Existence of a critical Gompertz parameter and its asymptotic expression

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Abstract

In this paper we provide an interval of existence of critical mortality rate parameters $A$ and $\alpha$ and its asymptotic expressions in Gompertz survival model, in the absence of age specific mortality data.

Keywords: Gompertz critical parameter; Gompertz equation; Mortality rates; Survival function

1. Introduction

Benjamin Gompertz (1825) proposed that the force of mortality—the hazard of death—increased exponentially with age for humans, at least as a serviceable approximation over the range of adult ages for which he had data. Various subsequent researchers, especially in biology and gerontology, have viewed Gompertz observation as a law that describes the process of senescence in almost all multicellular animals at all ages after the onset of reproduction. As a rough approximation at younger adult ages, Gompertz exponential formula does capture the rise in mortality in a great variety of species [1].

Until recently, it was impossible to determine whether this exponential rise continued to advanced ages. For humans, the scattered data available suggested mortality decelerated at the highest ages, but questions about data reliability precluded strong conclusions. For other species, virtually nothing was known about mortality at advanced ages because the populations studied had been too small to permit dependable estimates of death rates at ages that only a small fraction of the starting cohort reached [2,3].

Beyond 30 years of age, the mortality rate for human populations can be approximated by the classic Gompertz exponential function

$$m(t) = Ae^{\alpha t},$$

where the positive parameter $A$ is the age-independent hazard rate coefficient and the positive parameter $\alpha$ is the age-dependent mortality rate coefficient [1,4,5].

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The corresponding Gompertz survival function can be obtained by integrating the mortality rate function equation (1):

\[ S(t) = e^{\frac{A}{\alpha}(1 - e^{\alpha t})}. \]  

(2)

On the basis of the Gompertz mortality rate model, one may estimate the MRD from the maximum lifespan and the overall population mortality rate.

By setting \( S(t) = \frac{1}{N} \), where \( N \) is the initial population size, we obtain the approximation

\[ t^*_m \sim t_m = \frac{\ln(1 + \alpha \ln(N)/A)}{\alpha}. \]  

(3)

where \( t^*_m \) is the “species maximum lifespan” and \( t_m \) is the time point at which only one organism remains in the population. The average mortality rate of a steady-state population subject to age specific mortality rate of Eq. (1) is [6]

\[ A_{av} = \frac{1}{\int_0^\infty S(t)dt}. \]  

(4)

Rearranging Eq. (3) yields

\[ A = \frac{\ln N}{e^{\alpha t_m} - 1}. \]  

(5)

A simple substitution in the integral (4) gives [7]

\[ \alpha = A_{av} e^{\frac{\ln N}{e^{\alpha t_m} - 1}} \int_{\frac{\ln N}{e^{\alpha t_m} - 1}}^{\infty} \frac{e^{-z}}{z} dz. \]  

(6)

Finch et al. [6,8] have solved Eqs. (4) and (5) numerically for \( A \) and \( \alpha \), for a given \( A_{av}, t_m \) and \( N \).

MRDs were calculated for a range of \( N \), because population sizes are unknown. With the Gompertz model MRD depends only on \( \alpha \). Solving Eq. (1) for MRD gives [6],

\[ \text{MRD} = \frac{\ln 2}{\alpha}. \]

2. Estimation of parameters

It is well known that among most mammals, mortality rates are generally lowest at puberty and then accelerate at a constant rate during the major phase of adult life. When examined from puberty onwards, the mortality rate accelerations during adult aging fit the Gompertz model, at least up through the average life span [1,9]. However, extensive deviations from the Gompertz model were recently documented, in which mortality rate accelerations slow markedly by the average life span, e.g., in laboratory populations of fruit flies [10,11], medflies [2], beetles [12], and nematodes [13]. In human populations, according to published studies [14], the acceleration of mortality rate slows after 85 years. After 105 years, the mortality rate appears to cease increasing and may even decrease at these extremely advanced ages. Decreasing mortality at advanced ages is described in detail for flies [10,11,2]. Current evidence indicates that human mortality rates at ages above 85 are less than that predicted by the Gompertz law. In extreme old age, mortality rates may level off or even decline [15–17].

Consider the initial population of size \( N \) at puberty. Following Witten [14], the age at which the mortality rate, \( Ae^{\alpha t} \) of initial population \( N \) has ceased increasing or, equivalently, it tends to a constant [18], is called a critical age, \( t_c \). The remaining population left from an original population size \( N \) surviving at this critical age is called critical population, \( N_c \), and the corresponding Gompertz parameter in \( Ae^{\alpha t} \) is called critical Gompertz parameter, \( \alpha_c \).

As an extension of our earlier work [19], in this paper we provide an estimation of critical Gompertz parameter. Our estimation is valid for the age (critical), or the population (critical), at which mortality rate has ceased increasing and has thus reached a steady state. We also provide the asymptotic solution to the critical Gompertz parameter for a critical population. To our great surprise, the asymptotic formula given for a large population in our earlier work [19]
we obtain

it follows that we have not yet diminished to one survivor, the above said

t and it increases as 

N since ln 

2.2. Critical population \( N_c \)

we obtain the asymptotic solution of \( A \) zero at \( \alpha \) of \( A \) showing that the mouse mortality function also approaches a constant value at higher age 

respectively. From 

given by (see also, Witten and Satzer [20])

To determine the critical Gompertz parameter, \( \alpha_c \), first we use the identity 

\[
\frac{\alpha A_{av} t_m}{\ln N} = \frac{\alpha t_m}{e^{\alpha t_m} - 1}.
\]

From (8) it follows that

\[
A_{av} t_m \left\{ \begin{array}{ll}
< 1 & \text{if } \alpha > 0, \\
= 1 & \text{if } \alpha = 0, \\
> 1 & \text{if } \alpha < 0.
\end{array} \right.
\]

The point \( \frac{A_{av} t_m}{\ln N} = 1 \) at which \( \alpha \) changes sign is said to be the critical point of \( \alpha \).

Quite recently Wang and co-workers disclosed an elegant experiment for senescence accelerated mouse (SAM), showing that the mouse mortality function also approaches a constant value at higher age [21].

To determine the critical Gompertz parameter, \( \alpha_c \), we use the identity \( \frac{A_{av} t_m}{\ln N} = 1 \) to obtain the critical values of \( \alpha \), namely, critical population, \( N_c \) and critical age, \( t_c \). Since the partials of \( \alpha \) with respect to \( N \), \( A_{av} \) and \( t_m \) become zero at \( \frac{A_{av} t_m}{\ln N} = 1 \). Note that the condition \( \frac{A_{av} t_m}{\ln N} = 1 \) is necessary to have a constant mortality rate function. Finally we obtain the asymptotic solution of (6), for the critical values, \( N_c \) and \( t_c \).

Now we shall prove the existence of critical population and critical age.

2.2. Critical population \( N_c \)

For a given \( A_{av} \), \( t_m \), and \( N \) with \( \frac{A_{av} t_m}{\ln N} < 1 \) there exists a critical population \( N_c \) and is given by \( e^{A_{av} t_m} \). Indeed, since \( \ln N > A_{av} t_m \) we can take \( \ln N_c = A_{av} t_m \), or, \( N_c = e^{A_{av} t_m} \). For instance, (see Table I in [19]) when (Human) \( A_{av} = 0.015 \), \( t_m = 105 \), 115, and 120, we find that \( N_c = 4.8 \), 5.6, and 6.04, respectively. Note that \( N_c \) is not an integer and it increases as \( t_m \). Thus the remaining population (critical population) approximates 5 or 6. Since the population has not yet diminished to one survivor, the above said \( t_m \) can be treated as critical life span, \( t_c \). From this, we obtained
\[ t_c = \frac{\ln N_c}{A_{av}} \]  
To study the mortality rate of the remaining critical population, we need to consider critical Gompertz parameter, \( \alpha_c \), because \( \frac{A_{av}t_m}{\ln N} = 1 \) when \( N = N_c \). Thus, we conclude that

\[
\text{when } \frac{A_{av}t_m}{\ln N} < 1, \quad t_c = t_m \quad \text{and} \quad N_c = e^{A_{av}t_c}.
\]  \( \text{(10)} \)

Clearly, when \( \frac{A_{av}t_m}{\ln N} < 1 \), both \( t_c \) and \( t_m \) are same.

2.3. Critical lifespan \( t_c \)

On the contrary, when \( \frac{A_{av}t_m}{\ln N} > 1 \) it is trivial to find the critical life span. As \( t_m > \frac{\ln N}{A_{av}} \), we can take \( t_c = \frac{\ln N}{A_{av}} \). For instance, (see Table I, [19]) when (European robin) \( A_{av} = 0.62, t_m = 12 \) and \( N = 10^3 \), we find that \( t_c = 11.14 \). Similarly, when (Starling) \( A_{av} = 0.52, t_m = 20 \) and \( N = 10^3, 10^4 \), we obtain \( t_c = 13.28, 17.712 \), respectively.

Thus, we conclude that

\[
\text{when } \frac{A_{av}t_m}{\ln N} > 1, \quad t_c = \frac{\ln N}{A_{av}}, \quad N_c = e^{A_{av}t_c} \quad \text{and} \quad t_c \neq t_m.
\]  \( \text{(11)} \)

Clearly, when \( \frac{A_{av}t_m}{\ln N} > 1 \), \( t_c < t_m \). Thus, in general, for any given \( A_{av}, t_m \) and \( N \), we get \( t_c \leq t_m \).

3. Interval of existence of critical solution

Recall that in [19] we obtained the asymptotic solution of Eq. (6) for a large \( N \) with \( \frac{A_{av}t_m}{\ln N} < 1 \). The asymptotic solution is given by

\[
\alpha = \frac{1}{t_m} \ln \left[ 1 + \exp \left( \frac{A_{av}t_mC}{A_{av}t_m - 1} \right) \left( \ln N \right)^{\frac{\ln N}{A_{av}t_m - 1}} \right]
\]  \( \text{(12)} \)

and

\[
\alpha = \frac{1}{t_m} \ln \left[ 1 + \frac{\ln N - C \ln N - 1}{\ln(\ln N) + C} \right]
\]  \( \text{(13)} \)

when \( A_{av}t_m \neq 1 \) and \( A_{av}t_m = 1 \) respectively, and \( C = 0.577215 \), Euler’s constant.

We remark that when \( \frac{A_{av}t_m}{\ln N} > 1 \) [the cases (European robin) with \( N = 10^3 \) and (Starling) with \( N = 10^3 - 4 \)] the above asymptotic formulae remain valid, provided \( \frac{A_{av}t_m}{\ln N} < 1 \) or \( N > N_c \).

Our immediate concern is to extend the above formulae to \( N \geq N_c \). It is amazing to learn that the formulae (12) and (13) derived for a large \( N \) equally hold good for \( N = N_c \). Here is the proof:

Upon substitution \( N = N_c \) into (6), we get

\[
\alpha_c = A_{av}e^{\frac{\ln N_c}{\ln N} - 1} \int_{\frac{\ln N_c}{\ln N} - 1}^{\infty} \frac{e^{-z}}{z} \, dz,
\]  \( \text{(14)} \)

where \( t_c = t_m \) when \( \frac{A_{av}t_m}{\ln N} < 1 \) and \( t_c = \frac{\ln N_c}{A_{av}} \) when \( \frac{A_{av}t_m}{\ln N} > 1 \). Eq. (14) gives

\[
\alpha_c \leq A_{av}e^{\frac{\ln N_c}{\ln N} - 1} \left( \frac{e^{\alpha_c t_c} - 1}{\ln N_c} \right) \int_{\frac{\ln N_c}{\ln N} - 1}^{\infty} e^{-z} \, dz
\]  

\[= A_{av}e^{\frac{\ln N_c}{\ln N} - 1} \left( \frac{e^{\alpha_c t_c} - 1}{\ln N_c} \right) e^{\frac{\ln N_c}{\ln N} - 1}.
\]

Hence

\[
\alpha_c t_c \leq A_{av}t_c \left( e^{\alpha_c t_c} - 1 \right).
\]
The above inequality ensures the existence of solution \( \alpha_{c} \), provided \( \frac{A_{av}t_{c}}{\ln N_{c}} \rightarrow 1 \). As a consequence, we get
\[
\alpha_{c}t_{c} \leq e^{\alpha_{c}t_{c}} - 1. \tag{15}
\]
Obviously, any \( \alpha_{c} \geq 0 \) will satisfy Eq. (15).

To obtain the solution of (14) for every fixed \( A_{av}, t_{c} \) and \( N_{c} \), choose any \( \alpha^{*} \) from the solution interval \([0, \infty)\). As every \( \alpha_{c} \geq \alpha^{*} \) satisfies (15), let us find the smallest possible of these. To achieve this, fix one such \( \alpha^{*} \). Substitute this into the right-hand side of (14) to get
\[
\alpha_{c} = \frac{\ln N_{c}}{A_{av}t_{c}} e^{\alpha_{c}} - 1 \int_{\ln N_{c} / e^{\alpha_{c}t_{c}}}^{\infty} \frac{e^{-z}}{z} dz, \tag{16}
\]
where \( \epsilon = \frac{e^{\alpha_{c}t_{c}} - 1}{e^{\alpha_{c}t_{c}} - 1} \leq 1 \), since \( \alpha_{c} \geq \alpha^{*} \).

We observe that by sending \( \epsilon \) to zero in (16), one can obtain the asymptotic value of \( \alpha_{c} \) for \( N = N_{c} \), as we have
\[
0 \leq \alpha_{c}t_{c} \leq \frac{A_{av}t_{c}}{\ln N_{c}} \left( \frac{e^{\alpha_{c}t_{c}} - 1}{\epsilon} \right)
\]
from (15).

If we send \( \epsilon \) to zero, on account of 8.212, 1. in [22], we get
\[
\alpha_{c} = \lim_{\epsilon \to 0} \frac{1}{t_{c}} \ln \left[ \frac{e^{\alpha_{c}t_{c}} - 1}{\epsilon} + 1 \right] = \lim_{\epsilon \to 0} \frac{1}{t_{c}} \ln \left[ \frac{e^{\alpha_{c}t_{c}} - 1}{\epsilon} \right] + \lim_{\epsilon \to 0} \frac{1}{t_{c}} \ln \left[ 1 + \frac{\epsilon}{e^{\alpha_{c}t_{c}} - 1} \right] = \lim_{\epsilon \to 0} A_{av} e^{\alpha_{c}t_{c}} - 1 \epsilon \int_{\ln N_{c} / e^{\alpha_{c}t_{c}}}^{\infty} \frac{e^{-z}}{z} dz = \lim_{\epsilon \to 0} A_{av} e^{\alpha_{c}t_{c}} - 1 \epsilon \left[ -C - \ln \left( \frac{\ln N_{c}}{e^{\alpha_{c}t_{c}} - 1} \right) + \int_{0}^{\ln N_{c} / e^{\alpha_{c}t_{c}}} \frac{\epsilon}{\tau} \right] \right],
\]
where \( C = 0.577215 \), Euler’s constant.

Retaining dominant \( \ln \frac{1}{\epsilon} \) term only, after a little algebra, we get
\[
(1 - A_{av}t_{c}) \ln \frac{1}{\epsilon} = -A_{av}t_{c}C - A_{av}t_{c} \ln \left[ \frac{\ln N_{c}}{e^{\alpha_{c}t_{c}} - 1} \right] - \ln \left[ e^{\alpha_{c}t_{c}} - 1 \right] \tag{17}
\]
or, equivalently,
\[
\epsilon = \exp \left\{ \ln \left[ e^{\alpha_{c}t_{c}} - 1 \right] - \frac{A_{av}t_{c}C}{A_{av}t_{c} - 1} \right\}.
\]
A simple substitution into \( \frac{1}{t_{c}} \ln \left[ \frac{e^{\alpha_{c}t_{c}} - 1}{\epsilon} + 1 \right] \) yields the required asymptotic formula
\[
\alpha_{c} = \frac{1}{t_{c}} \ln \left[ 1 + e^{\frac{\ln N_{c}}{A_{av}t_{c}} - 1} \left( \ln N_{c} \right)^{A_{av}t_{c} - 1} \right] \tag{18}
\]
of (14) for \( N = N_{c} \) with \( \frac{A_{av}t_{c}}{\ln N_{c}} = 1 \).

Strangely, on account of (17), the above formula fails at \( A_{av}t_{c} = 1 \).

But when \( A_{av}t_{c} = 1 \) we have
\[
0 \leq \alpha_{c} \leq \frac{1}{t_{c} \ln N_{c}} (e^{\alpha_{c}t_{c}} - 1)
\]
from (15). Now setting \( (e^{\alpha_c} - 1) \frac{1}{\ln N_c} = \frac{1}{\epsilon} \), substitute this into Eq. (14), to get

\[
\alpha_c = \frac{1}{t_c} e^\epsilon \int_{\epsilon}^{\infty} \frac{e^{-z}}{z} dz.
\]

If we send \( \epsilon \) to zero, on account of 8.212, 1. in [22], we get

\[
\alpha_c = \lim_{\epsilon \to 0} \frac{1}{t_c} \ln \left[ \frac{\ln N_c}{\epsilon} + 1 \right]
= \lim_{\epsilon \to 0} \frac{1}{t_c} \ln \left[ \frac{\ln N_c}{\epsilon} \right] + \lim_{\epsilon \to 0} \frac{1}{t_c} \ln \left[ 1 + \frac{\epsilon}{\ln N_c} \right]
= \lim_{\epsilon \to 0} \frac{1}{t_c} \int_{\epsilon}^{\infty} \frac{e^{-z}}{z} dz
= \lim_{\epsilon \to 0} \frac{1}{t_c} e^\epsilon \left[ -C - \ln \epsilon + \int_{0}^{\epsilon} \frac{(1 - e^{-\tau})}{\tau} d\tau \right].
\]

As the dominant \( \frac{1}{\epsilon} \) term vanishes, we consider only \( \epsilon \)-order term. After a little algebra, we obtain

\[
\frac{1}{t_c} \ln(\ln N_c) + \frac{\epsilon}{t_c \ln N_c} = \frac{1}{t_c} [(1 + \epsilon)(-C + \epsilon)]
\]

or,

\[
\epsilon = \frac{\ln(\ln N_c) + C}{1 - C - \frac{1}{\ln N_c}}.
\]

A simple substitution into \( \frac{1}{t_c} \ln \left[ \frac{\ln N_c}{\epsilon} + 1 \right] \) yields the required asymptotic formula

\[
\alpha_c = \frac{1}{t_c} \ln \left[ 1 + \frac{\ln N_c - C \ln N_c - 1}{\ln(\ln N_c) + C} \right]
\]

of (14) for \( N = N_c \) when \( A_{av} t_c = 1 \).

To sum up, we have

**Theorem.** For every fixed \( A_{av} \), \( t_m \) and \( N \) satisfying either (10) or (11), let \( I_1 \) and \( I_2 \) be intervals defined by

\[
I_1 = \left[ 0, \frac{1}{t_c} \ln \left[ 1 + \frac{\ln N_c}{\ln(\ln N_c) + C} \right] \right]
I_2 = \left[ 0, \frac{1}{t_c} \ln \left[ 1 + \frac{\ln N_c - C \ln N_c - 1}{\ln(\ln N_c) + C} \right] \right]
\]

where \( C = 0.577215 \), Euler's constant.

Suppose there exists a unique solution of (6) in \( I_1 \) and \( I_2 \) respectively, when \( A_{av} t_c \neq 1 \) and \( A_{av} t_c = 1 \). Then it is necessary that \( \frac{A_{av} t_m}{\ln N_c} = 1 \).

Moreover, the asymptotic solution of (6) for \( N = N_c \) is given by (18) and (19) respectively, when \( A_{av} t_c \neq 1 \) and \( A_{av} t_c = 1 \).

Note that care must be taken while using the asymptotic solution (18) and (19), since the conditions (10) and (11) are crucially dependent on whether given \( \frac{A_{av} t_m}{\ln N} < 1 \) or \( \frac{A_{av} t_m}{\ln N} > 1 \).

4. Conclusion

The above theorem states that the asymptotic solution \( \alpha \) of (6) is a continuous function in the variables \( A_{av} \), \( t_m \) and \( N \) from puberty through critical life span (or, \( N \geq N_c \)). What happens to solution \( \alpha \) when \( N \leq N_c \) (below the critical population)? As we have already equated \( t_m \) with \( t_c \) when \( \frac{A_{av} t_m}{\ln N} < 1 \), what then is the actual (species) maximum life span, \( t_m^* \)? Such \( t_m^* > t_m \) exists, since the critical population \( N_c \) has not yet diminished to one survivor. How to
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determine this $t_m^*$? This in turn, leads us to consider negative Gompertz parameter, since $\frac{A_{aVt}}{\ln N} \geq 1$ for $N \leq N_c$. All these questions remain to be addressed.

**Remark 1.** From (5) it is easy to get the asymptotic formula of $A$ using (18) and (19) for $N = N_c$.

**Remark 2.** We listed in Table 1 the numerical values of asymptotic solution (18) for $N = N_c$ for a comparison with that for a large $N$.

**Remark 3.** There were no recorded samples with $A_{aVt} = 1$ to compare with the asymptotic solution (19) for $N = N_c$.

**Remark 4.** We note that for humans the critical MRD is three times larger than the asymptotic MRD. In other words, the age-dependent parameter $\alpha$ at critical age is three times smaller than that from puberty through average life span.

**Acknowledgement**

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**References**


