Finite-Dimensional Attractors for a General Class of Nonautonomous Wave Equations

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Abstract—Our aim in this note is to construct attractors and exponential attractors for a general class of nonautonomous semilinear wave equations. Following the approach described in [1], we define a semigroup $S(t)$ associated to an autonomous system, and then prove, using an energy functional, that $S(t)$ is an $\alpha$-contraction and satisfies the squeezing property. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this note, we study the long time behavior of a general wave equation of the form

$$Pu'' + Qu' + Au + F(t, u) = h(t),$$

in a Hilbert space $H$, where $A$, $P$, and $Q$ are linear positive self-adjoint operators. This class of equations contains different types of wave equations previously studied (see [1-5] and the references therein). For instance, this class of equations contains the Sine-Gordon equation (see [5])

$$u'' + \alpha u' - \Delta u + \beta \sin u = f(t), \quad \alpha > 0,$$

which models the dynamics of a Josephson junction driven by a current source. The forcing term $f$ is proportional to the current intensity applied to the junction, it is thus, natural to
consider forcing terms that depend explicitly on the time. One can also consider generalized beam equations (see [6]) of the form

\[ u'' + (\delta + \gamma A) u' + \alpha A^2 u + f \left( |A^{1/2}u| \right)^2 A u = g(t), \quad \alpha, \gamma, \delta > 0, \] (1.3)

with time dependent forcing terms. Here, we restrict ourselves to the case where \( F \) and \( h \) are quasiperiodic with respect to the time. Some of our results can, however, be extended to the other types of time dependence considered in [1].

Using the general framework developed by Chepyzhov and Vishik in [1], we study in fact a family of equations, which enables us to construct a semigroup \( S(t) \) on an extended space (see Section 2 below, we also refer the readers to [7,8], and the references therein for another approach of nonautonomous equations (actually, these are the first known results for nonautonomous systems) based on the so-called skew product flow). We then study the long time behavior of \( S(t) \).

After having established the existence of an absorbing set, we prove, using an appropriate energy functional, that \( S(t) \) is an \( \alpha \)-contraction in the sense of [9], and thus, obtain the existence of the global attractor which is a compact invariant set which attracts uniformly the trajectories as time goes to infinity. We also prove that \( S(t) \) satisfies the squeezing property (see [2]), and obtain the existence of an exponential attractor which is a compact positively invariant set which has finite fractal dimension, contains the global attractor, and attracts exponentially the trajectories (see [2]).

2. SETTING OF THE PROBLEM

Let \( H \) be a separable Hilbert space which we endow with the scalar product \((\cdot,\cdot)\) and the associated norm \(|\cdot|\). We consider the following equation in \( H \):

\[
P u'' + Q u' + A u + F(t, u) = h(t), \quad t \geq \tau, \quad \tau \in \mathbb{R}. \tag{2.1}
\]

\[
u(\tau) = u_0, \quad u'(\tau) = u_1, \tag{2.2}
\]

We assume that \( A, P, \) and \( Q \) are linear positive self-adjoint operators and that \( A \) has compact inverse. We also assume that \( D(A) \subset D(Q) \subset D(P) \) with continuous injections and that there exist three positive constants \( \alpha, \beta, \) and \( \lambda \) such that

\[
\alpha |u| \leq |Q^{1/2}u| \leq \beta |A^{1/2}u|, \quad \forall u \in D\left(A^{1/2}\right), \tag{2.3}
\]

\[
|P^{1/2}u| \leq \lambda |Q^{1/2}u|, \quad \forall u \in D\left(Q^{1/2}\right). \tag{2.4}
\]

Finally, we assume that \( AP = PA, QA = AQ, \forall u \in D(A) \).

We set \( X = D\left(A^{1/2}\right) \times D\left(P^{1/2}\right) \) and we endow this space with the scalar product (which we still denote \((\cdot,\cdot)\) defined by \((u,v), (w,z)) = (A^{1/2}u, A^{1/2}w) + (P^{1/2}v, P^{1/2}z)\).

We assume that \( F \in C_b^1\left(R, C_b(D(A^{1/2}), D(Q^{-1/2}))\right)\), where the index \( b \) means that we consider bounded functions, and that there exists \( G \in C_b^2\left(R, C_b(D(A^{1/2}), R)\right) \) such that \( d_u G(t, u) \cdot v = (F(t, u), v), \forall t \in R, \forall u, v \in D(A^{1/2}) \). We also assume that there exists a positive constant \( c_0 \) such that

\[
(F(t, u), u) - G(t, u) \geq c_0, \tag{2.5}
\]

\[
G(t, u) \geq -c_0. \tag{2.6}
\]

\( \forall t \in R, \forall u \in D(A^{1/2}) \). Finally, we assume that \( h \in C_b^1\left(R, D(Q^{-1/2})\right) \).
We now assume that $F$ and $h$ are quasi-periodic with respect to the time, and more precisely that $(F(t, \cdot), h(t)) = (F(at, \cdot), h(at))$, where $a = (a_1, \ldots, a_k)$, and $\omega_i \mapsto (F(\omega_1, \ldots, \omega_k, \cdot), h(\omega_1, \ldots, \omega_k))$ is $2\pi$-periodic, $i = 1, \ldots, k$; the $\alpha_i$ being rationally independent.

Following the construction given in [1], we study in fact the family of equations

$$Pu'' + Qu' + Au + F(\omega(t), u) = h(\omega(t)), \quad t \geq \tau, \quad \tau \in \mathbb{R},$$

where $\omega(t) = \alpha t + \sigma \pmod{T^k}$, $\sigma \in T^k$, $T^k$ being the $k$-dimensional torus.

We associate to (2.7), (2.8) the following autonomous system:

$$Pu'' + Qu' + Au + F(\omega, u) = h(\omega),$$

(2.9)

$$\omega = \alpha,$$

(2.10)

$$u(0) = u_0, \quad u'(0) = u_1,$$

(2.11)

$$\omega(0) = \sigma.$$  

(2.12)

In order to obtain the existence of solutions for (2.7), (2.8) (and consequently, for (2.9)–(2.12)), we consider the equation satisfied by

$$E_1(t) = \frac{1}{2} \left| P^{1/2}u \right|^2 + \frac{1}{2} \left| A^{1/2}u \right|^2 + G(t, u),$$

(2.13)

and we justify the formal estimates by considering Galerkin approximations. To have uniqueness of solutions, we make an assumption of the form

$$Q^{-1/2}(F(at + \sigma, u) - F(at + \sigma, v)) \leq M \left( \left| A^{1/2}u \right|, \left| A^{1/2}v \right| \right) (|A\gamma(u - v)| + |\sigma - \sigma'|),$$

(2.14)

$\forall u, v \in D(A^{1/2})$, $\forall \sigma, \sigma' \in T^k$, where $|\sigma|^2 = \sum_{i=1}^k \sigma_i^2$ if $\sigma = (\sigma_1, \ldots, \sigma_k)$ and $0 < \gamma < 1/2$.

This allows us to define the family of processes $U_\sigma(t, \tau)$ on $X$ defined by $U_\sigma(t, \tau)(u_0, u_1) = (u(t), u'(t))$, where $u$ is the solution of (2.7), (2.8), and the semigroup $S(t)$ on $X \times T^k$ defined by $S(t)(u_0, u_1, \sigma) = (u(t), u'(t), \omega(t))$, where $(u, \omega)$ is the solution of (2.9)–(2.12).

3. CONSTRUCTION OF FINITE-DIMENSIONAL ATTRACTORS

We first give the following definitions (see [1,10]).

**Definition 3.1.**

(a) A compact set $A_{T^k} \subset X$ is called the uniform attractor for the family of processes $U_\sigma(t, \tau)$ acting on $X$ if:

(i) $\forall B \subset X$ bounded, $\limsup_{t \to +\infty} \sup_{\sigma \in T^k} \text{dist}_X(U_\sigma(t, \tau)B, A_{T^k}) = 0$,

(ii) (minimality property) $\forall A' \subset X$ closed and bounded satisfying (i), $A_{T^k} \subset A'$.

(b) A compact set $M_{T^k} \subset X$ is a uniform exponential attractor for the family of processes $U_\sigma(t, \tau)$ acting on $X$ if:

(i) $A_{T^k} \subset M_{T^k} \subset X$,

(ii) $M_{T^k}$ has finite fractal dimension,

(iii) $\forall B \subset X$ bounded, there exist two constants $c_1$ and $c_2$ that depend only on $B$ such that $\sup_{\sigma \in T^k} \text{dist}_X(U_\sigma(t, \tau)B, M_{T^k}) \leq c_1 e^{-c_2(t-\tau)}$, $\forall t \geq \tau$, $\tau \in \mathbb{R}$.

In order to prove the existence of the uniform attractor and of uniform exponential attractors for the family of processes associated to (2.7), (2.8), we first prove the existence of the global attractor and of exponential attractors for the semigroup $S(t)$ associated to (2.9)–(2.12). We then project on $X$ (see [11]).
We first prove the existence of a uniform (with respect to $\sigma$) bounded absorbing set $B$ for $U_\sigma(t,T)$, which then yields the existence of an absorbing set ($B \times T^k$ works here) for $S(t)$. To do so, we consider the functional $\Phi_\eta(u,v) = (1/2)|P^{1/2}u|^2 + (1/2)|A^{1/2}u|^2 + G(t,u) - (h(t),u) + \eta(Pu,v) + (\eta/2)|Q^{1/2}u|^2$ defined on $X$ (see [3]). If $u$ is a solution of (2.7), (2.8), we have if $0 < \delta < \eta, \eta$ small enough (see [3]),

$$\frac{d}{dt} \Phi_\eta(u,u') + \delta \Phi_\eta(u,u') \leq K(c_0,\lambda,\beta,h) + \frac{dG}{dt}(t,u) - (h'(t),u),$$

(3.1)

which yields, since $|(h',u)| \leq \epsilon|Q^{1/2}u|^2 + c(\epsilon)|Q^{-1/2}h'|^2, \forall \epsilon > 0$, and taking $\epsilon$ small enough so that $\epsilon|Q^{1/2}u|^2$ can be absorbed in the term $(\delta/2)\Phi_\eta(u,u')$,

$$\frac{d}{dt} \Phi_\eta(u,u') + \frac{\delta}{2} \Phi_\eta(u,u') \leq K(c_0,\lambda,\beta,G,h).$$

(3.2)

We then consider two solutions $(u,\omega)$ and $(v,\bar{\omega})$ of (2.9)-(2.12). We set $w = u - v$. The function $w$ satisfies the following equation:

$$Pw'' + Qw' + Aw + F(\alpha t + \sigma, u) - F(\alpha t + \sigma, v) = h(\alpha t + \sigma) - h(\alpha t + \sigma).$$

(3.3)

We set $E_2(t) = |P^{1/2}w'|^2 + |A^{1/2}w|^2 + |Q^{1/2}w|^2 + 2(Pw',w) + |\sigma - \bar{\sigma}|^2$. We prove as in [3], using (2.14), that

$$\frac{d}{dt}E_2(t) + \eta E_2(t) \leq c\left(|A^{1/2}w|^2 + |\sigma - \bar{\sigma}|^2\right).$$

(3.4)

if $\eta > 0$ is properly chosen. It then follows by equivalence of norms that

$$\frac{d}{dt}E_2(t) + \eta E_2(t) \leq c' E_2(t).$$

(3.5)

This last inequality allows us to prove that $S(t)$ is an $\alpha$-contraction in the sense of [9]. This yields the existence of the global attractor $A$ for $S(t)$ on $X \times T^k$.

Furthermore, following the same steps, except that we first project the equations (we thus assume that there exists an orthonormal basis of $H$, which we denote $(w_n)_{n \in \mathbb{N}}$ such that the orthogonal projectors $\Pi_n : H \rightarrow \text{Vect}(w_1,\ldots,w_n)$ commute with $A, P$, and $Q$), we can prove the squeezing property. To do so, we need an assumption of the form

$$|A^{1/2}Q^{-1/2}(F(\alpha t + \sigma, u) - F(\alpha t + \sigma, v))| \leq M\left(|A^{1/2}u|,|A^{1/2}v|\right)\left(|A^{1/2}(u - v)| + |\sigma - \bar{\sigma}|\right),$$

(3.6)

$\forall u,v \in D(A^{1/2}), \forall \sigma,\bar{\sigma} \in T^k, \delta > 0$ (in practical situations, it suffices to verify this assumption on a bounded absorbing set, and $\delta$ is small, see [3]). Moreover, we consider the projectors $\mathcal{P}_n$ defined by $\mathcal{P}_n(u,v,\sigma) = (\Pi_n u, \Pi_n v, \sigma)$. These projectors are thus, orthogonal projectors with finite rank on $X \times T^k$. We then have the following theorem.

**Theorem 3.1.** The semigroup $S(t)$ possesses an exponential attractor $M$ on a proper positively invariant compact subset $Y$ of $X \times T^k$. Therefore, the global attractor $A$ has finite dimension.

We refer the reader to [3] where different types of wave equations (and in particular those presented in the introduction) that can be written in the form (1.1) are presented. We can easily adapt the proofs performed in the autonomous case to the present situation.

**Remark 3.1.**

(a) We could only consider quasiperiodic in time functions in order to prove the squeezing property. Indeed, we needed to construct orthogonal projectors with finite rank on the extended space (and more precisely to treat the $T^k$-component). This assumption was not
necessary in the first part of the proof (i.e., to obtain an $\alpha$-contraction) and more general
time dependence (see [1]) can be considered.

(b) We can easily adapt the proofs presented in this note to the case where $A$ varies slowly
with respect to the time (this will be precised below), i.e., we consider an equation of the form

$$Pu'' + Qu' + A(t)u + F(t, u) = h(t),$$

where $A(t)$ is also quasi-periodic (with $k$ modes), and to which we associate the au-
tonomous system

$$Pu'' + Qu' + A(\omega)u + F(\omega, u) = h(\omega),$$

$$\omega' = \alpha.$$  

In this case, we assume that $A(\omega)$ is a linear self-adjoint positive operator with compact inverse,
$\forall \omega \in T^k$. Furthermore, we assume that $\forall s > 0$, $\exists D_A \subset H$ with compact injection such that
$D(A(\omega)^s) = D_A$, $\forall \omega \in T^k$, and that there exist two positive constants $c_1$ and $c_2$ such that

$$c_1|u|_{D_A^s} \leq |A(\omega)^s u| \leq c_2 |u|_{D_A^s},$$  

$\forall \omega \in T^k$. Finally, we assume that if $A(t)^s : V_1^s \rightarrow V_2^s$, then $[A(t)^s]' : V_1^s \rightarrow V_2^s$ and

$$\left|[A(t)^s]' u\right|_{V_2^s} \leq c_s |A(t)^s|_{V_2^s},$$

where $c_s$ does not depend on $t$. We then make an assumption similar to (2.3) (with constants
that do not depend on the time) and consider the functionals

$$E_1(t) = \frac{1}{2} \left|P^{1/2}u\right|^2 + \frac{1}{2} \left|A(t)^{1/2}u\right|^2 + G(t, u),$$

$$E_2(t) = \left|P^{1/2}w\right|^2 + A(t)^{1/2}w^2 + Q^{1/2}w^2 + 2 (Pw, w) + |\sigma - \sigma|^2.$$  

The only difficulty here is to prove the existence of a bounded absorbing set. Using $\Phi_\eta$ defined
in (3.13), we have (for $\eta$ and $\delta$ small enough) an inequality of the form

$$\frac{d}{dt} \Phi_\eta (u, u') + \delta \Phi_\eta (u, u') \leq -\epsilon \left|A(t)^{1/2}u\right|^2 + \left|[A(t)^{1/2}]' u\right| A(t)^{1/2}u + K,$$

where $\epsilon$ is small. We can thus, conclude if we have an inequality of the form

$$\left|[A(t)^{1/2}]' u\right| \leq \epsilon \left|A(t)^{1/2}u\right|.$$  

Finally, in order to prove the squeezing property and construct projectors, we assume that $D_A$
is the domain of an operator $\widetilde{A}$, and we consider the spectrum of $\widetilde{A}$.

(c) The case where $P$ and $Q$ also depend on the time is more involved and will be treated in [12].
REFERENCES


