</tr>
<tr>
<td>1</td>
<td>0.409</td>
<td>0.373</td>
<td>0.280</td>
<td>0.197</td>
</tr>
<tr>
<td>1/2</td>
<td>0.289</td>
<td>0.246</td>
<td>0.192</td>
<td>0.149</td>
</tr>
<tr>
<td>1/4</td>
<td>0.178</td>
<td>0.132</td>
<td>0.101</td>
<td>0.075</td>
</tr>
<tr>
<td>1/10</td>
<td>0.061</td>
<td>0.047</td>
<td>0.035</td>
<td>0.025</td>
</tr>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>λ</th>
<th>i. Optimal Rebalance</th>
<th>ii. Welfare cost</th>
<th>iii. $E(c(1-\xi))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>0.593</td>
<td>0.593</td>
<td>0.593</td>
</tr>
<tr>
<td>10</td>
<td>0.501</td>
<td>0.500</td>
<td>0.499</td>
</tr>
<tr>
<td>4</td>
<td>0.495</td>
<td>0.490</td>
<td>0.489</td>
</tr>
<tr>
<td>2</td>
<td>0.433</td>
<td>0.439</td>
<td>0.441</td>
</tr>
<tr>
<td>1</td>
<td>0.391</td>
<td>0.398</td>
<td>0.383</td>
</tr>
<tr>
<td>1/2</td>
<td>0.244</td>
<td>0.245</td>
<td>0.248</td>
</tr>
<tr>
<td>1/4</td>
<td>0.114</td>
<td>0.121</td>
<td>0.129</td>
</tr>
<tr>
<td>1/10</td>
<td>0.026</td>
<td>0.038</td>
<td>0.045</td>
</tr>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Unless noted otherwise, the table is computed using the following parameter values: $\zeta = 6$, $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$. 
The figure plots the stationary distribution of allocation to the illiquid asset as a fraction of total wealth, \( x = \frac{X}{X + W} \). The vertical solid gray line corresponds to the value of the optimal rebalancing point \( x^*/(1 + x^*) \), which is the desired allocation to the illiquid asset as a fraction of total wealth at the time of rebalancing. The figure uses \( \gamma = 6, \mu = \nu = .12, r = .04, \lambda = 1, \sigma = \psi = .15, \) and \( \rho = 0 \).
The figure in panel a plots the curvature of the value function with respect to liquid wealth, $-F_{WW}W/F_W$. The figure in panel b plots the elasticity of substitution in the value function between liquid and illiquid wealth, $F_{WX}X/F_{WW}W$. The solid lines represent the case where $\lambda = 1$. The dotted lines correspond to the Merton case $\lambda \to \infty$. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$. 
Figure 3: Optimal Allocation to the Liquid Risky Asset

The figure displays the optimal allocation to the liquid assets. The solid black lines represent allocation to the liquid risky asset taken as a fraction of total wealth, $\theta/(1 + x)$, whereas the dashed lines represent the allocation to the liquid risky asset as a fraction of liquid wealth only, $\theta$. The gray horizontal line corresponds to the allocation to the risky asset in the one- and/or two-asset Merton economy. The vertical gray line is the point $x^*/(1 + x^*)$, which is the optimal holding of illiquid assets relative to total wealth at the arrival of the trading time. The dashed line can be above one because the investor can use the liquid (but not illiquid) risky asset as collateral, as in the standard Merton problem. The solid line must remain between zero and one because the illiquid asset cannot be so used. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = 0.12$, $r = 0.04$, $\lambda = 1$, $\sigma = \psi = 0.15$, and $\rho = 0$. 

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We plot the optimal consumption policy. The solid black line is the consumption policy as a fraction of total wealth, $c/(1 + x)$, and the dashed line depicts consumption policy as a fraction of liquid wealth only, $c$. The horizontal gray lines correspond to consumption in the one- and two-asset Merton benchmarks (consumption is higher in the two-asset case). The vertical solid gray line corresponds to the value of the optimal rebalancing point, $x^*/(1 + x^*)$, which is the desired allocation to the illiquid asset as a fraction of total wealth at the time of rebalancing. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$. 
Figure 5: The Effect of Correlation on Asset Holdings

We plot the optimal allocations to the liquid risky asset (---) and the illiquid risky asset as a fraction of total wealth (---) at the rebalancing time, both as a function of \( \rho \). The remainder is allocated to the riskless asset. The curves are plotted with the following other parameter values: \( \gamma = 6, \mu = .12, \nu = .20, r = .04, \lambda = 1, \) and \( \sigma = \psi = .15 \).
Figure 6: Deterministic versus Stochastic Liquidity, Welfare

A. Stochastic Trading Time

The figure compares the value function ($H(\xi)$) as a function of the portfolio allocation across different expected times until next trade: $E(T) = 1/2$ (grey line) and $E(T) = 2$ (black dotted line). In panel A the trading time is stochastic, so $E(T) = 1/\lambda$; in Panel B the time until the next trade is deterministic, so $E(T) = T$. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$. 

B. Deterministic Trading Time
The figure compares portfolio (Panels A and B) and consumption (Panel C) policies across the liquid (solid line) and illiquid (dotted line) regimes for different degrees of frequency ($\chi_I$), average duration ($1/\chi_L$) and severity ($1/\lambda$) of liquidity crises. Unless noted otherwise, the curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, $\rho = 0$, $\chi_I = 0.1$, $\chi_L = 1/1.5$ and $\lambda = 1$
Figure 8: The Welfare Cost of Illiquidity and the Cost of Insurance

A. Welfare cost of illiquidity

The figure shows the welfare cost of illiquidity (Panel A) and the risk premium associated with liquidity insurance \( \hat{\chi}_I - \chi_I \) (Panel B) for different degrees of frequency \( (\chi_I) \), average duration \( (1/\chi_L) \) and severity \( (1/\lambda) \) of liquidity crises. Unless noted otherwise, the curves are plotted with the following parameter values: \( \gamma = 6, \mu = \nu = .12, r = .04, \sigma = \psi = .15, \rho = 0, \chi_I = 0.1, \chi_L = 1/1.5 \).