Optimal Mechanism for Selling a Set of Commonly-Ranked Objects

Juan Feng*

(352) 392-0648

jane.feng@cba.ufl.edu

Warrington College of Business

University of Florida, Gainesville, FL

March 6, 2007

*University of Florida (jane.feng@cba.ufl.edu). I am grateful to Kalyan Chatterjee for his invaluable guidance and support. I also thank Hemant Bhargava, Anthony Kwasnica, Tomas Sjostrom, David Sappington, Jinhong Xie, Max Shen, Paul Zipkin and Ira Horowitz for their insightful comments. The remaining errors are mine.

Optimal Mechanism for Selling a Set of Commonly-Ranked Objects

March 6, 2007

Abstract

This paper designs the optimal mechanism for selling a set of commonly-ranked objects. While buyers rank these objects in the same order, the rates at which their valuations change for a less-preferred object might be different. Four stylized cases are identified according to this difference: "parallel," "convergent," "divergent," and "convergent then divergent." The optimal mechanism imposes a reserve price for each of the positions. Depending on which of the four stylized cases is considered, a higher-type bidder may be allocated a higher-ranked or lower-ranked position. There is also a positive probability that a higher-ranked object is not allocated while a lower-ranked one *is* allocated. In a departure from the extant mechanism-design literature, the individual-rationality (IR) constraint for a mid-range type of bidder can be binding.

Keywords: slot allocation, optimal mechanism, common ranking, auction

Optimal Mechanism for Selling a Set of Commonly-Ranked Objects

1 Introduction

Selling a set of commonly-ranked objects is ubiquitous in the business world. For example, grocery stores often allocate shelf spaces to manufacturers by accepting "slotting allowances"(Kim 2005), where the shelf space at the more easily accessible height or location is usually the more highly valued. The FTC (FTC 2003) has estimated that it takes $1 to over $2 million in slotting allowances to introduce a new grocery product nationwide, and the practice has spread to such industries as computer software, compact discs, books, magazines, apparel, over-the-counter drugs, and tobacco products (Bloom, Gundlach, and Cannon 2000).

Another example is the popular practice of selling advertisement positions on search pages, which is commonly adopted by leading Internet players such as Yahoo! (Overture), Google (AdWords and Adsense), MSN (AdCenter), and eBay (AdContext). Since it is commonly accepted that a higher placement on a search page leads to higher traffic, and, eventually, to an increased financial payoff (Cotriss 2002), a higher slot is more valuable to advertisers than its successors. These keyword auctions generate more than $5 billion, constitute 41% of the total internet advertising revenue in 2005, [1], and are widely credited for the revitalization of the search engine business.[2]

These practices attract academic researchers in various areas. In the Marketing literature Bloom, Gundlach, and Cannon (2000), Sudhir and Rao (2006), and Kim (2005) explore whether slotting allowances enhance efficiency or hinder competitions. Retailers can use slotting allowances to screen a high-demand product from a low-demand one (Chu 1992), and manufacturers might use them to signal its potential high demand to a retailer (Desai 2000). Whether slotting allowances are used as signals depends on the retailer's cost structure and information asymmetry in the distribution channel (Lariviere and Padmanabhan 1997). Rao and Mahi (2003) identify factors that might influence the relative magnitude of slotting allowances. These papers, however, do not consider the allocation of multiple shelf slots to manufacturers, which is the focus of this paper. Other related allocation problems include the one-sided matching problem such as assigning

[1]http://www.iab.net/news/pr_2006_04_20.asp

[2]http://www.economist.com/displaystory.cfm?story_id=1932434.

dormitory to college students (Bogomolnaia and Moulin 2002), where the preferences of the students are known to the allocator. In the present paper, however, the buyer preferences are only known through a distribution function; and the "right-to-choose" auction, where bidders bid for the right to choose their ideal products in a sequence of auctions (Eliaz, Offerman, and Schotter 2005). There are also papers studying how to allocate the advertising spaces on a search page through the keyword auctions (Feng, Bhargava, and Pennock forthcoming, Feng, Shen, and Zhan forthcoming, Liu and Chen forthcoming). Different from these papers which study a given mechanism, the purpose of the present paper is to design an *optimal* mechanism.

The most closely related literature addresses optimal auction design. The optimal auction mechanism for selling a single object was first established in Myerson (1981), and extended to the case of multiple identical objects by Maskin and Riley (1990). Auctions of multiple heterogenous objects are usually complex to analyze. Many papers focus on modifying a small set of design parameters for a given mechanism, such as whether or not to sell the items separately or in bundles (Palfrey 1983, Avery and Hendershott 2000, Armstrong 2000), or whether a uniform-price auction generates more revenue than a "pay-as-bid" auction when consumers demand multiple items (Ausubel and Cramton 2002). Krishna (2002) provides an excellent survey of the literature in this stream.

This paper extends the optimal auction literature to study a set of commonly-ranked objects. While the valuations of buyers for these objects are ranked in the same order, the rates at which these valuations change may not be the same for all buyers. For example, a buyer with the highest valuation for a top-ranked object, may have a higher or lower valuation for a lower-ranked object than another buyer, depending on the identities of the buyers concerned. When competing for advertising positions in a search engine, for example, Wal-Mart might not see much difference between obtaining the first slot and the second. For a small firm that relies on these links to catch the consumer's eye, however, the difference between the two slots might be significant, and it may even attach a higher value to a higher-ranked slot than does Wal-Mart.

To address this property, the ranked-item environment is separated into four stylized cases. I find that the optimal allocation and payment rules are quite different in each case. In the optimal mechanism, there is a reserve price for each position. Depending on the case, first, a higher type buyer can be allocated a higher-ranked or lower-ranked object; second, there is a positive probability that a higher-ranked object is not allocated while a lower-ranked one *is* allocated; third, the payment for a winning object is at least as high as the valuation of the highest losing buyer for that object; and, fourth, in a departure from the extant mechanism-design literature, the individual-rationality (IR) constraint for a mid-range type of bidder can be binding.

This paper is organized as follows. In section 2 the model and notation are introduced. Then the optimal mechanism under four stylized cases is discussed in section 3. The summary and some implementation issues are discussed in section 4. Section 5 concludes with discussions about future research.

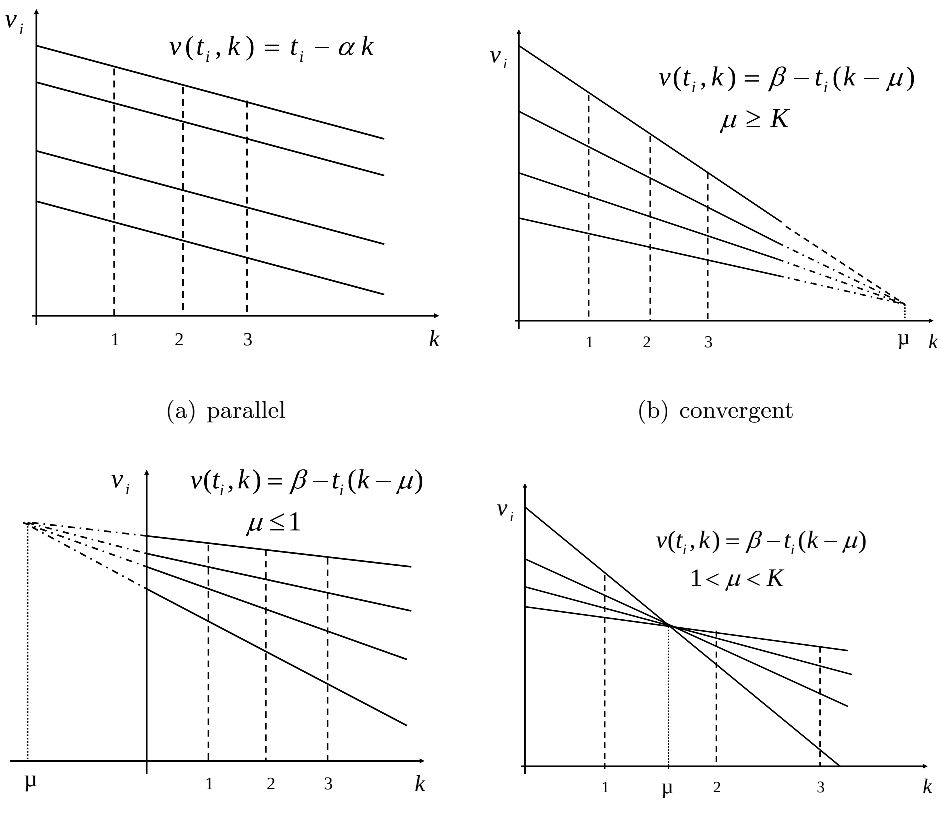
2 Model

Assume that a set of n risk-neutral buyers $N = \{1, 2, ...n\}$ compete for $K \leq n$ objects/positions. Buyer types $t_i, i = 1, ..., n$ are independent and drawn from a common distribution function F over the interval $T = [a, b]$ $(a \geq 0)$, with associated density function f. Let $\mathcal{T} = T^n$ denote the product of the sets of buyer types, and let $\mathcal{T}_{-i} = T^{n-1}$. Let $\mathbf{t}$ denote the vector of buyer valuations $(t_1, t_2, ..., t_n)$, and $f(\mathbf{t})$ be the joint density of vector $\mathbf{t}$. Similarly, let $f(\mathbf{t_{-i}})$ denote the joint density of $\mathbf{t_{-i}} = (t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)$. Thus, $f(\mathbf{t}) = \prod_{i \in N} f(t_i)$, and $f(\mathbf{t_{-i}}) = \prod_{j \in N, j \neq i} f(t_j)$.

Assume the seller has a zero reservation values for each of the positions. Let $v(t_i, k)$ represent buyer i's valuation for the kth-ranked object. For simplicity, let $v(t_i, k) = \beta(t_i) - \alpha(t_i)k$ for $k = 1, 2, ..., K$, where $\beta(t_i)$ and $\alpha(t_i)$ are positive for all $t_i \in \mathcal{T}$. Thus, a buyer's valuation drops for a lower-ranked object. More importantly, the rate at which it drops, $\alpha(t_i)$, can be type-specific. Moreover, the difference between any two types $(t_i - t_j)$ is separable from the difference between the two values $v(t_i, k) - v(t_j, k)$. For simplicity, the difference between any two values is assumed to be linear in the difference between the two types. Thus, $v(t_i, k) - v(t_j, k) = (t_i - t_j)S(k)$, $\forall k$, and $\forall i \neq j$. Here, $S(k)$ represents the relationship between $v(t_i, k) - v(t_j, k)$ and $(t_i - t_j)$, which can be different for a different rank k, and independent of t_i. For example, if $v(t_i, k) = 5 - t_i k$, then $v(t_i, k) - v(t_j, k) = (t_i - t_j)(-k)$, and $S(k) = -k$.

Based on how one buyer's valuation drops relative to that of the others' (they may drop at the same rate, in which case $\alpha(t_i)$ is independent of t_i, or some may drop faster/slower than others depending on their types), the environment can be categorized into four stylized cases: "parallel,"

Figure 1: Different cases of buyer preferences with respect to the ranking of the positions.



(a) parallel

(b) convergent

(c) divergent

(d) convergent then divergent

"convergent," "divergent," and "convergent then divergent." In the parallel case, let $\beta(t_i) = t_i$ and $\alpha(t_i) = \alpha$, where $\alpha > 0$ is a constant, so that the valuation of each buyer drops for a less-preferred object at the same rate. Thus, $v(t_i, k) = t_i - \alpha k$. In the three other cases, let $\alpha(t_i) = t_i$, which allows the buyer valuations for less-preferred objects to drop at different rates depending on the type of buyer. Also let $\beta(t_i) = \beta + t_i\mu$, where μ is a constant. Thus $v(t_i, k) = \beta - t_i(\mu - k)$, and $\mu \geq K$ in the "convergent" case, $\mu \leq 1$ in the "divergent" case, and $1 < \mu < K$ in the "convergent then divergent" case. Figure 1 illustrates these four cases, where cases (b), (c), and (d) are differentiated by the location of μ.

Given the density function f, the number of available objects K, the number of buyers n, and the valuation function $v(t_i, k)$ (which is different in different cases), the seller's problem under each of the four cases, is to select a mechanism that maximizes the seller's expected revenue, subject to the incentive-compatibility (IC) and individual-rationality (IR) constraints. By the "Revelation Principle" ((Allen 1973, Green and Laffont 1977, Myerson 1979)), without loss of generality, only direct mechanisms, in which the buyers simultaneously report their types to the auctioneer are considered. The auctioneer then decides who gets which position, and how much each bidder must pay. Thus, a direct mechanism can be characterized by two rules: an allocation rule and a payment rule. Let $P : \mathcal{T} \rightarrow \{1, 2, .., K\}$ represent the allocation rule, and let $X : \mathcal{T} \rightarrow \mathcal{R}^n$ represent the payment rule. The goal is to identify the optimal rules (P, X) that are incentive compatible and individually rational. Let $p_i(\mathbf{t})$ represent the probability that buyer i wins an object, and let $x_i(\mathbf{t})$ be the expected amount that buyer i will pay the seller. More specifically, let $p_i^k(\mathbf{t})$ represent the probability that buyer i wins the kth object, and $x_i^k(\mathbf{t})$ be buyer i's expected payment for the kth object. Then, $p_i(\mathbf{t}) = \sum_{k=1}^{k=K} p_i^k(\mathbf{t})$, and $x_i(\mathbf{t}) = \sum_{k=1}^{k=K} x_i^k(\mathbf{t})$.

Suppose the seller uses the direct mechanism (P, X). Then buyer i's expected utility is:

$$U(p, x, t_i) = \int_{\mathcal{T}_{-i}} \left(\sum_{k=1}^{k=K} \left[v(t_i, k) p_i^k(t_i, \mathbf{t_{-i}}) - x_i^k(t_i, \mathbf{t_{-i}}) \right] \right) f(\mathbf{t_{-i}}) d\mathbf{t_{-i}}. \tag{1}$$

The seller's expected utility is:

$$U_0(p, x) = \int_{\mathcal{T}} \left(\sum_{k=1}^{k=K} \sum_{N} x_i^k(\mathbf{t}) \right) f(\mathbf{t}) d\mathbf{t}, \tag{2}$$

where

$$p_i^k(\mathbf{t}) \geq 0 \;\; \forall i, \;\; \forall k, \;\; \forall \mathbf{t} \in \mathcal{T}, \tag{3}$$

$$\sum_{N} p_i^k(\mathbf{t}) \leq 1 \;\; \forall k, \;\; \forall \mathbf{t} \in \mathcal{T}, \tag{4}$$

$$\sum_{k=1}^{k=K} p_i^k(\mathbf{t}) \leq 1 \;\; \forall i, \;\; \forall \mathbf{t} \in \mathcal{T}, \tag{5}$$

where Eq. (4) states that a position cannot be allocated to more than one bidder, and Eq. (5) states that a buyer cannot win more than one position.

In addition, an "Individual Rationality" (IR) condition ensures that by not participating, buyers guarantee themselves a payment of zero:

$$U(p, x, t_i) \geq 0 \;\; \forall i, \;\; \forall t_i, \tag{6}$$

The "Incentive Compatibility" (IC) condition ensures that every buyer's true type is reported:

$$U(p,x,t_i;t_i) \geq U(p,x,s;t_i) = \int_{\mathcal{T}_{-i}} \left(\sum_{k=1}^{k=K} \left[v(t_i,k) p_i^k(s,\mathbf{t_{-i}}) - x_i^k(s,\mathbf{t_{-i}}) \right] \right) f(\mathbf{t_{-i}}) d\mathbf{t_{-i}} \;\; \forall i, \;\; \forall t_i, \;\; \forall s \neq t_i, \tag{7}$$

Thus, the seller's goal is to identify the optimal $p_i^k(\mathbf{t})$ and $x_i^k(\mathbf{t})$ that maximizes his expected payoff. That is,

$$\max (2)$$

$$\text{subject to} \quad (3), (4), (5), (6), \text{and } (7).$$

A feasible mechanism is one that satisfies all the five constraints. Lemma 1 presents a simplified characterization of such a mechanism. Define $Q(p,t_i) = \int_{\mathcal{T}_{-i}} \sum_{k=1}^{k=K} S(k) p_i^k(t_i,\mathbf{t_{-i}}) f(\mathbf{t_{-i}}) d\mathbf{t_{-i}}$ as the *adjusted* probability of winning, weighted by $S(k)$ for different positions k.[3]

Lemma 1 *When $S(k) \geq 0$, an allocation mechanism is feasible if and only if:*

$$\textit{if } s \leq t_i, \quad \textit{then } Q(p,s) \leq Q(p,t_i), \tag{8}$$

$$U(p,x,t_i) = U(p,x,a) + \int_a^{t_i} Q(p,s) ds, \tag{9}$$

$$U(p,x,a) \geq 0, \tag{10}$$

and (3), (4), (5).

Throughout this paper, I present all the proof in the Appendix. The case of $S(k) < 0$ is discussed in greater detail for the "divergent" case.

Rearranging the objective function, this maximization problem can be transformed into:

$$\max \int_{\mathcal{T}} \left(\sum_N \sum_{k=1}^{k=K} \left[v(t_i,k) - S(k) \frac{1-F(t_i)}{f(t_i)} \right] p_i^k(\mathbf{t}) \right) f(\mathbf{t}) d\mathbf{t} - N \cdot U(p,x,a), \tag{11}$$

subject to (3), (4), (5),(8), (9) and (10).

[3]Consequently, condition 8 can be understood as the *adjusted monotonicity condition*, which reduces to the standard monotonicity condition in (Myerson 1981) (when $S(k) = 1 \ \forall k$).

The details are provided in the Appendix.

To facilitate the subsequent discussions, define buyer i's contribution to the seller's objective function, if winning position k, as buyer i's "adjusted virtual value" for position k, denoted $c(t_i, k)$. For example, for the objective function (11), $c(t_i, k) = v(t_i, k) - S(k)\frac{1-F(t_i)}{f(t_i)}$. Assume that both the distribution function F and $1 - F$ are log-concave. Distributions satisfying these conditions are a sub-class of the class of *IFR* (Increasing Failure Rate) distribution. This assumption is satisfied by many distributions such as the Uniform, Normal, Logistic, Chi-squared, Exponential, Gamma, and Beta ($a \geq 1, b \geq 1$) distribution. Each of the four stylized cases is now discussed in section 3.

3 The Four Stylized Cases

3.1 The Parallel Case

First consider the case in which each buyer's valuation for a lower-ranked object drops at the same rate. This might happen when the bidders have similar backgrounds, tastes, or purpose for the objects. For example, the valuations of Coca-Cola and Pepsi, the two major national Cola brands, are likely to change at the same rate for a set of retail shelf slots. Let bidder i's valuation for position k, $v(t_i, k)$, be represented by $\beta(t_i) - \alpha(t_i)k = t_i - \alpha k$. The objective function, Eq.(11), now becomes:

$$\max \int_{\mathcal{T}} \left(\sum_{N} \sum_{k=1}^{k=K} \left(t_i - \frac{1 - F(t_i)}{f(t_i)} - \alpha k \right) p_i^k(\mathbf{t}) \right) f(\mathbf{t}) d\mathbf{t} - N \cdot U(p, x, a). \tag{12}$$

It is straightforward to set $U(p, x, a) = 0$ since it has a negative sign. The adjusted virtual value, $c(t_i, k)$, is then defined as $t_i - \frac{1-F(t_i)}{f(t_i)} - \alpha k$. Since $t_i - \frac{1-F(t_i)}{f(t_i)}$ is strictly increasing in t_i, Eq.(12) is maximized when the seller picks the collection of K buyers with the highest types, given that their virtual values are non-negative.

There is a reserve price $r(k)$ for each position k, which is determined by solving $c(r(k), k) = r(k) - \frac{1-F(r(k))}{f(r(k))} - \alpha k = 0, \forall k$. It turns out, however, that the reserve price conditions are not "strict," in the sense that once the winners are determined, the allocation of the objects among them does not matter. This unique property can be shown from the objective function, where the expressions of t_i and k are separable, which implies that the allocation among the winners is inconsequential.

With reserve prices, it is possible to have some positions left unallocated. To determine the number of winners (represented by $\tilde{K} < K$), rank the bidders from the highest to the lowest, compare the highest bid to the highest reserve price, and the second-highest bid to the second-highest reserve price, and so on. If the reserve-price condition is not satisfied for a certain position, delete one position at the bottom, and shift all bids up for one rank. Repeat this comparison until all the reserve-price conditions are satisfied for consecutive $\tilde{K}$ positions starting from the most-preferred object. Then the first $\tilde{K}$-highest buyers are the winners. The un-allocated position will only be at the bottom, because in this case it is always better for the seller to sell a higher-ranked object for a higher price, while it is easier for a buyer to meet the reserve-price condition for a higher position than for a lower one.

Define $\tau_j(\mathbf{t_{-i}})$ as the jth-highest type among all the buyers except i. The following proposition specifies the allocation rule:

Proposition 1 (Allocation Rule — Parallel) *In the "parallel" case, the optimal allocation rule is to allocate one object to each of the K bidders of the highest types, given that the reserve price conditions are satisfied. The allocation of the positions among the winners is inconsequential. Formally,*

$$p_i(s, \mathbf{t_{-i}}) = \begin{cases} 1 & \textit{if } \exists k \leq K \;\; s.t \;\; \{s \geq \tau_k(\mathbf{t_{-i}}) \;\; \textit{and} \;\; \exists i \leq k \;\; s.t. \;\; c(s,i) \geq 0\} \\ 0 & \textit{otherwise}, \end{cases} \tag{13}$$

where both k and i are integers.

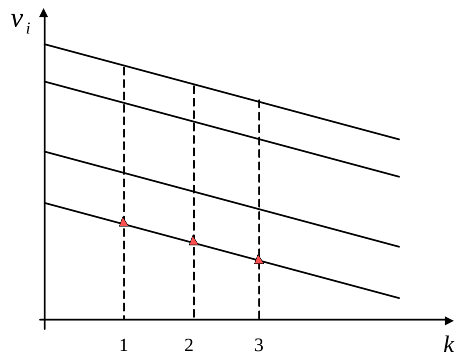
The proof is straightforward from the discussion above, and is thus omitted. Proposition 1 shows that if buyers' valuations change in a similar way for a set of objects, the actual allocation of the objects to the buyers is inconsequential. Formally, this is because in the objective function, the expressions of t_i and k are separable. How does the payment rule $x_i(t)$ make this allocation incentive compatible? Define $Z_k^j(\mathbf{t_{-i}}) = \tau_k(\mathbf{t_{-i}}) - \alpha j$, representing type $\tau_k^\prime$s valuation for the jth position. The optimal expected payment function is thus determined by Eq.(9) and Eq.(1). Formally,

Proposition 2 (Payment Rule — Parallel) *In the "parallel" case, the winner of position k, $k = 1, ..., \tilde{K}$, pays the valuation of the highest rejected buyer's valuation for position k. That is, $x(t_i, k) = Z_{\tilde{K}}^k$.*

This payment rule confirms the result that the allocation among the winners is inconsequential: as long as a winner pays the $\tilde{K}+1$'s buyer's valuation for the object won, the winner's utility is the same no matter which object is won.

This mechanism is illustrated in Figure 2, with four buyers and three objects. The payment scheme is represented by $\triangle$. The allocation rule is not marked specifically, because after the highest three buyers are identified as the winners, the allocations of the positions among them can be randomized.

Figure 2: In the "parallel" case, the three highest-type buyers win, and pay the fourth-highest-type buyer's valuation for the winning position.



3.2 Non-parallel Cases

In some scenarios, bidder valuations for objects at different ranks change at different rates. For example, when acquiring shelf spaces, a well-known national brand may not care about the placement of its product as much as would a small manufacturer with a new product of unknown quality. Depending on the budget, or confidence about the market potential of its product, a small manufacturer's valuation for its most desirable placement may be either higher or lower than that of the established brand, and its valuation for a less-preferred placement may drop either faster or slower than that of the established brand, too. These scenarios are discussed in the following three stylized cases.

3.2.1 The Convergent Case

Suppose a higher-type buyer's valuation, while remaining higher for each position, drops faster for a lower position than does a lower-type buyer's valuation. In this case, a higher-type buyer is more sensitive to which object is won than a lower-type buyer. This may happen when a national brand strongly needs a preferable placement to promote an important product. Represent bidder i's valuation for position k by $v(t_i, k) = \beta - t_i(k - \mu)$, where $\mu \geq K$. Since $v(t_i, k) - v(t_j, k) = (t_i - t_j)(\mu - k)$, it follows that $S(k) = (\mu - k) \geq 0$. The objective function, Eq.(11), becomes:

$$\max \int_{\mathcal{T}} \left(\sum_{N} \sum_{k=1}^{k=K} \left(\beta + \left(t_i - \frac{1 - F(t_i)}{f(t_i)} \right) (\mu - k) \right) p_i^k(\mathbf{t}) \right) f(\mathbf{t}) d\mathbf{t} - N \cdot U(p, x, a). \qquad (14)$$

Again in this case, $U(p, x, a) = 0$ and the buyer with the lowest possible type (a) has a binding IR constraint. The "adjusted virtual value" is defined as $c(t_i, k) = \beta + \left(t_i - \frac{1-F(t_i)}{f(t_i)} \right)(\mu - k)$. To maximize (14), the seller needs to identify the set of the highest non-negative adjusted virtual values, which is again the K bidders with the highest types. There is a reserve price $r(k)$ for each of the K positions, which is determined by solving $c(r(k), k) = 0$, $\forall k$.

How, then, should the positions be allocated among the winners? The optimal allocation should assign a more-preferred object (larger $\mu - k$) to a buyer of a higher type (larger $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$). Define $\tau_j(\mathbf{t_{-i}})$ as in section 3.1, the jth highest type among all the buyers except i. Also, define

$$z_k(\mathbf{t_{-i}}) = inf\{s | c(s, k) \geq 0 \; and \; s \geq \tau_k(\mathbf{t_{-i}})\}, \quad \text{for} \;\; k = 1, 2, ..., K, \qquad (15)$$

and let $z_0(\mathbf{t_{-i}}) = b$, where b is the upper bound of the buyer type. Thus, $z_k(\mathbf{t_{-i}})$ is the smallest type that is higher than $\tau_k(\mathbf{t_{-i}})$, while satisfying the reserve price condition for position k. Similarly, let $Z_k^j(\mathbf{t_{-i}}) = \beta - z_k(j - \mu)$ represent type $z_k(\mathbf{t_{-i}})'$s valuation for the jth position. Thus,

Proposition 3 (Allocation Rule — Convergent) *In the "convergent" case, the optimal allocation rule is to allocate a more-preferred object to a buyer of a higher type, as long as the reserve-price conditions are satisfied. In other words,*

$$p_i^k(s, \mathbf{t_{-i}}) = \begin{cases} 1 & if \;\; z_k(\mathbf{t_{-i}}) \leq s \leq z_{k-1}(\mathbf{t_{-i}}), \;\; k = 1, 2, ..., K \\ 0 & otherwise. \end{cases} \qquad (16)$$

Proposition 3 shows that when a high-type buyer's valuation drops faster than that of a low-type buyer, in order to maximize his revenue, the auctioneer should allocate a higher-ranked object to a higher-type buyer. This allocation rule satisfies two out of the three criteria that Menezes and Monteiro (1998) mention: zero expected payoff for the lowest type; and the K highest-valued bidders win. As opposed to Menezes and Monteiro (1998), however, with this mechanism, it is possible that a higher position is not allocated, while a lower position is filled, due to the restriction of the reserve prices.

The payment function is again determined by taking Eq.(16) and Eq.(9). Then,

Proposition 4 (Payment Rule — Convergent) *The optimal payment function for the first position is* $x_i^1(\mathbf{t}) = Z_1^1(\mathbf{t_{-i}})$ *if* $p_i^1 = 1$*, and for the kth position* ($k > 1$)*:*

$$\begin{aligned} x_i^k(\mathbf{t}) &= v(t_i,k) - \big(v(t_i,k) - Z_k^k(\mathbf{t_{-i}}|z_k \le t_i \le z_{k-1})\big) \cdot prob(z_k \le t_i \le z_{k-1}) \\ &= v(t_i,k)\left(1 - \frac{\int_0^{t_i} \frac{(n-1)!}{(n-1-k)!(k-1)!}F(y)^{n-1-k}(1-F(t_i))^{k-1}f(y)dy}{\int_0^{t_i} \frac{(n-1)!}{(n-1-k)!(k-1)!}F(y)^{n-1-k}(1-F(y))^{k-1}f(y)dy}\right) + \frac{\int_0^{t_i} y\frac{(n-1)!}{(n-1-k)!(k-1)!}F(y)^{n-1-k}(1-F(t_i))^{k-1}f(y)dy}{\int_0^{t_i} \frac{(n-1)!}{(n-1-k)!(k-1)!}F(y)^{n-1-k}(1-F(y))^{k-1}f(y)dy} \end{aligned} \tag{17}$$

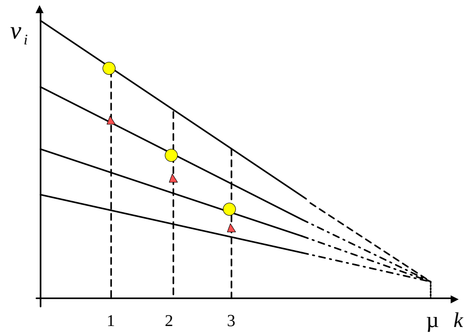
if $p_i^k = 1, k = 2, ...K$.

The derivation of the payment function is in the Appendix. Thus, the winner of the most-preferred object only needs to pay the second-highest bidder's valuation of it. The payment for the remaining objects is higher than that in the traditional "highest-rejected-bid" auction. This can be easily shown by Eq. (17), because the payment for position k in a "highest-rejected-bid" auction is $Z_k^k(\mathbf{t_{-i}}|z_k \le t_i \le z_{k-1})$, which is smaller than $v(t_i, k)$. In summary

Corollary 1 *In the convergent case, a winner will pay at least as much as the next-highest-type bidder's valuation for that position.*

The intuition of this result is that, when valuations of different buyers for less-preferred objects fall at different rates, the seller has an incentive to optimally match objects to buyers in order to maximize his expected revenue. Charging a higher price than the highest rejected bid for a less-preferred object is the consequence of the "truth-telling" (IC) constraint. It prevents a higher-type buyer from shading her bid in order to win a less-preferred object, thus increases the seller's expected payoff. As a result, this mechanism also performs better than a typical second-price (or highest-rejected-bid) sequential auction.

Figure 3: In the "Convergent" case, a higher-type buyer is allocated a more-preferred object, and pays no less than the next-highest-buyer's valuation for the object won.



3.2.2 The Divergent Case

The "divergent" case describes a situation in which a higher-type buyer's valuation drops slower for a less-preferred object than does a lower-type buyer's valuation. In this case a lower-type buyer is more sensitive to which object is won than is a higher-type buyer. This may happen when a small firm strongly needs a preferable placement to catch the consumer's eye when promoting a new product that is crucial for the firm's survival. Let $\mu \leq 1$ in the valuation function $v(t_i, k) = \beta - t_i(k - \mu)$. Now, as opposed to the "convergent" case, the smaller the t_i, the higher the values. Since $v(t_i, k) - v(t_j, k) = (t_i - t_j)(\mu - k)$, it follows $S(k) = (\mu - k) < 0$.[4] Thus, the IC conditions (9) and (10) are rewritten as follows:

$$U(p, x, t_i) = U(p, x, b) - \int_{t_i}^{b} Q(p, s)ds. \tag{18}$$

$$U(p, x, b) \geq 0. \tag{19}$$

Re-arranging the objective function (see Appendix for the details), it becomes:

$$\max \int_{\mathcal{T}} \left(\sum_{N} \sum_{k=1}^{k=K} \left(\beta + \left(t_i + \frac{F(t_i)}{f(t_i)} \right) S(k) \right) p_i^k(\mathbf{t}) \right) f(\mathbf{t}) d\mathbf{t} - N \cdot U(p, x, b). \tag{20}$$

Since $U(p, x, b) = 0$, the buyer of the highest possible type (b) has a binding IR constraint. Again define $c(t_i, k) = \beta + \left(t_i + \frac{F(t_i)}{f(t_i)} \right)(\mu - k)$ as bidder i's "adjusted virtual value" for position k. Since $S(k)$ is negative, to maximize the set of adjusted virtual values, the K smallest types (t_i)

[4]Lemma 1 can not be directly applied

should be selected, given that their adjusted virtual values $c(t_i, k)$ are non-negative for the winning objects. Since β is a constant, this is equivalent to finding the best allocation that minimizes $t_i + \frac{F(t_i}{f(t_i)}(-S(k))$. It is shown in Appendix A.4 that the optimal allocation is to allocate a lower position (larger k) to a lower type (smaller $t_i + \frac{F(t_i)}{f(t_i)}$).

Again there is a unique reserve price $r(k)$ for each object k, which is determined by solving $c(r(k), k) = 0$, $\forall k$. The number of actual winners ($\tilde{K} \leq K$) can be determined by calculating how many buyer types satisfy the reserve-price conditions, as in the "parallel" case. As a result, the unfilled position will only be at the bottom, as in the "parallel" case.

More specifically, define

$$d_k(\mathbf{t_{-i}}) = sup\{s | c(s, k) \geq 0 \; and \; s \leq \tau_{K-k}(\mathbf{t_{-i}})\}, \quad \text{for} \;\; k = 1, 2, ..., K \tag{21}$$

and $d_0(\mathbf{t_{-i}})$ equal to a, the lower bound of the buyers' value distribution. Thus $d_k(\mathbf{t_{-i}})$ represents the largest type that is smaller than $\tau_{K-k}(\mathbf{t_{-i}})$, or the kth-lowest type among other bidders, and that also satisfies the reserve-price condition. The following proposition summarizes the results above:

Proposition 5 (Allocation Rule — Divergent) *In the "divergent" case, the optimal allocation mechanism is to allocate a lower position to a buyer with a smaller t_i, given that the reserve-price condition for that position is satisfied. In other words,*

$$p_i^k(s, \mathbf{t_{-i}}) = \begin{cases} 1 & if \;\; d_{\tilde{K}-k}(\mathbf{t_{-i}}) \leq s \leq d_{\tilde{K}-k+1}(\mathbf{t_{-i}}), \\ 0 & otherwise \end{cases} \tag{22}$$

where $\tilde{K}$ is the number of allocated slots.

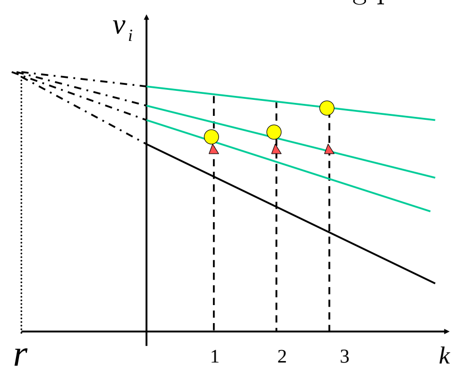
Proposition 5 shows that when a lower-type buyer's valuation drops faster for a lower-ranked object than a higher-type buyer's does, the auctioneer should allocate a higher-ranked object to a lower-type buyer, given that the buyer's type is high enough to win one object. This special allocation is due to the negative coefficient of t_i, $-(k - \mu)$. It is counter-intuitive that among the winners, a lower-type bidder gets a higher-ranked object. This is because in this case, the high-type buyer cares less about which object is won than does a low type buyer, thus this allocation generates the maximum revenue for the auctioneer. How does the payment scheme make this allocation incentive compatible?

Proposition 6 (Payment Rule — Divergent) *The optimal payment function for the last position ($\tilde{K}$) is $x_i^1(\mathbf{t}) = D_1^{\tilde{K}}(\mathbf{t_{-i}})$ if $p_i^1 = 1$, while for the kth position ($k \leq \tilde{K}$):*

$$\begin{aligned} x_i^k(\mathbf{t}) &= v(t_i,k)(1 - prob(d_{K-k} \leq t_i \leq d_{K-k+1}) + D_k^k(t_i | d_{K-k} \leq t_i \leq d_{K-k+1}) \\ &\cdot prob(d_{K-k} \leq t_i \leq d_{K-k+1}) \\ &= v(t_i,k)\left(1 - \frac{\int_{t_i}^1 \frac{(n-1)!}{(n-1-k)!(k-1)!} F(t_i)^{k-1}(1-F(y))^{n-1-k} f(y)dy}{\int_{t_i}^1 \frac{(n-1)!}{(n-1-k)!(k-1)!} F(y)^{k-1}(1-F(y))^{n-1-k} f(y)dy}\right) + \frac{\int_{t_i}^1 y \frac{(n-1)!}{(n-1-k)!(k-1)!} F(t_i)^{k-1}(1-F(y))^{n-1-k} f(y)dy}{\int_{t_i}^1 \frac{(n-1)!}{(n-1-k)!(k-1)!} F(y)^{k-1}(1-F(y))^{n-1-k} f(y)dy} \end{aligned} \tag{23}$$

if $p_i^k = 1, k = 1, ..., \tilde{K} - 1.$

Figure 4: In the "Divergent" case, a higher-value buyer is allocated a lower position, and pays no less than the next-highest buyer's valuation for the winning position.



Thus, other than the one who wins the lowest position (the buyer with the least-steep slope), all other winners are paying more than the next-highest buyer's valuation for that winning position. Again, this property comes from the "truth-telling" (IC) constraint. By increasing the payment for a higher-ranked position, it prevents a buyer with a higher type from pretending to be a lower one to win a higher position—buyers have to pay more for higher positions, which results in lower profit under the divergent utility functions. This mechanism works better than a simple second-price sequential auction, where the winners pay the next rejected buyer's valuation for their winning positions.

A numerical example is given in the Appendix, and Figure 4 illustrates the optimal mechanism: given the payment scheme, the highest-type bidder is better off winning the lowest-ranked object.

3.2.3 The Convergent-then-Divergent Case

Combining the "convergent" and "divergent" cases, suppose a high-type buyer's valuation for a lower position drops so dramatically that it can be lower than that of the competitor's. Let $1 < \mu < K$ in the valuation function $v(t_i, k) = \beta - t_i(k - \mu)$. As $v(t_i, k) - v(t_j, k) = (t_i - t_j)(\mu - k)$, it is straightforward that $S(k) = (\mu - k) \geq 0$ when $k \leq \lfloor \mu \rfloor$, and $S(k) < 0$ when $k > \lfloor \mu \rfloor$.

Again one of the IC conditions (9) should be checked because the sign of $S(k)$ changes before and after $k = \lfloor \mu \rfloor$. More specifically, there exists a $w \in (a, b)$ such that Eq. (9) can be rewritten as:

$$\begin{gathered} U(p,x,t_i) = U(p,x,w) + \textstyle\int_w^{t_i} Q(p,s)ds \quad \text{if} \quad t_i \geq w, \\ \text{and} \\ U(p,x,t_i) = U(p,x,w) - \textstyle\int_{t_i}^{w} Q(p,s)ds \quad \text{if} \quad t_i < w, \end{gathered} \tag{24}$$

Accordingly

$$U(p, x, w) \geq 0.$$

Re-arranging the objective function (See A.5):

$$\begin{aligned} &\max \textstyle\int_a^w \left(\sum_N \sum_{k=1}^{k=K} \left(\beta + \left(t_i + \frac{F(t_i)}{f(t_i)}\right)(\mu - k)\right) p_i^k(\mathbf{t})\right) f(\mathbf{t}) d\mathbf{t} \\ &+ \textstyle\int_w^b \left(\sum_N \sum_{k=1}^{k=K} \left(\beta + \left(t_i - \frac{1-F(t_i)}{f(t_i)}\right)(\mu - k)\right) p_i^k(\mathbf{t})\right) f(\mathbf{t}) d\mathbf{t} - N \cdot U(p, x, w). \end{aligned} \tag{25}$$

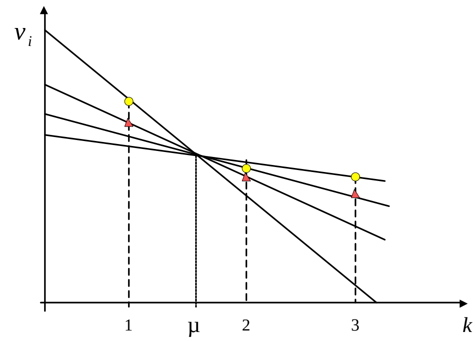
With a negative sign in the objective function, $U(p, x, w) = 0$. Thus, different from most of the mechanism-design literature, a buyer type in the middle ($w \in (a, b)$) can have a binding IR constraint. When $t_i \geq w$, $\mu - k \geq 0$, it is optimal to follow the "convergent" case, and to allocate the $\lfloor \mu \rfloor$ highest-ranked objects to the buyers with the highest types (t_i); when $t_i < w$, $\mu - k < 0$, it is optimal to follow the "divergent case," and to allocate the $K - \lfloor \mu \rfloor$ lowest-ranked objects to the lowest-type buyers.

Proposition 7 summarizes this allocation rule.

Proposition 7 (Allocation Rule — Convergent then Divergent) *For a given $w \in (a, b)$, the optimal allocation mechanism for the "convergent then divergent" case is: for each $k < \lfloor \mu \rfloor$, allocate the highest remaining position to the buyers with the highest remaining t_i, as long as $t_i \geq w$ and the adjusted virtual value is non-negative ($\beta + \left(t_i - \frac{1-F(t_i)}{f(t_i)}\right)(\mu - k) \geq 0$). For each $k > \lfloor \mu \rfloor$, allocate the remaining lowest position to the buyers with the remaining lowest t_i, as long as $t_i < w$ and the adjusted virtual value is non-negative ($\beta + \left(t_i + \frac{F(t_i)}{f(t_i)}\right)(\mu - k) \geq 0$).*

Proposition 7 shows that the optimal allocation rule in this case is a combination of that in the "convergent" case when $k \leq \lfloor \mu \rfloor$ and in the "divergent" case when $k > \lfloor \mu \rfloor$. The same combination applies to the payment rule. The proof follows the cases of $\mu - k \geq 0$ and $\mu - k < 0$. Figure 5 illustrates this case.

Figure 5: The "Convergent then Divergent" case combines the "Convergent" and "Divergent" cases.



Proposition 7 gives the optimal mechanism for a given $w \in (a, b)$. A higher-level decision is to choose the optimal w^*, which is a function of μ, n, and K. For example, the ideal w^* should have the property of having at least $\lceil K - \mu \rceil$ buyers whose types are below w, and at least $\lfloor \mu \rfloor$ buyers whose types are above w. This property indicates that, given K, the optimal w should be non-increasing in μ. But to complete this mechanism, w should be pre-announced. Hence, there is a positive probability that if allocated an object, a certain buyer whose type satisfies the reserve-price condition, will give the seller a higher profit, but this buyer cannot win because her type falls on the "wrong" side of w. Thus this mechanism is more *inefficient* in addition to the inefficiency created by the existence of a reserve price for each position.

4 Implementation

This mechanism can be implemented as an auction with the following properties:

1. Bidders only need to report one non-rank-specific bid, but the object that a bidder wins, as well as the payment for the winning position, depends on the rank of her bid among all the other bids;

2. The seller imposes a reserve price for each of the objects;

3. As a result, certain positions may remain unfilled, if the reserve-price condition is not met.

In terms of performance, the optimal mechanism generates a higher expected revenue than does a standard "second price" setting. This is a consequence of the IC constraint, as simply charging a second price will give a buyer an incentive not to bid truthfully, and to win other positions with a possibly lower price.

The single-bid property greatly simplifies the communication of the auction rules to the participating bidders, as the auctioneer only needs to pre-announce the allocation rule, with the corresponding payment functions for each object, and requires each bidder to submit a single bid. To apply it to the allocation of shelf space using slotting allowances, manufacturers need to submit a lump-sum payment for display, and the space is allocated in accordance with the rank of bids.

This mechanism can also be adopted to apply to the keyword auctions hosted by the major search engines. As current technologies make it possible to accurately monitor the click-through rate of each advertisement,[5] advertisers only need to submit their willingness to pay for each "click" received from the search engine, which can be understood as bidders' type t_i. The rank of the advertisement slots also depends on the rank of the bids received.

As opposed to the current practice used by Yahoo! and Google, where only a single reserve price is used, this optimal mechanism imposes a reserve price for each position. Consequently, the mechanism requires the auctioneer to fully commit not to sell a position if the reserve price is not met. This full-commitment assumption is widely observed in practice. For example, in the airline industry, passengers in Coach Class are not allowed to sit in First Class without paying extra, even when the First Class is not full. In contests, the highest award given is sometimes the second prize, while the first prize remains un-assigned. When allocating retail spaces, if no manufacturer is willing to pay a required amount for a certain position, the retailer can display its own private label to occupy the slot. This is a "credible threat" to enforce the reserve price, as the private labels tend to have higher retail margins than those of national brands (Cappo 2003). For keyword auctions, this can be done similarly by inserting an advertisement of the search engine itself.

[5]such internet technologies make it possible to apply some advanced pricing schemes (Liu and Zhang 2006).

5 Conclusion

This paper shows how the earlier work on optimal auctions (Myerson 1981, Maskin and Riley 1990) can be extended and applied to the allocation of a set of commonly-ranked objects, where bidders are permitted to purchase at most one item. The optimal way to sell such a ranked set of objects is quite different when preferences of buyers for different objects change in different ways. Thus, understanding buyer preferences is vital in determining the optimal mechanism. This mechanism works better than a simple second-price sequential auction. In terms of efficiency issues, however, aside from the inefficiency created by the reserve prices, this optimal allocation mechanism in the "convergent then divergent" case can be more inefficient due to the choice of the "pivot" type among the buyers.

This paper assumes that every buyer's type is drawn from a common distribution function. It is straightforward to extend the setting to that of an asymmetric distribution. However, as in many optimal auctions with asymmetric value distribution functions, inefficiency may arise when the distribution functions have different domains. The optimal mechanism might favor buyers whose valuation is drawn from a tighter domain (Myerson 1981). The linearity assumption about user valuation functions may also be relaxed, by adopting some non-linear functions satisfying $v(t_i, k) - v(t_j, k) = (t_i - t_j)S(k)$.

This paper assumes that a bidder's valuation for a particular object does not depend on the allocation of other objects. This assumption may not fully convey the preferences of the bidders. As Dholakia and Simonson (2005) points out, the willingness-to-pay of bidders may be affected by the prices of auctions in the adjacent placementand it would be interesting to study the optimal mechanism under the more general cases. The case in which an auctioneer cannot commit to hold an object is another interesting extension. Finally, when more than one seller competes for bidders, a more appropriate mechanism should aim to enhance the benefit of both the seller and the buyers (Shugan 2005).

References

Allen, G. (1973): "Manipulation of Voting Schemes: A General Result," *Econometrica*, 41, 587–602.

Armstrong, M. (2000): "Optimal Multi-Object Auctions," *Review of Economic Studies*, 67, 455–481.

Ausubel, L. M., and P. Cramton (2002): "Demand Reduction and Inefficiency in Multi-Unit Auctions," Working paper.

Avery, C., and T. Hendershott (2000): "Bundling and Optimal Auctions of Multiple Products," *Review of Economic Studies*, 67, 483–497.

Bloom, P. N., G. T. Gundlach, and J. P. Cannon (2000): "Slotting Allowances and Fees: Schools of Thought and the Views of Practicing Managers," *Journal of Marketing*, 64(2), 92–108.

Bogomolnaia, A., and H. Moulin (2002): "A Simple Random Assignment Problem With a Unique Solution," *Economic Theory*, 19(3), 623–635.

Cappo, J. (2003): "How Retailer Power Changes Marketing," *Advertising Age*, 72(29), 16–17.

Chu, W. (1992): "Demand Signalling and Screening in Channels of Distribution," *Marketing Science*, 11(4), 327–347.

Cotriss, D. (2002): "Marketers report high ROI with paid listings," *B to B*, 87.

Desai, P. S. (2000): "Multiple Messages to Retain Retailers: Signaling New Product Demand," *Marketing Science*, 1999(4), 381–389.

Dholakia, U. M., and I. Simonson (2005): "The Effect Of Explicit Reference Points On Consumer Choice And Online Bidding Behavior," *Marketing Science*, 24(2), 206217.

Eliaz, K., T. Offerman, and A. Schotter (2005): *Creating Competition Out of Thin Air: Market Thickening and Right-to-Choose Auctions*Working Paper, New York University.

Feng, J., H. K. Bhargava, and D. Pennock (forthcoming): "Implementing Sponsored Search in Web Search Engines: Computational Evaluation of Alternative Mechanisms," *Informs Journal on Computing*.

Feng, J., Z. M. Shen, and R. L. Zhan (forthcoming): "Ranked Items Auctions and Online Advertisement," *Special Issue on E-Auctions for Procurement Operations, Production and Operations Management*.

FTC (2003): "Slotting Allowances in the Retail Grocery Industry: Selected Case Studies in Five Product Categories," An ftc staff study, FEDERAL TRADE COMMISSION, http://www.ftc.gov/os/2003/11/slottingallowancerpt031114.pdf.

Green, J., and J. J. Laffont (1977): "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods," *Econometrica*, 45(2), 427–438.

Kim, S. (2005): *Shelf-Access Payments: Slotting Fees, Pay-To-Stay Fees And Exclusivity Deals*http://www.sen.ca.gov/sor/reports/reports_by_subj/economy_employment/slotting.pdf.

Krishna, V. (2002): *Auction Theory.* Elsevier Science.

Lariviere, M. A., and V. Padmanabhan (1997): "Slotting Allowances and New Product Introductions," *Marketing Science*, 16(2), 112–128.

Liu, D., and J. Chen (forthcoming): "Designing Online Auctions with Past Performance Information," *Decision Support Systems*.

Liu, Y., and Z. J. Zhang (2006): "The Benefits of Personalized Pricing in a Channel," *Marketing Science*, 25(1), 97105.

Maskin, E., and J. Riley (1990): "Optimal Multi-unit Auction," in *The Economics of Missing Markets, Information and Games*, ed. by F. Hahn, chap. 14, pp. 312–335. Oxford University Press.

Menezes, F. M., and P. K. Monteiro (1998): "Simultaneous Pooled Auctions," *The Journal of Real Estate Finance and Economics*, 17(3), 219–232.

Myerson, R. (1979): "Incentive Compatability and the Bargaining Problem," *Econometrica*, 47, 61–73.

——— (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Palfrey, T. R. (1983): "Bundling Decisions by a Multiproduct Monopolist with Incomplete Information," *Econometrica*, 51, 463–484.

Rao, A., and H. Mahi (2003): "The Price of Launching a New Product: Empirical Evidence on Factors Affecting the Relative Magnitude of Slotting Allowances," *Marketing Science*, 22(2), 246268.

Shugan, S. M. (2005): "Editorial: Marketing And Designing Transaction Games," *Marketing Science*, 24(4), 525530.

Sudhir, K., and V. R. Rao (2006): "Do Slotting Allowances Enhance Efficiency or Hinder Competition?," *Journal of Marketing Research*, 43(2), 137–155.

A Appendix

A.1 Model

Proof of Lemma 1. To show the "only if" part,

$$\begin{aligned} & U(p,x,s;t_i) \\ = \ & \sum_K \int_{\mathcal{T}_{-i}} \left[v(t_i,k) p_i^k(s,\mathbf{t_{-i}}) - x_i^k(s,\mathbf{t_{-i}}) \right] f(\mathbf{t_{-i}}) d\mathbf{t_{-i}} \\ = \ & \sum_K \int_{\mathcal{T}_{-i}} \left[\left(v(s,k) + (t_i - s) S(k) \right) p_i^k(s,\mathbf{t_{-i}}) - x_i^k(\mathbf{s},\mathbf{t_{-i}}) \right] f(\mathbf{t_{-i}}) d\mathbf{t_{-i}} \\ = \ & U(p,x,s) + \sum_K \int_{\mathcal{T}_{-i}} \left((t_i - s) S(k) \right) p_i^k(s,\mathbf{t_{-i}}) f(\mathbf{t_{-i}}) d\mathbf{t_{-i}} \\ = \ & U(p,x,s) + (t_i - s) Q(p,s). \end{aligned}$$

The incentive-compatibility constraint implies that:

$$U(p,x,t_i;t_i) \geq U(p,x,s;t_i) + (t_i - s)Q(p,s) \ \ \forall s. \tag{26}$$

Use (26) twice to get:

$$(t_i - s)Q(p,s) \leq U(p,x,t_i) - U(p,x,s) \leq (t_i - s)Q(p,t_i). \tag{27}$$

Then when $s \leq t_i$,

$$Q(p,s) \leq Q(p,t_i).$$

which is Condition (8).

Let $t_i - s = \delta$. Then (27) can also be written as:

$$\delta Q(p,s) \leq U(p,x,s+\delta) - U(p,x,s) \leq \delta Q(p,s+\delta). \tag{28}$$

$Q(p,s)$ is integrable and can be written as: $\int_a^{t_i} Q(p,s)ds = U(p,x,t_i) - U(p,x,a)$. Hence

$$U(p,x,t_i) = U(p,x,a) + \int_a^{t_i} Q(p,s)ds.$$

which is Condition (9). From condition (8), (9), and (6), we got $U(p,x,a) \geq 0$.

From the other direction (the "if" part), to show (26), assume $s \leq t_i$, and using (8) and (9), we get:

$$\begin{aligned} U(p,x,t_i) &= U(p,x,s) + \int_s^{t_i} Q(p,r)dr \\ &\geq U(p,x,s) + \int_s^{t_i} Q(p,s)dr \\ &= U(p,x,s) + (t_i - s)Q(p,s). \end{aligned}$$

If $s > t_i$, then

$$\begin{aligned} U(p,x,t_i) &= U(p,x,s) - \int_{t_i}^{s} Q(p,r)dr \\ &\geq U(p,x,s) - \int_{t_i}^{s} Q(p,s)dr \\ &= U(p,x,s) + (t_i - s)Q(p,s). \end{aligned}$$

Given conditions (8), (9), and $U(p,x,a) \geq 0$, we can recover (6).

Thus when $S(k) \geq 0$, $(p_i(\mathbf{t}), x_i(\mathbf{t}))$ is feasible if it satisfies (8), (9), (10), (3), (4), and (5). $\diamond$

Re-arranging the Objective Function

Re-arrange the objective function (2):

$$\begin{aligned} U_0(p,x) &= \int_{\mathcal{T}} \sum_K \left[\sum_N x_i^k(\mathbf{t})\right] f(\mathbf{t})d\mathbf{t} \\ &= \int_{\mathcal{T}} \left(\sum_N \sum_K p_i^k(\mathbf{t}) v(t_i,k)\right) f(\mathbf{t})d\mathbf{t} \\ &\quad + \int_{\mathcal{T}} \left(\sum_N \sum_K \left(x_i^k(\mathbf{t}) - p_i^k(\mathbf{t}) v(t_i,k)\right)\right) f(\mathbf{t})d\mathbf{t}. \end{aligned} \tag{29}$$

For the last term of Eq.(29), using Eq.(9),

$$\begin{aligned} & \int_{\mathcal{T}} \left(\sum_N \sum_K \left(x_i^k(\mathbf{t}) - p_i^k(\mathbf{t}) v(t_i,k)\right)\right) f(\mathbf{t})dt \\ =\ & -\int_a^b \sum_N U(p,x,t_i) f(t_i)dt_i \\ =\ & -\int_a^b \sum_N \left(U(p,x,a) + \int_a^{t_i} Q(p,s)ds\right) f(t_i)dt_i \\ =\ & -N \cdot U(p,x,a) - \int_a^b \sum_N \left(\int_a^{t_i} Q(p,s)ds\right) f(t_i)dt_i \\ =\ & -N \cdot U(p,x,a) - \int_a^b \sum_N \left(\int_s^b Q(p,s)\right) f(t_i)dt_i ds \\ =\ & -N \cdot U(p,x,a) - \int_a^b \sum_N (1 - F(s))\, Q(p,s)ds \\ =\ & -N \cdot U(p,x,a) - \int_{\mathcal{T}} \sum_N \sum_K \left(S(k)\tfrac{1-F(t_i)}{f(t_i)}\right) p_i^k(\mathbf{t}) f(\mathbf{t})d\mathbf{t}. \end{aligned}$$

Substituting this back into Eq.(29):

$$\begin{aligned} & U_0(p,x) \\ =\ & -N \cdot U(p,x,a) + \int_{\mathcal{T}} \sum_N \sum_K \left[(v(t_i,k) - S(k)\tfrac{1-F(t_i)}{f(t_i)}\right] p_i^k(\mathbf{t}) f(\mathbf{t})d\mathbf{t}. \end{aligned} \tag{30}$$

Thus, maximizing equation (30) is equivalent to:

$$\max \int_{\mathcal{T}} \left(\sum_N \sum_K \left[v(t_i,k) - S(k)\frac{1-F(t_i)}{f(t_i)}\right] p_i^k(\mathbf{t})\right) f(\mathbf{t})dt - N \cdot U(p,x,a),$$

which is Eq (11).

A.2 The Parallel Case

Proof of Proposition 2.

From Eq.(9) and Eq.(1),

$$U(p,x,t_i) = \int_{\mathcal{T}_{-i}} \sum_K \left[v(t_i,k)p_i^k(t_i,\mathbf{t_{-i}}) - x_i^k(t_i,\mathbf{t_{-i}})\right] f(\mathbf{t_{-i}})d\mathbf{t_{-i}} = U(p,x,a) + \int_a^{t_i} Q(p,s)ds.$$

The optimal expected-payment function is thus determined by:

$$\textstyle\sum_K x_i^k(\mathbf{t}) \quad = \sum_K v(t_i,k)p_i^k(\mathbf{t}) - \sum_K \int_a^{t_i} S(k)p_i^k(s)f(s)ds. \tag{31}$$

Thus, bidder i has to pay only when winning an object, (if $p_i^k = 0,\ \forall k$, then the RHS of Eq (31) is zero, and $x_i(\mathbf{t}) = 0$, so it is not an all-pay auction). If she wins position k, the payment will be $x_i^k = \sum_{\tilde{K}} v(t_i,k)p_i^k(\mathbf{t}) - S(k)(t_i - z_{\tilde{K}}(\mathbf{t_{-i}})) = v(t_i,k) - v(t_i,k) + Z_{\tilde{K}}^k$, where $Z_{\tilde{K}}^k$ is buyer $z_{\tilde{K}}(\mathbf{t_{-i}})'$s valuation for position k. ◇

A.3 The Convergent Case

To show it is optimal to allocate a higher position (larger $\mu - k$) to a buyer with a higher type (larger $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$).

Consider two ranked lists of y_j and $\mu - k$. The objective is to find the best allocation rule $r(j) : j \to k$ to allocate y_j to $\mu - k$ to generate the largest sum of the product $(\sum y_j(\mu - r(j)))$. First, consider only two terms in each list. Let y_j be the j'th highest value of $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$, comparing $y_j(\mu-k) + y_{j+1}(\mu-k-1)$ and $y_{j+1}(\mu-k) + y_j(\mu-k-1)$. The difference between these two expressions are: $y_j - y_{j+1} > 0$. Thus the summation of a higher-ranked y multiplied by a higher-valued $\mu - k$ generates the largest sum. This can be generalized to the case where there are more than two terms in each list.

◇

Proof of Proposition 4. The payment function is determined by taking Eq. (16) into Eq.(9):

$$\sum_K x_i^k = \sum_K v(t_i,k)p_i^k - \sum_K \int_a^{t_i} S(k)p_i^k f(s)ds. \tag{32}$$

If buyer i is allocated the first object, then Eq.(32) becomes:

$$x_i^1 = v(t_i,1) - \int_a^{t_i} S(1)p_i^1 f(s)ds.$$

Define $Z_k^j(\mathbf{t_{-i}})$ as buyer z_k's valuation for the j'th position (in Sec.3.2.1). Then, $\int_a^{t_i} S(1)p_i^1 f(s)ds = S(1)(t_i - z_1(\mathbf{t_{-i}}) = v(t_i,1) - Z_1^1(\mathbf{t_{-i}})$. Thus, the optimal payment for the first object is $x_i^1 = Z_1^1(\mathbf{t_{-i}})$.

Now consider the second object. Buyer i can win the second object only if t_i is between the first-highest and second-highest buyers' type other than t_i. Thus, we get $\int_a^{t_i} S(2)p_i^2 f(s)ds = S(2)(t_i - z_2(\mathbf{t_{-i}}|z_2 \leq t_i \leq z_1) prob(z_2 \leq t_i \leq z_1) = (v(t_i,2) - Z_2^2(\mathbf{t_{-i}}|z_2 \leq t_i \leq z_1)) \cdot prob(z_2 \leq t_i \leq z_1)$. It follows from Eq. (32) that the optimal payment for the remaining object is:

$$\begin{aligned} x_i^2 &= v(t_i,2) - \big(v(t_i,2) - Z_2^2(\mathbf{t_{-i}}|z_2 \leq t_i \leq z_1)\big) \cdot prob(z_2 \leq t_i \leq z_1) \\ &= v(t_i,2)(1 - prob(z_2 \leq t_i \leq z_1) + Z_2^2(\mathbf{t_{-i}}|z_2 \leq t_i \leq z_1) \cdot prob(z_2 \leq t_i \leq z_1) \\ &= v(t_i,2)\left(1 - \frac{\int_0^{t_i} \frac{(n-1)!}{(n-3)!(1)!} F(y)^{n-3}(1-F(t_i))f(y)dy}{\int_0^{t_i} \frac{(n-1)!}{(n-3)!(1)!} F(y)^{n-3}(1-F(y))f(y)dy}\right) + \frac{\int_0^{t_i} y \frac{(n-1)!}{(n-3)!(1)!} F(y)^{n-3}(1-F(t_i))f(y)dy}{\int_0^{t_i} \frac{(n-1)!}{(n-3)!(1)!} F(y)^{n-3}(1-F(y))^1 f(y)dy}. \end{aligned}$$

The optimal payment for the rest of the positions can be obtained in the same way. ◇

A Numerical Example of "convergent" allocation

Assume that buyers' types follow $U(0,1)$. The reserve price for position k is $\frac{1}{2} - \frac{\beta}{2(\mu-k)}$, which is decreasing in k. If there are three positions available, and $\mu = 6$, $\beta = 1$, then the reserve prices for positions 1, 2, and 3 are $\frac{2}{5}$, $\frac{3}{8}$, and $\frac{1}{3}$, respectively. If there are four buyers with realized types $t_1 = 0.8$, $t_2 = 0.6$, $t_3 = 0.4$, and $t_4 = 0.2$, then positions 1, 2, and 3 will be allocated to buyers 1, 2, and 3, respectively, with expected payments for those positions being 4, 2.868 and 1.999. Thus the revenue is 8.867. An illustration of this example is depicted in Fig. 3, where ∘ represents the allocation, and △ represents the payment. In a sequential auction, however, while the exact payment is hard to determine, its upper bound is simply the sum of the highest-rejected-bidders' valuations for the winning positions, which is only 8.2.

A.4 The Divergent Case

Re-arrange the objective function Re-arrange the last term of Eq.(29),

$$
\begin{aligned}
& \int_{\mathcal{T}} \sum_K \left(x_i^k(\mathbf{t}) - p_i^k(\mathbf{t}) v(t_i, k) \right) f(\mathbf{t}) dt \\
= \; & - \int_a^b U(p, x, t_i) f(t_i) dt_i \\
= \; & -U(p, x, b) + \int_a^b \left(\int_a^{t_i} Q(p_i, s) ds \right) f(t_i) dt_i \\
= \; & -U(p, x, b) + \int_a^b \left(\int_a^s Q(p, s) \right) f(t_i) dt_i ds \\
= \; & -U(p, x, b) + \int_a^b (F(s)) Q(p, s) ds \\
= \; & -U(p, x, b) + \int_a^b \left(F(t_i) \int_{T_{-i}} \sum_K S(k) p_i^k(\mathbf{t}) f(\mathbf{t_{-i}}) d\mathbf{t_{-i}} \right) dt_i \\
= \; & -U(p, x, b) + \int_{\mathcal{T}} F(t_i) \sum_K S(k) p_i^k(\mathbf{t}) f(\mathbf{t}) dt.
\end{aligned}
$$

To show it is optimal to allocate a lower position (larger k) to a smaller t_i.

Let A represent the list of $t_i + \frac{F(t_i}{f(t_i)}$, and B represent the list of $k - \mu$ (which is $-S(k)$). Assume first there are only three terms in each of the lists, and $0 < A_1 < A_2 < A_3$, $0 < B_1 < B_2 < B_3$. The objective is to minimize $\sum_{i,j} A_i B_j$. First $A_1B_3 + A_2B_2 + A_3B_1 < A_1B_2 + A_2B_3 + A_3B_1$ because $A_1(B_3 - B_2) + A_2(B_2 - B_3) < 0$; Second, $A_1B_3 + A_2B_2 + A_3B_1 < A_1B_1 + A_2B_2 + A_3B_3$, because $A_1(B_3 - B_1) + A_3(B_1 - B_3) = (A_1 - A_1)(B_3 - B_1) < 0$. This shows $A_1B_3 + A_2B_2 + A_3B_1$ is the smallest among all the possible combinations. Thus to allocate a lower position to a smaller $t_i + \frac{F(t_i}{f(t_i)}$ (lower type) is optimal. This result can be generalized to the case where each list contains more than three entries. ◇

A Numerical Example of "divergent" allocation

Assume that buyers' types follow a uniform distribution between $[0, 1]$. Then, the reserve-price condition for position k is: $t_i \leq \frac{\beta}{2(k-\mu)}$. If there are three positions available, $\mu = 0$, and $\beta = 3$, and the reserve prices for positions 1, 2, and 3 are 1, 0.75, and 0.6, respectively. If there are four buyers with realized types $t_1 = 0.8$, $t_2 = 0.6$, $t_3 = 0.4$, and $t_4 = 0.2$, then positions 1, 2, and 3 will be allocated to buyers 2, 3, and 4. The expected payment for those positions are 0.8694, 0.8668, and 1, while the winning bidders' values for the winning positions are 1.2, 1.4, and 2, respectively. If, however, the realized types are $t_1 = 0.9$, $t_2 = 0.85$, $t_3 = 0.8$, and $t_4 = 0.2$, then t_3 is allocated to position 1, and t_4 is allocated to position 2, while position 3, the bottom position, is not allocated.

A.5 Convergent then Divergent

Re-arrange the objective function Re-arrange the last term of Eq.(29),

$$\begin{aligned}
& \int_{\mathcal{T}} \sum_N \sum_K \left(x_i^k(\mathbf{t}) - p_i^k(\mathbf{t}) v(t_i,k)\right) f(\mathbf{t}) dt \\
= & -\int_a^w \sum_N U(p,x,t_i) f(t_i) dt_i - \int_w^a \sum_N U(p,x,t_i) f(t_i) dt_i \\
= & -\int_a^w \sum_N \left(U(p,x,w) - \int_a^w \sum_N Q(p,s) ds\right) f(t_i) dt_i \\
& -\int_w^b \sum_N \left(U(p,x,w) + \int_w^b \sum_N Q(p,s) ds\right) f(t_i) dt_i \\
= & -N \times U(p,x,w) + \int_a^w \sum_N Q_i(p_i,s) ds f(t_i) dt_i - \int_w^b \sum_N Q_i(p,s) ds f(t_i) dt_i \\
= & -N \times U(p,x,w) + \int_a^w \sum_N \left(\int_w^s Q(p,s)\right) f(t_i) dt_i ds - \int_w^a \sum_N \left(\int_w^s Q(p,s)\right) f(t_i) dt_i ds \\
= & -N \times U(p,x,w) + \int_a^w \sum_N \left(F(s)\right) Q(p,s) ds - \int_w^a \sum_N \left(1 - F(s)\right) Q(p,s) ds \\
= & -N \times U(p,x,w) + \int_a^w \sum_N \left(F(t_i) \int_{\mathcal{T}_{-i}} \sum_K S(k) p_i^k(t_i, \mathbf{t_{-i}}) f(\mathbf{t_{-i}}) d\mathbf{t_{-i}}\right) dt_i \\
& -\int_w^a \sum_N \left(1 - F(t_i) \int_{\mathcal{T}_{-i}} \sum_K S(k) p_i^k(t_i, \mathbf{t_{-i}}) f(\mathbf{t_{-i}}) d\mathbf{t_{-i}}\right) dt_i
\end{aligned}$$

where $\int_a^w U(p,x,t_i) f(t_i) dt_i$ can be written as $\int_a^w \left(U(p,x,w) - \int_a^w Q_i(p,s) ds\right) f(t_i) dt_i$ (the second "=" sign), which implies that $\int_a^w Q(p,s) ds f(t_i) dt_i$ is negative. Substitute this back into Eq.(29), and replace $S(k)$ by $\mu - k$, it is Eq.(25).