Approximation Algorithms for Low-Density Graphs

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Abstract

We study the family of intersection graphs of low density objects in low dimensional Euclidean space. This family is quite general, and includes planar graphs. We prove that such graphs have small separators. Next, we present efficient \((1 + \varepsilon)-\)approximation algorithms for these graphs, for \textit{Independent Set}, \textit{Set Cover}, and \textit{Dominating Set} problems, among others. We also prove corresponding hardness of approximation for some of these optimization problems, providing a characterization of their intractability in terms of density.

1. Introduction

\textbf{Realistic inputs and low density.} Motivated by the discrepancy between worst-case analysis and real-world behavior of geometric algorithms, more realistic models of input were developed, together with algorithms that take advantage of such properties. Informally, a set of objects is \textit{low-density} if no ball can intersect too many objects that are larger than it. This notion was introduced by van der Stappen \textit{et al.} [vdSOdBV98], although weaker notions involving a single resolution were studied earlier, see for example the work by Schwartz and Sharir [SS85].

Another property that turns out to be useful, in defending against the curse of unrealistic inputs, is \textit{fatness}. Informally, an object is fat if it contains a ball, and is contained inside another ball, that up to constant scaling are of the same size. Fat objects have low union complexity, and significant research was done to understand their behavior [APS08]. In particular, disjoint fat objects have low density [Sta92], and this is the key property that leads to efficient algorithms for such objects.

For more information about realistic input models and low-density, see [SV96, Vle97, BKS02, dBR12, Dri13].

\textbf{Low density graphs.} In this work we take a different tack – we study the intersection graphs arising out of low-density scenes. The purpose of this work is to understand this family of graphs, both as far as what kind of graphs belong to this family, and what optimization problems can be solved efficiently for these graphs.

Specifically, a set \(F\) of objects in \(\mathbb{R}^d\) induces an \textit{intersection graph} \(G_F\) having \(F\) as its the set of vertices, and two objects \(f, g \in F\) connected by an edge if and only if \(f \cap g \neq \emptyset\). Without any restrictions,
intersection graphs can represent any graph. In this work, we assume that the density of the objects of $\mathcal{F}$ is bounded; that is, for any ball $b$ the number of objects of $\mathcal{F}$ that intersect it and have a larger diameter is a constant.

**Related graph families.** The circle packing theorem [Koe36, And70, PA95] implies that every planar graph can be realized as a coin graph, where the vertices are interior disjoint disks, and there is an edge connecting two vertices if their corresponding disks are touching. This implies that planar graphs are low density. Miller et al. [MTTV97] studied the intersection graphs of balls (or fat convex object) of bounded depth (i.e., every point is covered by a constant number of balls), and these intersection graphs are readily low density.

There is much work on intersection graphs, from interval graphs, to unit disk graphs, and more. For example, some results related to our work include: (i) planar graphs are the intersection graph of segments [CG09], and (ii) string graphs (i.e., intersection graph of curves in the plane) have small separators [Mat14].

**Maximum independent set.** For a graph $G = (V, E)$, the task is to compute the largest vertex set $I \subseteq V$ such that no two vertices in $I$ are connected by an edge. This problem is $\text{NP-Hard}$ [Kar72], and Håstad showed that there is no $n^{1-\varepsilon}$-approximation for any $\varepsilon > 0$, in polynomial time, unless $P = ZPP$ [Has96].

The problem is easier if $G$ has some additional properties. Lipton and Tarjan obtained $O\left(\sqrt{\log \log n}\right)$ approximation algorithm for planar graphs using planar separators [LT79, LT80]. An analogous separator and approximation algorithm was found by Alon et al. [AST90] for graphs excluding a fixed minor. Baker [Bak94] found a $\text{PTAS}^\text{\$}$ for planar graphs by decomposing planar graphs into $k$-outerplanar subgraphs. This approach was the basis of Eppstein’s $\text{PTAS}$ for graphs with bounded local treewidth, a class that includes bounded genus graphs [Epp00].

**Independent set in geometric settings.** Clark et al. [CCJ91] showed that maximum independent set remains $\text{NP-Hard}$ in the intersection graph of unit disks (i.e., unit disk graphs) [CCJ91]. Hunt et al. [HMR+98] designed a $\text{PTAS}$ for unit disk graphs when a geometric representation of the disks is given, and Neiberg et al. [NHK04] designed a robust $\text{PTAS}$ for unit disk graphs that does not require geometric representation. Chan obtained a $\text{PTAS}$ for fat objects using shifted quadtrees and a separator theorem, and Erlebach et al. obtained a $\text{PTAS}$ when the objects are disks, squares, regular polygons, or rectangles with bounded ratio of height to width [Cha03, EJS05]. Recently, Chan and Har-Peled showed that local search is a $\text{PTAS}$ for pseudo-disks in the plane, and the analysis is centered on an application of the planar separator theorem [CH12]. This algorithm also works for planar graphs.

**Separators.** Loosely speaking, a balanced separator is a subset of vertices of a graph, whose removal splits a graph into two roughly equal parts with no edges between them. For a connectivity problem (e.g., independent set), a balanced separator effectively divides the problem, as long as the separator is small compared to the optimal solution. A separator for the intersection graph of disks/balls of bounded depth is implied by the work of Miller et al., which in turn implies an alternative geometric proof of the planar separator theorem [MTTV97, Har13]. For further constructions of separator via geometry, see Smith and Wormald [SW98]. Separators have played a key role in developing geometric optimization

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$^\text{\$}$A $\text{PTAS}$ is a polynomial time approximation scheme: for every constant $\varepsilon > 0$, the algorithm computes a $(1 + \varepsilon)$-approximation in polynomial time. $\text{QPTAS}$ is a “$\text{PTAS}$” with running time of the form $n^{\text{poly} \left( \log n, 1/\varepsilon \right)}$, where $\text{poly}(\ldots)$ is some fixed degree polynomial in its arguments.
algorithms, including a PTAS for piercing half-spaces and pseudo-disks [MR10], a QPTAS for maximum weighted independent sets of polygons [AW13, AW14, Har14], and a QPTAS for Set Cover by pseudodisks [MRR14a].

1.1. Our results

In Section 2, we formally define low-density graphs, and quickly survey some of their properties (some of which we prove in subsequent sections). In Section 3, we prove that low-density graphs have balanced separators and supply a linear-time algorithm to compute them. The result is somewhat more general: Given an integer \( k \), there is a ball \( b \) that intersects \( \Theta(k) \) objects, of which \( o(k) \) are not wholly contained in \( b \). For \( k = \Theta(n) \), where \( n \) is the total number of objects, this implies a balanced separator.

In Section 4, we prove that local search is a PTAS for some geometric optimization problems on low-density graphs. The geometric representation is not required for the algorithm itself, and is only used in the analysis, where the separator theorem is applied recursively to argue about the existence of beneficial local exchanges. Our analysis covers both packing and covering problems, where the latter requires additional structural properties. In Section 4.4, we show that connected dominating set can be approximated by local search despite its global connectivity constraint.

An interesting property of our PTAS for the Independent Set problem, is that it works for any graph family that has heredity sublinear separators. This observation is implicit in the previous work of Chan and Har-Peled [CH12]. The PTAS for the Dominating Set problem requires in addition that the graph remains in the family (i.e., still has a small separator), after shrinking local clusters into single vertices.

In Section 5, we give hardness results for geometric set cover, hitting set, and independent set. The geometric objects at play are characterized by simple and ostensibly useful geometric properties (e.g. fat, triangular, interior disjoint, and in low-dimensions), but notably lack a fixed density. The results demonstrate the significance of density in our algorithms, and is tight as some recent PTAS and QPTAS results demonstrate [MR10, AW13, AW14, Har14, MRR14a]. Some of the results of Section 5 appeared before in an unpublished manuscript [Har09].

Implications. Given a set of points and a set of fat triangles of the same size in the plane, Lemma 5.3 shows that there is no PTAS (or QPTAS) for the Set Cover problem of approximating the minimum number of triangles needed to cover all the points. Here, the set of triangles realizing this hardness has unbounded density. On the other hand, if the set of triangles has bounded density (which for fat triangles, is equivalent to bounded depth) then there is a PTAS for Set Cover in this setting, see Corollary 4.9. Furthermore, if the density is polylogarithmic then our results imply a QPTAS, but there is no PTAS under the ETH (exponential time hypothesis). Figure 5.3p19 and Figure 5.4p20 summarizes these results. Thus, density seems to characterize the hardness of geometric set-cover and hitting-set problems fairly well.

Exposed segments. In Appendix C we prove a condition for segments that implies that they have low density. Generally speaking, the condition forbids a segment to be too close to too many segments that are longer and nearly parallel to it. Line segments satisfying this condition are an example of a low-density collection of objects each with zero volume. We feel this characterization may be of independent interest.

Paper organization. In Section 2 we formally define low density graphs and summarize their properties. In Section 3, we prove the existence of separators for such graphs, and provide an algorithm
2. Low density graphs: Definition and properties

Consider a set of objects $\mathcal{F}$. The intersection graph of $\mathcal{F}$, denoted by $G_{\mathcal{F}}$, is the graph having $\mathcal{F}$ as its set of vertices, and an edge between two objects $f, g \in \mathcal{F}$ if they intersect; that is, formally $G_{\mathcal{F}} = (\mathcal{F}, \{fg | f, g \in \mathcal{F} \text{ and } f \cap g \neq \emptyset\})$.

Definition 2.1. A set of objects $\mathcal{F}$ in $\mathbb{R}^d$ (not necessarily convex or connected) has density $\rho$ if any ball $b$ intersects at most $\rho$ objects in $\mathcal{F}$ with diameter larger than the diameter of $b$. If $\rho$ is a constant, then $\mathcal{F}$ has low density. We denote the density of $\mathcal{F}$ by $\text{density}(\mathcal{F})$.

In particular, a graph $G$ that is the intersection graph of low density objects in constant dimensions is a low density graph. Low density graphs have the following properties:

(P1) **Bounded fatness and depth $\Rightarrow$ low density.** A set of $\alpha$-fat convex objects in $\mathbb{R}^d$ with bounded depth had bounded density, where a set had depth $k$ if every point in $\mathbb{R}^d$ is covered at most $k$ times. See Section A.1.

(P2) **Low density graphs are degenerate.** Given a set $\mathcal{F}$ of low density objects in $\mathbb{R}^d$, there is always a vertex in $G_{\mathcal{F}}$ of constant degree. This is easy to see if one considers the object in $\mathcal{F}$ with the smallest diameter.

(P3) **Low density graphs are not minor free.** Surprisingly, even for a set $\mathcal{F}$ of convex and fat shapes in the plane, with constant density, their intersection graph $G_{\mathcal{F}}$ can have arbitrarily large cliques as minors. See Figure 2.1 for an example.

(P4) **Low density is preserved even if one merges shapes locally.** For a low density family of shapes $F$, a $t$-patch is a subset $F' \subseteq F$ such that $G_{F'}$ is connected and has graph diameter $\leq t$. A $t$-patch of objects $F'$ can be replaced by the merged object $\bigcup F' = \bigcup_{f \in F'} f$. Now, given a partition of $\mathcal{F}$ into $t$-patches, for bounded $t$, the resulting set of objects from merging each patch into a single object, is still low density. See Section A.2 for the proof.

(P5) **Low density graphs are nowhere dense.** See Section A.3 for details.
3. Separators and divisions for low density graphs

3.1. Definitions: Separators and breakable graphs

Definition 3.1. Let \( G = (V, E) \) be an undirected graph. Two sets \( X, Y \subseteq V \) are **separate** in \( G \) if (i) \( X \) and \( Y \) are disjoint, and (ii) there is no edge between the vertices of \( X \) and \( Y \) in \( G \). A set \( Z \) is a **separator** of \( X \) and \( Y \), if \( X \setminus Z \) and \( Y \setminus Z \) are separate. A set \( Z \subseteq V \) is a **balanced separator** for \( U \), if \( |Z| = o(|U|) \), and \( U \setminus Z \) can be partitioned into two separate sets \( X \) and \( Y \) with \( |X|, |Y| = \Omega(|U|) \).

Definition 3.2. For constants \( \alpha \in [0,1) \) and \( c > 0 \), a graph \( G = (V, E) \) is **(\( \alpha, c \))-breakable** (or just **\( \alpha \)-breakable**), if for any \( m \), and any subset \( X \subseteq V \) of size \( m \), the **induced** subgraph \( G_X = (X, \{ uv \in E \mid u, v \in X \}) \) has a balanced separator of size \( \leq cm^\alpha \log m \).

Trees have a balanced separator that is a single vertex, and as such all trees are \( 0 \)-breakable. Planar graphs are \( 1/2 \)-breakable, and graphs that do not have a clique of size \( h \) as a minor are also \( (1/2, h^{3/2}) \)-breakable [AST90]. Similarly, the intersection graph of fat convex objects in \( \mathbb{R}^d \) with bounded depth, are \( (1-1/d) \)-breakable [SW98]. It is easy to verify that breakable graphs are degenerate; that is, they always contain a vertex of constant degree. We next prove that low-density graphs are also \( (1-1/d) \)-breakable.

3.2. Separating low-density objects

**Theorem 3.3.** Let \( F \) be a set of \( n \) objects in \( \mathbb{R}^d \) with density \( \rho > 0 \) (see Definition 2.1, p4), and let \( k \leq n \) be some prespecified number. Then, one can compute, in expected \( O(n) \) time, a sphere \( S \) that intersects \( O\left( \rho + \rho^{1/d} k^{-1/d} \log^{1/d} k \right) \) objects of \( F \). Furthermore, the number of objects of \( F \) strictly inside \( S \) is at least \( k - o(k) \), and at most \( O(k) \).

*Proof:* For ease of exposition, we assume that each object in \( F \) is connected. The proof also works for the more general case by carefully defining what it means for an object to “intersect” a sphere.

For every object \( f \in F \), choose an arbitrary representative point \( p_f \in f \). Let \( P \) be the resulting set of points. Next, let \( b(c, r) \) be the smallest ball containing \( k \) points of \( P \). As in [Har13], randomly pick \( R \) uniformly in the range \([r, 2r]\). We claim that the sphere \( S = S(c, R) \) bounding the ball \( b = b(c, R) \) is the desired separator.

The ball \( b' = b(c, 3r) \) can be covered by \((c_{dbl})^2 \) balls of radius \( r \), and as such it can contain at most \( k' = (c_{dbl})^2 k \) points of \( P \), which implies that \( b \) can contain at most \( O(k) \) objects of \( F \) inside it, where \( c_{dbl} \) is the doubling constant of \( \mathbb{R}^d \) (see Definition A.2, p21).
For $i \geq 0$, let $\ell_i = r / (4 \cdot 2^i)$. For any such $i$, cover $b'$ by a small number of balls of radius $\ell_i$, and let $B_i$ be the resulting set of balls. By careful construction of $B_i$, we have that $\text{vol}(B_i) = \sum_{b'' \in B_i} \text{vol}(b'') = O(\text{vol}(b')) = O(r^d)$.

Next, let $G_0$ be all the objects of $F$ that intersects $b(c, 2r)$ and have diameter $\geq 2\ell_0$. It is easy to verify that $|G_0| = O(\rho)$. Similarly, for $i > 0$, let $G_i$ be all the objects of $F$ that intersects $b(c, 2r)$ and have diameter in the range $[2\ell_i, 2\ell_{i-1}]$. Observe that every ball of $B_i$ can intersect at most $\rho$ objects of $G_i$. Furthermore, all the objects of $G = \bigcup_{i=1}^{\infty} G_i$ are contained in $b'$, and thus $|G| = O(k)$.

For $i > 0$, we have that

$$\sum_{f \in G_i} (\text{diam}(f))^d = \sum_{b'' \in B_i} \sum_{f \in G_i, f \cap b'' \neq \emptyset} (\text{diam}(f))^d \leq \sum_{b'' \in B_i} O\left(\sum_{f \in G_i} \rho \ell_{i-1}^d\right) = O\left(\rho \sum_{b'' \in B_i} \text{vol}(b'')\right) = O(\rho r^d).$$

In particular, for $M = 10 \lceil \log_2 k \rceil$, we have that

$$\sum_{f \in G} (\text{diam}(f))^d = \sum_{i=1}^{M} \sum_{f \in G_i} (\text{diam}(f))^d + \sum_{i=M+1}^{\infty} \sum_{f \in G_i} (\text{diam}(f))^d = O\left(\rho (3r)^d M + k(\ell_M)^d\right)$$

$$= O(\rho r^d \log k),$$

as $|G| = O(k)$ and $\ell_M \leq r / k^9$.

Consider an object $f \in G$, and let $q$ and $q'$ be the points of $f$ closest and farthest from $c$, respectively. Clearly, the probability of $S$ to intersect $f$ is the probability of $r \in I = [\|c - q\|, \|c - q'\|]$. By the triangle inequality, $|I| \leq \|q - q'\| \leq \text{diam}(f)$. As such, the probability that $f$ intersects $S$ is $\leq \text{diam}(f) / r$.

For an object $f$, let $X_f$ be an indicator variable that is one if $f$ intersects $S$. By linear of expectations, and Hölder’s inequality, we have that the expected number of objects of $G$ that intersects $S$ is

$$\alpha = \mathbb{E}\left[\sum_{f \in G} X_f\right] = \sum_{f \in G} \mathbb{P}[f \text{ intersects } S] \leq \sum_{f \in G} \frac{\text{diam}(f)}{r} = \frac{1}{r} \sum_{f \in G} 1 \cdot \text{diam}(f)$$

$$\leq \frac{1}{r} \left(\sum_{f \in G} 1^{d/(d-1)}\right)^{(d-1)/d} \left(\sum_{f \in G} \text{diam}(f)^d\right)^{1/d} \leq O\left(\frac{|G|^{(d-1)/d}}{r} (\rho \cdot r^d \log k)^{1/d}\right)$$

$$= O\left(\rho^{1/d} k^{(d-1)/d} \log^{1/d} k\right).$$

Overall, in expectation, the sphere $S$ intersects at most $\alpha' = |G_0| + \alpha = O\left(\rho + \rho^{1/d} k^{(d-1)/d} \log^{1/d} k\right)$, objects of $F$.

As for the running time, it is sufficient to find a two approximation to the smallest ball that contains $k$ points of $P$, and this can be done in linear time [HR13]. Using such an approximation slightly deteriorates the constants in the bounds. By Markov’s inequality, $S$ intersects at most $2\alpha'$ objects of $F$ with probability $\geq 1/2$. If this is not true, we rerun the algorithm. Clearly, in expectation, after a constant number of iterations the algorithm would succeed in finding a sphere that intersects at most $2\alpha'$ objects of $F$.

\begin{observation}
Mark de Berg (personal communication) pointed out that the bound of Theorem 3.3 can be improved (by removing the log factor), and the proof itself can be simplified. To this end, consider the distance $\ell = t \cdot r$, where $t \in (0, 1)$ is some real number to be specified shortly. The ball $b' = b(c, 3r)$ intersects at most $O(\rho / t^d)$ objects that have diameter larger than $\ell$. Furthermore, a sphere with a radius
\end{observation}
between \( r \) and \( 3r \) can intersect only \( O(\rho/\ell^{d-1}) \) objects with diameter \( \geq \ell \) – indeed, cover the sphere with \( O(1/\ell^{d-1}) \) balls of radius \( \ell/2 \), and let \( \mathcal{B} \) be this set of balls. Now, charge each object intersecting the sphere to the ball of \( \mathcal{B} \) that intersect it. Each ball of \( \mathcal{B} \) get changed \( \rho \) times at most.

On the other hand, the ball \( b_r(c, 3r) \) intersects at most \( O(k) \) smaller objects. In particular, there at most \( O(k) \) such object with diameter \( < \ell \). Each such object of diameter \( < \ell \), has probability at most \( \ell/r = t \) to be in the separator. As such, we get that the separator size in expectation is \( O(\rho + \rho/\ell^{d-1} + kt) \). As such, for \( \rho/\ell^{d-1} = kt \) we get \( t = (\rho/k)^{1/d} \), and the resulting separator is in expectation of size \( O((\rho + \rho^{1/d}k^{1-1/d})/d) \).

3.2.1. Separating weighted low-density objects

It is easy to extend the above to the weighted settings, yielding the following result. For the sake of completeness, the proof is provided in Appendix B.1.

**Lemma 3.5.** Let \( \mathcal{F} \) be a set of \( n \) objects in \( \mathbb{R}^d \) with density \( \rho \), and weights \( w : \mathcal{F} \to \mathbb{R} \). Let \( W = \sum_{f \in \mathcal{F}} w(f) \) be the total weight of all objects in \( \mathcal{F} \). Then one can compute, in expected linear time, a sphere \( \mathbb{S} \) that intersects \( O(\rho + \rho^{1/d}n^{1-1/d} \log^{1/d} n) \) objects of \( \mathcal{F} \). Furthermore, the total weight of objects of \( \mathcal{F} \) strictly inside/outside \( \mathbb{S} \) is at most \( cW \), where \( c \) is a constant that depends only on \( d \).

**Corollary 3.6.** Let \( \mathcal{F} \) be a set of \( n \) objects in \( \mathbb{R}^d \) with density \( \rho \), and \( G = \mathcal{G}_\mathcal{F} \) be its intersection graph.

(A) For any \( k \leq n \), one can compute, in expected \( O(n) \) time, a separator \( Z \subseteq V(G) \), of size \( O(\rho + \rho^{1/d}k^{1-1/d} \log^{1/d} k) \), that separates \( G \) into two sets \( L \) and \( R \) such that \( k - k/c \leq |L| \leq ck \); where \( c > 1 \) is a constant that depends only on \( \rho \) and \( d \).

(B) A balanced separator \( Z \subseteq V(G) \), of size \( O(\rho + \rho^{1/d}n^{1-1/d} \log^{1/d} n) \), can be computed, in expected linear time. It separates \( G \) into two sets of size \( \Omega(n) \).

4. Approximation algorithms for low density graphs

4.1. Recursive separations

4.1.1. Definitions

Consider a set \( V \), and a family \( \mathcal{C} \) of sets \( C_1, \ldots, C_k \subseteq V \), such that \( V = \bigcup_{i=1}^k C_i \). A set \( C_i \) is a cluster, and \( \mathcal{C} = \{C_1, \ldots, C_k\} \) is a cover. A cover of a graph \( G = (V, E) \) is a cover of its vertices. Given a cover \( \mathcal{C} \), the excess of a vertex \( v \in V \) that appears in \( j \) clusters is \( j - 1 \). The excess of a vertex in a cover is always nonnegative. The total excess of the cover \( \mathcal{C} \) is the sum of excesses of all the vertices in \( V \).

A cover \( \mathcal{C} \) of a graph \( G \) is a \( \lambda \)-division if (i) for all \( C, C' \in \mathcal{C} \), \( C \setminus C' \) and \( C' \setminus C \) are separated, and (ii) for all \( C \in \mathcal{C} \), \(|C| \leq \lambda \). A vertex \( v \in V \) is an interior vertex of a cover \( \mathcal{C} \), if it appears in exactly one cluster of \( \mathcal{C} \) (and then its excess is zero), and it is a boundary vertex otherwise. The boundary vertices and interior vertices are complimentary sets, and the clusters overlap only in the boundary vertices.

4.1.2. Breakable graphs have divisions with little excess

Henzinger et al. [HKRS97] remarked that \( \lambda \)-divisions exist for any \( n^\gamma \) separator (where \( \gamma < 1 \)). The salient feature of such \( \lambda \)-divisions is that the excess can be made to be arbitrarily small by increasing \( \lambda \).
Lemma 4.1. Let $\alpha \in [0, 1), c \in \mathbb{R}^+$ be constants, $G = (V, E)$ be an $(\alpha, c)$-breakable graph with $n$ vertices, and $\varepsilon > 0$ be a parameter. Then, there exists a constant $c_\varepsilon$, such that for any $\lambda > c_\varepsilon$, there is a $\lambda$-division of $G$ with excess $\leq \varepsilon n$. Here $c_\varepsilon = O\left(\frac{\varepsilon \ln \frac{1}{\varepsilon}}{\varepsilon \ln \frac{1}{\varepsilon}}\right)$.

For the sake of completeness, we provide a proof of Lemma 4.1 in Appendix B.2.

4.1.3. Low density graphs have divisions with little excess

Lemma 4.1 implies that low density graphs have divisions with little excess, but one can also prove it directly from the lopsided separator result (Theorem 3.3), yielding divisions with slightly better constants.

Lemma 4.2. Let $F$ be a set of $n$ objects in $\mathbb{R}^d$, with density $\rho > 0$, and let $\varepsilon > 0$ and $\lambda$ be parameters, such that $\lambda \geq c\rho^{(d+1)/d}/\varepsilon^{d+1}$, where $c$ is some sufficiently large constant. Then, there exists a $\lambda$-division of $F$ with total excess at most $\varepsilon n$.

Proof: Let $\lambda \geq \rho$ be a parameter to be specified shortly, and let $G$ the intersection graph induced by $F$. Applying Corollary 3.6 to $G$ with $k = \Theta(\lambda)$, we get a set $W_1$ such that $|W_1| \leq \lambda$, $|W_1| = \Omega(\lambda)$, and with $O(\rho^{1/d}\lambda 1/d \log 1/d \lambda)$ boundary vertices. Removing the interior vertices of $W_1$ from $G$, we repeat the extraction until the set $F$ is exhausted, resulting in a $\lambda$-division $W = \{W_1, \ldots, W_m\}$, with $m = \Theta(n/\lambda)$. For some constants $c, c_1$, the total excess is bounded by

$$cm\rho^{1/d}\lambda^{1-1/d} \log^{1/d} \lambda = n \frac{c_1\rho^{1/d}\lambda^{1-1/d} \log^{1/d} \lambda}{\lambda}.$$ 

For this quantity to be smaller than $\varepsilon n$, we need that

$$\frac{c_1\rho^{1/d}\lambda^{1-1/d} \log^{1/d} \lambda}{\lambda} \leq \varepsilon \iff \frac{c_1\rho \log \lambda}{\lambda} \leq \varepsilon^d,$$

which holds if $\lambda = \Omega\left(\frac{\rho}{\varepsilon^d} \log \frac{\rho}{\varepsilon^d}\right) = \Omega\left(\rho^{(d+1)/d}/\varepsilon^{d+1}\right)$. 

Remark 4.3. One can also obtain $\lambda$-divisions in the same manner as [Fre87], using the weighted balanced separator in Lemma 3.5 in place of Lipton and Tarjan’s weighted planar separator. More exactly, by following the proof of Theorem 1 of [Fre87], we can obtain a $\lambda$-division of $F$ such that each cluster contains at most $O\left(\rho + \rho^{1/d}\lambda^{(d-1)/d} \log^{1/d} b\right)$ boundary vertices in time $O(n \log n)$. This in turn implies Theorem 3.3.

4.2. PTAS for independent set and packing problems

In the following, we describe how to get a PTAS for computing sparse structures in breakable graphs, and in low density graphs. The following definitions are from [Har14], and the basic idea of the analysis of the local search algorithm is from Chan and Har-Peled [CH12].

Definitions. Let $\Pi \subseteq 2^V$ be a property defined over subsets of vertices of a graph $G = (V, E)$. The property $\Pi$ is hereditary if for every nested pair of sets $S \subseteq T \subseteq V$, if $T$ satisfies $\Pi$, then $S$ satisfies $\Pi$. The property $\Pi$ is mergeable if for any two sets $S, T \subseteq V$ that are separate in $G$, if $S$ and $T$ each satisfy $\Pi$, then $S \cup T$ satisfies $\Pi$. We assume that whether or not $S \in \Pi$ can be checked in polynomial time. A packing problem consists of a property $\Pi$, and the objective is to find the largest subset of $F$ satisfying $\Pi$. Some geometric flavors of packing problems are:
(A) Given a collection of objects \( F \), find the maximum independent subset of \( F \).

(B) Given a collection of objects \( F \), find the maximum subset of \( F \) with bounded depth.

(C) Given a collection of objects \( F \), find the maximum subset of \( F \) whose intersection graph is planar or otherwise excludes a graph minor.

(D) Given a point set \( P \), a constant \( k \), and a collection of objects \( F \), find the maximum subset of \( F \) such that each point in \( P \) is contained in at most \( k \) objects in \( F \).

Here, we show how to solve all these problems by local search if the objects have low density.

### 4.2.1. The local search algorithm

Let \( G = (V, E) \) be a breakable graph, \( \Pi \) a property defined over subsets of vertices, and \( \lambda \) a fixed integer. For two sets, \( S \) and \( T \), their \textit{symmetric difference} is \( S \triangle T = (S \setminus T) \cup (T \setminus S) \). Two vertex sets \( S \) and \( T \) are \( \lambda \)-\textit{close} if \( |S \triangle T| \leq \lambda \); that is, if one can transform \( S \) into \( T \) by adding and removing at most \( \lambda \) vertices from \( S \). A vertex set \( S \in \Pi \) is \( \lambda \)-\textit{locally optimal} in \( \Pi \) if there is no \( T \in \Pi \) that is \( \lambda \)-close to \( S \) and “improves” upon \( S \). In a maximization problem \( T \) \textit{improves} \( S \) \( \iff \) \( |T| > |S| \). In a minimization problem, an improvement decreases the cardinality.

A \( \lambda \)-\textit{local search algorithm} starts with a solution \( S \in \Pi \) and repeatedly makes \( \lambda \)-close improvements if one exists, by repeatedly trying all \( \lambda \)-close sets, terminating at a \( \lambda \)-locally optimal solution. Each improvement in a maximization (resp., minimization) problem increases (resp., decreases) the cardinality of the set, so there are at most \( n \) rounds of improvements. Within a round we can exhaustively try all exchanges in time \( n^{O(\lambda)} \), bounding the total running time by \( n^{O(\lambda)} \).

### 4.2.2. Analysis

#### Lemma 4.4.

Let \( \varepsilon > 0, \alpha \in [0, 1) \) and \( c \in \mathbb{R}^+ \) be fixed parameters, \( G = (V, E) \) be a \((\alpha, c)\)-breakable graph, and \( \Pi \) be a mergeable and hereditary property defined over \( V \). Then the local search algorithm (see Section 4.2.1) computes a \((1 - \varepsilon)\)-approximation for the maximum size subset of \( F \) satisfying \( \Pi \), in time \( n^{O(\lambda)} \), assuming that membership in \( \Pi \) can be tested in polynomial time. Here \( \lambda \) is a constant that depends on \( \alpha, c \) and \( \varepsilon \). In particular, this algorithm provides a \( \text{PTAS} \) for independent set in \( G \).

**Proof:** Let \( O \in \Pi_F \) be the optimal maximal set in \( \Pi \), and \( L \in \Pi_F \) a \( \lambda \)-locally maximal set in \( \Pi \). Consider the induced subgraph \( H = G_{|L \cup O|} \), by Lemma 4.1 there exists a \( \lambda \)-division \( W = \{W_1, \ldots, W_m\} \) of \( H \), with boundary vertices \( B \), and total excess \((\varepsilon/2)|L \cup O| \leq \varepsilon|O|\), as \(|L| \leq |O|\). For \( i = 1, \ldots, w \), let

1. \( O_i = (O \cap W_i) \setminus B, a_i = |O_i| \)
2. \( L_i = L \cap W_i, l_i = |L_i| \)
3. \( B_i = B \cap W_i, b_i = |B_i| \)

Fix \( i \), and consider the set \( L' = (L \setminus (L_i \cup B_i)) \cup O_i \). Since \( \Pi \) is hereditary, \( L \setminus L_i \in \Pi \), and since \( O_i \) and \( L \setminus L_i \) are separated, \( L' \in \Pi \). Since \( L \) is \( \lambda \)-locally optimal, the exchange replacing \( L_i \) by \( O_i \) can not improve the size; that is, \( l_i \geq a_i \). Summing over all \( i \), we have

\[
|L| \geq \sum_{i=1}^{m} l_i - \sum_{i=1}^{m} b_i \geq \sum_{i=1}^{m} a_i - \sum_{i=1}^{m} b_i \geq |O| - \text{excess}(W) \geq (1 - \varepsilon)|O|.
\]

Plugging the bounds of Lemma 4.2 into Lemma 4.3 implies the following.

#### Corollary 4.5.

Let \( \varepsilon > 0 \) be given, \( F \) a collection of objects in \( \mathbb{R}^d \), and let \( \Pi \) be a mergeable and hereditary property defined over \( F \) such that any subset of \( F \) satisfying \( \Pi \) has density \( \rho \). Then the local search algorithm (see Section 4.2.1) computes a \((1 - \varepsilon)\)-approximation for the maximum size subset of \( F \) satisfying \( \Pi \), in time \( n^{O(\rho^{d+1/4 \varepsilon d+1})} \), assuming that membership in \( \Pi \) can be tested in polynomial time.
Figure 4.1: (A) A set of triangles covered by two flowers. (B) The first flower $F$. (C) The second flower -- note that flowers might shave triangles. (D) The first flower $F$ induces a polygon $\cup F$.

Remark 4.6. (A) Corollary 4.5 requires low density to hold only for the set being output, and the input set does not have to be low density.

(B) The basic argument of using separator to compute a sparse (i.e., independent set) subset using local search was used by Chan and Har-Peled [CH12]. In particular, Lemma 4.4 is generalizing this result by pointing out that the only necessary property is that the graph is breakable.

4.3. Covering problems

Let $\mathcal{F}$ be a collection of objects in $\mathbb{R}^d$. A dominating set is a subset $\mathcal{D}$ of $\mathcal{F}$ such that every object $f \in \mathcal{F}$ has a nonempty intersection with some object $g \in \mathcal{D}$. Given demands $\delta : \mathcal{F} \to \mathbb{N}$, $\mathcal{D} \subseteq \mathcal{F}$ is a $\delta$-dominating set if object $f \in \mathcal{F}$ intersects at least $\delta(f)$ objects in $\mathcal{D}$ (importantly, $\delta(f)$ might be zero if it requires no domination). In the dominating set problem, we are given $\mathcal{F}$, $\delta$, and a $\delta$-dominating set $\mathcal{D}$ of $\mathcal{F}$, and the task is to compute the minimum cardinality subset $\mathcal{E} \subseteq \mathcal{D}$ that $\delta$-dominates $\mathcal{F}$.

This problem encompasses geometric covering problems in addition to dominating set, such as:

(A) For a set of objects $\mathcal{F}$ and a set of points $\mathcal{P}$, the set cover problem is to compute a minimum cardinality subset $\mathcal{E} \subseteq \mathcal{F}$ such that $\mathcal{P} \subseteq \bigcup_{f \in \mathcal{E}} f$.

(B) The hitting set problem is to compute the minimum cardinality set $\mathcal{Q} \subseteq \mathcal{P}$ such that for every $f \in \mathcal{F}$, we have $\mathcal{Q} \cap f \neq \emptyset$.

(C) Given a fixed integer $k$ and a set of objects $\mathcal{F}$, the $k$-dominating set problem is to compute the minimum cardinality subset $\mathcal{D} \subseteq \mathcal{F}$ such that each object in $\mathcal{F}$ intersects at least $k$ objects in $\mathcal{D}$.

To prove that local search is a PTAS for such problems we again make a local exchange argument, but first require some structural observations regarding density.

4.3.1. PTAS for dominating set using local search in low-density graphs

Flowers. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a flower in $\mathcal{G}$ is a vertex set $F \subseteq \mathcal{V}$ whose induced subgraph is, up to some extra edges, a star; that is, there is a common vertex $h \in F$ (called the head) that all other vertices in $F$ (called the petals) are connected by an edge. A flower is also a 2-patch, see Section A.2. For a set of objects $\mathcal{F}$ in $\mathbb{R}^d$, a flower in $\mathcal{F}$ is a subset of objects $F \subseteq \mathcal{F}$ that forms a flower in the intersection graph of $\mathcal{F}$, see Figure 4.1.

If $\mathcal{C} = \{F_1, \ldots, F_c\}$ is a collection of flowers in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the flower graph $\mathcal{G}_\mathcal{C}$ is the graph with vertex set $\mathcal{C}$ and an edge between two flowers $F_i$ and $F_j$ if either $F_i \cap F_j \neq \emptyset$ or there are vertices $u \in F_i$ and $v \in F_j$ connected by an edge in $\mathcal{G}$.
**Theorem 4.7.** Let $\mathcal{F}$ be a collection of $n$ objects in $\mathbb{R}^d$ with density $\rho$, let $\mathcal{D} \subseteq \mathcal{F}$ be a $\delta$-dominating set for demands $\delta : \mathcal{F} \to \{0, \ldots, \beta\}$, and let $\varepsilon > 0$ be a prespecified parameter. Then the local search algorithm (see Section 4.2.1) computes a $(1 + \varepsilon)$-approximation for the smallest cardinality subset of $\mathcal{D}$ that $\delta$-dominates $\mathcal{F}$ in time $n^{O((\beta \rho)^{(d+1)/d} / \varepsilon^{d+1})}$.

**Proof:** Let $\lambda = O((\beta \rho)^{1+1/d} / \varepsilon^{d+1})$. Let $O$ be the optimal (minimum) subset and $L$ a $\lambda$-locally optimal subset of $\mathcal{D}$ that $\delta$-dominates $\mathcal{F}$. Let $O = \{F_s \mid s \in O\}$ be a cover of $\mathcal{F}$ by flowers such that each $f \in \mathcal{F}$ is contained in exactly $\delta(f)$ flowers, where $s \in O$ is the head of the flower $F_s$. Similarly, let $L = \{F_g \mid g \in L\}$ be the cover by flowers induced by $L$. By Lemma A.6, $O$ and $L$ are of density $O(\beta \rho)$, and therefore $O \cup L$ has density $O(\beta \rho)$.

By Lemma 4.2, there is a $\lambda$-division of $O \cup L$ with clusters $\mathcal{W} = \{W_1, \ldots, W_m\}$, boundary vertices $B$, and total excess $(\varepsilon/8)|O \cup L| \leq (\varepsilon/4)|L|$, as $|O| \leq |L|$. For $i = 1, \ldots, m$, let

(A) $O_i = O \cap W_i$, $O_i = \{s \in O \mid F_s \in O_i\}$, and $o_i = |O_i|$.

(B) $L_i = (L \cap W_i) \setminus B$, $L_i = \{g \in L \mid F_g \in L_i\}$, and $l_i = |L_i|$.

(C) $B_i = B \cap W_i$, and $b_i = |B_i|$.

Fix $i$, and consider the set $L' = (L \setminus L_i) \cup O_i$, and consider an object $f \in \mathcal{F}$. If for some $k > 0$, $f$ is dominated by only $\delta(f) - k$ objects in $L \setminus L_i$, then $f$ is contained in $k$ flowers of $L_i$, which are all interior to the cluster $W_i$. As such, any flower of $O$ that contains $f$, must be in the cluster $W_i$; that is $O_i$ must cover $f$ at least $\delta(f)$ times, and as such it is covered at least $\delta(f)$ times by $L'$. Namely, $L'$ is a valid solution.

Since $L$ is $\lambda$-locally minimal, and $|L_i \cup O_i| \leq \lambda$, this exchange does not decrease the total cardinality; that is, $l_i \leq o_i$. Summing over all $i$, we have

$$|L| \leq \sum_{i=1}^{m} (l_i + b_i) \leq \sum_{i=1}^{m} (o_i + b_i) \leq |O| + 2 \text{excess}(\mathcal{W}) \leq |O| + \frac{\varepsilon}{2} |L|$$

As such, we have $|L| \leq |O| / (1 - \varepsilon/2) \leq (1 + \varepsilon)|O|$.  

**Corollary 4.8 (PTAS for geometric hitting set for low density objects).** Let $\mathcal{F}$ be a collection of $m$ objects in $\mathbb{R}^d$ with density $\rho$, and let $P$ be a set of $n$ points in $\mathbb{R}^d$ such that every object in $\mathcal{F}$ contains at least one point in $P$. Then the local search algorithm (see Section 4.2.1), for exchanges of size $\lambda = O(\rho^{(d+1)/d} / \varepsilon^{d+1})$, computes a subset $Q \subseteq P$ that is a $(1 + \varepsilon)$-approximation for the smallest cardinality subset of $P$ that is a hitting set for $\mathcal{F}$. The running time of the algorithm is $nm^{O(\rho^{(d+1)/d} / \varepsilon^{d+1})}$.

**Proof:** We apply Theorem 4.7 to $\mathcal{F} \cup P$ with $\mathcal{D} = P$, and $\delta$ defined by $\delta(f) = 1$ for all $f \in \mathcal{F}$, and $\delta(p) = 0$ for all $p \in P$.

**Corollary 4.9 (PTAS for geometric set cover for low-density objects).** Let $\mathcal{F}$ be a collection of $m$ objects in $\mathbb{R}^d$ with density $\rho$, and let $P$ be a set of $n$ points in $\mathbb{R}^d$ covered by $\mathcal{F}$. Then the local search algorithm (see Section 4.2.1), for exchanges of size $\lambda = O(\rho^{(d+1)/d} / \varepsilon^{d+1})$, computes a subset $G \subseteq \mathcal{F}$ that is a $(1 + \varepsilon)$-approximation for the smallest cardinality subset of $\mathcal{F}$ that covers $P$. The running time of the algorithm is $nm^{O(\rho^{(d+1)/d} / \varepsilon^{d+1})}$.

**Proof:** We apply Theorem 4.7 to $\mathcal{F} \cup P$ with $\mathcal{D} = \mathcal{F}$ and $\delta$ defined by $\delta(p) = 1$ for all $p \in P$ and $\delta(f) = 0$ for all $f \in \mathcal{F}$.  

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Remark 4.10. The proof of Theorem 4.7 only requires that the graph is recursively separable and closed under shallow minors. In particular, a similar argument shows that the local search algorithm also works for obtaining a PTAS for dominating set in minor free graphs (when the demands are only 0 or 1). Indeed, such a result was recently obtained by Cabello and Gajser [CG14]. However, the algorithm of Grohe [Gro03] is faster, and as such we provide no further details.

4.4. Connected dominating set

Let $G = (V, E)$ be an undirected graph. In the connected dominating set problem, we want to find the smallest vertex set $D \subseteq V$ such that (i) every vertex in $V$ is either in $D$ or connected by an edge to $V$, and (ii) the induced subgraph of $D$ is connected. Unlike the problems considered so far, whose constraints were local in nature, the connected dominating set problem has to satisfy a global connectivity constraint.

Connected dominating set is of interest in wireless network design, in which a connected dominating set forms a “backbone” for the network. In wireless ad-hoc networks, the network may be induced by disks representing the signal range of each device.

Again we lift our algorithm to a more generalized form. Let $F$ be a collection of objects, $\delta : F \rightarrow \{1, \ldots, k\}$ a set of positive demands, and $D \subseteq F$ a connected dominating set. In the connected dominating set problem, we want to find the smallest vertex subset $E \subseteq D$ that is both connected and satisfies the demands.

We start with a simple observation.

Lemma 4.11. Let $G = (V, E)$ be a connected graph and $X, Y \subseteq V$ two sets, such that their union is a dominating set of $G$. Then, $d_G(X, Y) \leq 3$, where $d_G(X, Y) = \min_{u \in X, v \in Y} d_G(u, v)$, and $d_G(u, v)$ is the minimum number of edges in the shortest path in $G$ between $u$ and $v$. In particular, a dominating set $U \subseteq V$ with $k$ connected components can be made to be connected by adding $2(k - 1)$ vertices to it.

Proof: Let $\pi$ be the shortest path in $G$ realizing $d_G(X, Y)$ with its endpoints being $u$ and $v$. If $\pi$ is made out of more than 3 edges, then consider any vertex $z \in \pi$ that is not adjacent to $u$ or $v$. Clearly, we have $d_G(u, z) \geq 2$, and $d_G(z, v) \geq 2$. However, $z$ is dominated by $X \cup Y$, and as such there must be a vertex (say) $z'$ in $X$ that is adjacent to $z$. But then there is a shorter path between $X$ and $Y$, by going from $z'$ to $z$, and then along $\pi$ to $Y$, a contradiction.

As for the second part, as long as the current dominating set $U$ is not connected, set $X$ to be one of its connected components, and $Y = U \setminus X$. Now, using the above, add the (at most) two middle vertices in the shortest path between $X$ and $Y$ to $U$. Repeat this process until $U$ is connected. ■

Theorem 4.12. Let $F$ be a collection of objects in $\mathbb{R}^d$ with density $\rho$, let $D \subseteq F$ a connected $\delta$-dominating set for positive demands $\delta : F \rightarrow \{1, \ldots, \beta\}$, and let $\varepsilon > 0$ be a prespecified parameter. Then the local search algorithm (see Section 4.2.1) computes a $(1 + \varepsilon)$-approximation for the smallest cardinality connected subset of $D$ that $\delta$-dominates $F$, in time $n^{O((\beta \rho)^{d+1}/\varepsilon^{d+1})}$ (i.e., the union of objects in $D$ is connected).

Proof: Let $\lambda = O((\beta \rho)^{d/(d+1)}/\varepsilon^{d+1})$. Let $O$ be the optimal (minimum) connected $\delta$-dominating set and $L$ an $O(\lambda)$-locally minimal connected $\delta$-dominating set of $F$. Let $O = \{F_s | s \in O\}$ be a cover of $F$ by flowers such that each $f \in F$ is contained in exactly $\delta(f)$ flowers, with $s \in O$ the head of the flower $F_s$. Similarly define $L = \{F_g | g \in L\}$ to be the cover of $F$ by flowers included by $L$. The set $O$ is a bounded-radius cover of $F$ in which each $f \in F$ repeats at most $\beta$ times, so $O$ has density $O(\beta \rho)$. Similarly, $L$ has density $O(\beta \rho)$, and $O \cup L$ has density $O(\beta \rho)$.  

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Let \( \mathcal{W} = \{ \mathcal{W}_1, \ldots, \mathcal{W}_m \} \) be an \( \lambda \)-division of \( \mathcal{O} \cup \mathcal{L} \) with boundary vertices \( \mathcal{B} \) and total excess \( \text{excess}(\mathcal{W}) \leq (\varepsilon/16) |\mathcal{O} \cup \mathcal{L}| \leq (\varepsilon/8) |\mathcal{L}| \). One can safely assume that every cluster is connected. For \( i = 1, \ldots, m \), let

(i) \( \mathcal{O}_i = \mathcal{O} \cap \mathcal{W}_i \), \( \mathcal{O}_i = \{ \mathcal{s} \in \mathcal{O} \mid F_s \in \mathcal{O}_i \} \), and \( \mathcal{o}_i = |\mathcal{O}_i| \).

(ii) \( \mathcal{L}_i = (\mathcal{L} \cap \mathcal{W}_i) \setminus \mathcal{B} \), \( \mathcal{L}_i = \{ \mathcal{g} \in \mathcal{L} \mid F_g \in \mathcal{L}_i \} \), and \( \mathcal{l}_i = |\mathcal{L}_i| \).

(iii) \( \mathcal{B}_i = \mathcal{B} \cap \mathcal{W}_i \), \( \mathcal{B}_i = \{ \mathcal{s} \in \mathcal{O} \cup \mathcal{L} \mid F_s \in \mathcal{B}_i \} \), and \( \mathcal{b}_i = |\mathcal{B}_i| \).

For an index \( i \), and consider the set \( \mathcal{L}' = (\mathcal{L} \setminus \mathcal{L}_i) \cup \mathcal{O}_i \). Arguing as in the proof of Theorem 4.7, it follows that \( \mathcal{L}' \) dominates \( \mathcal{F} \), but \( \mathcal{L}' \) may not be connected.

However, \( \mathcal{L} \) is connected, and all the edges between vertices of \( \mathcal{L} \setminus \mathcal{L}_i \) and \( \mathcal{L}_i \) are edges between vertices of \( \mathcal{L} \cap \mathcal{B}_i \) and \( \mathcal{L}_i \). In other words, if \( \mathcal{B}_i \cap \mathcal{L} \) is connected, then \( \mathcal{L} \setminus \mathcal{L}_i \) is also connected. As such, every connected component of \( \mathcal{L} \setminus \mathcal{L}_i \) contains a vertex of \( \mathcal{B}_i \). Arguing similarly every connected component of \( \mathcal{O}_i \) contains a vertex of \( \mathcal{B}_i \).

The connected components of \( \mathcal{L}' = (\mathcal{L} \setminus \mathcal{L}_i) \cup \mathcal{O}_i \) are unions of connected components of \( \mathcal{L} \setminus \mathcal{L}_i \) and \( \mathcal{O}_i \), hence every connected component of \( \mathcal{L}' \) contains a vertex of \( \mathcal{B}_i \). In particular, there are at most \( \mathcal{b}_i \) connected components in \( \mathcal{L}' \), as every such connected component must contain at least one vertex of \( \mathcal{B}_i \). Now, \( \mathcal{L}' \) dominates the whole graph, and in particular \( \mathcal{L}' \) dominates the (connected) set \( \mathcal{D} \). By Lemma 4.11 one can reconnect \( \mathcal{L}' \) by including at most \( 2(\mathcal{b}_i - 1) \) additional objects from \( \mathcal{D} \).

Since \( \mathcal{L} \) is \( O(\lambda) \)-locally optimal, this exchange must increase the overall cardinality, so \( \mathcal{l}_i \leq \mathcal{o}_i + 2\mathcal{b}_i \). Summing over \( i = 1, \ldots, m \), we have

\[
|\mathcal{L}| \leq \sum_{i=1}^{m} (\mathcal{l}_i + \mathcal{b}_i) \leq \sum_{i=1}^{m} (\mathcal{o}_i + 3\mathcal{b}_i) \leq |\mathcal{O}| + 4 \sum_{i=1}^{m} \mathcal{b}_i = |\mathcal{O}| + 4 \cdot \text{excess}(\mathcal{W}) \leq |\mathcal{O}| + \frac{\varepsilon}{2} |\mathcal{L}| \leq (1 + \varepsilon) |\mathcal{O}|,
\]

as desired.

\[\blacksquare\]

**Corollary 4.13.** Let \( \mathcal{F} \) be a collection of \( m \) connected objects in \( \mathbb{R}^d \) with density \( \rho \), and let \( \mathcal{P} \) be a set of \( n \) points in \( \mathbb{R}^d \) covered by \( \mathcal{F} \) such that every object in \( \mathcal{F} \) contains at least one point in \( \mathcal{P} \). Then the local search algorithm (see Section 4.2.1), for exchanges of size \( \lambda = O(\rho^{(d+1)/|\mathcal{E}|^{d+1}}) \), computes a subset that is a \( (1 + \varepsilon) \)-approximation to the smallest cardinality subset \( \mathcal{D} \subseteq \mathcal{F} \) that is connected and covers all the points of \( \mathcal{P} \). The running time of the algorithm is \( nm^{O(\rho^{(d+1)/|\mathcal{E}|^{d+1}})} \).

**Remark 4.14.** As with (unconnected) dominating set, a similar argument proves that local search is a PTAS for connected dominating set in minor-free families of graphs for binary demands. Note that our general form of connected dominating set permits variants such as connected vertex cover and connected edge cover (with a little care). More specifically, we split each edge by a vertex, and argue that we can still “reconnect” dominating sets after local exchanges by adding vertices in proportion to the number of boundary vertices.

**Proof:** Since every object \( \mathcal{F} \) contains at least one point in \( \mathcal{P} \), a set cover for \( \mathcal{P} \) doubles as a dominating set for \( \mathcal{F} \). We apply Theorem 4.7 to \( \mathcal{F} \cup \mathcal{P} \) with \( \delta = \mathcal{F} \) and \( \delta = 1 \) uniformly.

\[\blacksquare\]

The preceding corollary is unsatisfying because we can not handle set cover when there are sets that do not contain any points and only serve as “bridges”. This more general problem reduces to Steiner tree, and we leave the problem of devising a PTAS for this variant as an open problem for further research.
5. Hardness of approximation

Some of the results of this section appeared in an unpublished manuscript [Har09].

5.1. Friendly set cover

Let $U$ be a set of $n$ elements, and $\mathcal{F}$ a set of subsets of $U$ each containing at most $k$ elements of $U$. In the minimum $k$-set cover problem, we want to find the smallest subcollection $\mathcal{G} \subseteq \mathcal{F}$ that covers $U$. The problem is MaxSNP-Hard for $k \geq 3$, meaning there is no PTAS unless $P = NP$ [ACG+99].

**Definition 5.1.** Let $P$ be a set of $n$ points in the plane, and $\mathcal{F}$ be a set of $m$ regions in the plane, such that

(I) the shapes of $\mathcal{F}$ are convex, fat, and of similar size,

(II) the boundaries of any pair of shapes of $\mathcal{F}$ intersect in at most 6 points,

(III) the union complexity of any $m$ shapes of $\mathcal{F}$ is $O(m)$, and

(IV) any point of $P$ is covered by a constant number of shapes of $\mathcal{F}$.

We are interested in finding the minimum number of shapes of $\mathcal{F}$ that covers all the points of $P$. This variant is the friendly geometric set cover problem.

We remind the reader that if the objects in $\mathcal{F}$ have bounded depth, then they have low density, and by Theorem 4.7 there is a PTAS for this problem. The following hardness proof reduces to an instance of friendly geometric set cover where the intersection graph of $\mathcal{F}$ is the furthest thing from being separable: a clique.

**Lemma 5.2.** *There is no PTAS for the friendly geometric set cover problem, unless $P = NP$.***

**Proof:** We will reduce an instance $(U, \mathcal{F})$ of the minimum $k$-set cover problem (for $k = 3$) into an instance of the friendly geometric set cover problem.

Figure 5.1: (i) A region $g$ constructed for the set $S_t = \{u_i, u_j, u_k\}$. Observe that in the construction, the inner disk is even bigger. As such, no two points are connected by an edge of the convex-hull when we add in the inner disk to the convex-hull. As such, each point “contribution” to the region $g$ is separated from the contribution of other points. (ii) How two such regions together. (iii) Their intersection.
Let $U = \{u_1, \ldots, u_n\}$ be a set of $n$ elements, and $\mathcal{F} = \{S_1, \ldots, S_m\}$ a collection of $m$ subsets of $U$. We place $n$ points equally spaced on the unit radius circle centered at the origin, and let $P = \{p_1, \ldots, p_n\}$ be the resulting set of points. For each point $u_i \in U$, let $f(u_i) = p_i$. For each set $S_i \in \mathcal{F}$ (of size at most 3), we define the region

$$g_i = \mathcal{CH}\left( \text{disk}\left(1 - \frac{i}{10n^2m}\right) \cup f(S_i) \right),$$

where $\mathcal{CH}$ is the convex hull, $f(S_i) = \bigcup_{x \in S_i} \{f(x)\}$, and $\text{disk}(r)$ denotes the disk of radius $r$ centered at the origin. Visually, $g_i$ is a disk with three (since $k = 3$) teeth coming out of it, see Figure 5.1. Note that the boundary of two such shapes intersects in at most 6 points.

It is now easy to verify that the resulting instance of geometric set cover $(P, \{g_1, \ldots, g_m\})$ is friendly, and clearly any cover of $P$ by these shapes can be interpreted as a cover of $U$ by the corresponding sets of $\mathcal{F}$. Thus, a PTAS for the friendly geometric set cover problem implies a PTAS for the minimum $k$-set cover, which is impossible unless $P = \text{NP}$.

5.2. Set cover by fat triangles

It is known that Vertex Cover is APX-HARD even for a graph with maximum degree 3 [ACG+99]. A problem that is APX-HARD does not have a PTAS unless $P = \text{NP}$.

In the fat-triangle set cover problem, specified by a set of points in the plane $P$ and a set of fat triangles $\mathcal{F}$, one wants to find the minimum subset of $\mathcal{F}$ such that its union covers all the points of $P$.

Lemma 5.3. There is no PTAS for the fat-triangle set cover problem, unless $P = \text{NP}$. Furthermore, one can prespecify an arbitrary constant $\delta > 0$, and the claim would hold true even if the following conditions hold on the given instance $(P, \mathcal{F})$:

(A) The minimum angle of all the triangles of $\mathcal{F}$ is larger than $45 - \delta$ degrees.

(B) No point of $P$ is covered by more than two triangles of $\mathcal{F}$.

(C) The points of $P$ are in convex position.

(D) All the triangles of $\mathcal{F}$ are of similar size. Specifically, each triangle has diameter in the range (say) $(2 - \delta, 2]$.

(E) Each triangle of $\mathcal{F}$ has two angles in the range $(45 - \delta, 45 + \delta)$, and one angle in the range $(90 - \delta, 90 + \delta)$.

(F) The vertices of the triangles of $\mathcal{F}$ are the points of $P$.

Proof: Consider a graph $G$ with maximum degree three, and observe that a Vertex Cover problem in such a graph can be reduced to Set Cover where every set is of size at most 3. Indeed, the ground set $U$ is the edges of $G$, and every vertex $v \in V(G)$ gives a rise to the set $S_v = \{uv \in E(G) \mid u \in V(G)\}$, which is of size at most 3. Clearly, any cover $C$ of size $t$ for the set system $\mathcal{X} = \left( U, \{S_v \mid v \in V(G)\} \right)$, has a corresponding vertex cover of $G$ of the same size. Thus, Set Cover with every set of size (at most) three is APX-HARD (this is of course well known). Note that in this set cover instance, every element participates in exactly two sets (i.e., the two vertices adjacent to the original edge).

The graph $G$ has maximum degree three, and by Vizing’s theorem [BM76], it is 4 edge-colorable\(^2\).

\(^2\)Vizing’s theorem states that a graph with maximum degree $\Delta$ can be edge colored by $\Delta + 1$ colors. In this specific case, one can reach the same conclusion directly from Brook’s theorem (which states that the chromatic number of a graph with maximum degree $\Delta > 2$ is $\Delta$, if it does contain a clique $K_\Delta$ as a subgraph). Indeed, in our case, the adjacency graph of the edges has degree at most 4, and it does not contain a clique of size 4. As such, this graph is 4-colorable, implying the original graph edges are 4-colorable.
With regards to the set problem, the ground set of the set system \( X \) can be colored by 4 colors such that no set in this set system has a color appearing more than once.

We are given an instance of the Vertex Cover problem for a graph with maximum degree 3, and we transform it into a set cover instance as mentioned above, denoted by \( X = (U, \mathcal{F}_X) \). Let \( n = |U| \), and color \( U \) (as described above) by 4 colors such that no set of \( X \) has the same color repeated twice, let \( U_1, \ldots, U_4 \) be the partition of \( U \) by the color of the points.

Let \( C \) denote the circle of radius one centered at the origin. We place four relatively short arcs on \( C \), placed on the four intersection points of \( C \) with the \( x \) and \( y \) axes, see figure on the right. Let \( I_1, \ldots, I_4 \) denote these four circular intervals. We equally space the elements of \( U_i \) (as points) on the interval \( I_i \), for \( i = 1, \ldots, 4 \). Let \( P \) be the resulting set of points.

For every set \( S \in \mathcal{F}_X \), take the convex hull of the points corresponding to its elements as its representing triangle \( T_S \). Note, that since the vertices of \( T_S \) lie on three intervals out of \( I_1, I_2, I_3, I_4 \), it follows that it must be fat, for all \( S \in \mathcal{F}_X \).

As such, the resulting set of triangles \( \mathcal{F} = \{ T_S \mid S \in \mathcal{F}_X \} \) is fat, and clearly there is a cover of \( P \) by \( t \) triangles of \( \mathcal{F} \) if and only if the original set cover problem has a cover of size \( t \).

Any triangle having its three vertices on three different intervals of \( I_1, \ldots, I_4 \) is close to being an isosceles triangle with the middle angle being 90 degrees. As such, by choosing these intervals to be sufficiently short, any triangle of \( \mathcal{F} \) would have a minimum degree larger than, say, \( 45 - \delta \) degrees, and with diameter in the range between \( 2 - \delta \) and \( 2 \).

This is clearly an instance of the fat-triangle set cover problem. Solving it is equivalent to solving the original Vertex Cover problem, but since it is APX-HARD, it follows that the fat-triangle set cover problem is APX-HARD.

\[ \text{Remark 5.4.} \] For fat triangles of similar size a constant factor approximation algorithm is known [CV07]. Lemma 5.3 implies that one can do no better. Naturally, it might be possible to slightly improve the constant of approximation provided by the algorithm of Clarkson and Varadarajan [CV07]. However, for fat triangles of different sizes, only a \( \log^* \) approximation is known [ABES14]. It is natural to ask if this can be improved.

5.2.1. Extensions

Lemma 5.5. Given a set of points \( P \) in the plane and a set of circles \( \mathcal{F} \), finding the minimum number of circles of \( \mathcal{F} \) that covers \( P \) is APX-HARD; that is, there is no PTAS for this problem.

\[ \text{Proof:} \] Slightly perturb the point set used in the proof of Lemma 5.3, so that no four points of it are co-circular. Let \( P \) denote the resulting set of points. For every set \( S \in \mathcal{F}_X \), we now take the circle passing through the three corresponding points. Clearly, this results in a set of circles (that are almost identical, but yet all different), such that finding the minimum number of circles covering the set \( P \) is equivalent to solving the original problem.

Lemma 5.6. Given a set of points \( Q \) in \( \mathbb{R}^3 \) and a set of planes \( \mathcal{F} \), finding the minimum number of planes of \( \mathcal{F} \) that covers \( Q \) is APX-HARD; that is, there is no PTAS for this problem.

\[ \text{Proof:} \] Let \( P \) be the point set and \( \mathcal{F} \) be the set of circles constructed in the proof of Lemma 5.5, and map every point in it to three dimensions using the mapping \( f : (x,y) \to (x, y, x^2 + y^2) \). This is a standard lifting map used in computing planar Delaunay triangulations via convex-hull in three dimensions, see [BCKO08]. Let \( Q = f(P) \) be the resulting point set.
It is easy to verify that a circle of $c \in \mathcal{F}$ is mapped by $f$ into a curve that lies on a plane. We will abuse notations slightly, and use $f(c)$ to denote this plane. Let $\mathcal{G} = f(\mathcal{F})$. Furthermore, for a circle $c \in \mathcal{F}$, we have that $f(c \cap P) = f(c) \cap Q$. Namely, solving the set cover problem $(Q, \mathcal{G})$ is equivalent to solving the original set cover instance $(P, \mathcal{F})$.

The recent work of Mustafa et al. [MR10] gave a QPTAS for set cover of points by disks (i.e., circles with their interior), and for set cover of points by half-spaces in three dimensions. Thus, somewhat surprisingly, the “shelled” version of these problems are harder than the filled-in version.

5.3. Discrete hitting set for fat triangles

In the **fat-triangles discrete hitting set problem**, we are given a set of points in the plane $P$ and a set of fat triangles $T$, and want to find the smallest subset of $P$ such that each triangle in $T$ contains at least one point in the set.

**Lemma 5.7.** There is no PTAS for the fat-triangle discrete hitting set problem, unless $P = NP$. One can prespecify an arbitrary constant $\delta > 0$, and the claim would hold true even if the following conditions hold on the given instance $(P, T)$:

(A) Every angle of every triangle in $T$ is between $60 - \delta$ and $60 + \delta$ degrees.

(B) No point of $P$ is covered by more than three triangles of $T$.

(C) The points of $P$ are in convex position.

(D) All the triangles of $T$ are of similar size. Specifically, each triangle has side length in the range (say) $(\sqrt{3} - \delta, \sqrt{3} + \delta)$.

(E) The points of $P$ are a subset of the vertices of the triangles of $T$.

**Proof:** Let $G = (V, E)$ be a connected instance of **Vertex Cover** which has maximum degree three, and it is not an odd cycle. We remind the reader that **Vertex Cover** is APX-HARD for such instances [ACG+99].

By Brook’s theorem [CR14], this graph is three colorable, and let $V_1, V_2, V_3$ be the partition of $V$ by their colors. Let $p_1, p_2, p_3$ be three points on the unit circle that form a regular triangle. For $i = 1, 2, 3$, place a circular interval $J_i$ centered at $p_i$ of length $\delta/100$. Now, for $i = 1, 2, 3$, we place the vertices of $V_i$ as distinct points in $J_i$.

Let $Q_0 = V$ and $m = |E(G)|$. For $i = 1, \ldots, m$, let $u_iv_i$ be the $i$th edge of $G$. Assume, for the sake of simplicity of exposition, that $u_i \in V_1$ and $v_i \in V_2$. Pick an arbitrary point $q_i$ in $J_3 \setminus Q_{i-1}$, and create the triangle $T_i = \triangle u_iv_iq_i$. Set $Q_i = Q_{i-1} \cup \{q_i\}$, and continue to the next edge.

At the end of this process, we have $m$ triangles $\mathcal{T} = \{T_1, \ldots, T_m\}$ that are arbitrarily close to being regular triangles, and all their edges are arbitrarily close to being of the same length, see Figure 5.2. It is easy to verify that a minimum cardinality set of points $U \subseteq V$ that hits all the triangles in $\mathcal{T}$ is a minimum vertex cover of $G$.

5.4. Independent set of triangles in 3D

Given a set $\mathcal{F}$ of $n$ objects in $\mathbb{R}^d$ (say, triangles in 3d), we are interested in computing a maximum number of objects that are **independent**: that is, no pair of objects in this set (i.e., independent set) intersects. This is the geometric realization of the **independent set** problem for the intersection graph induced by these objects.

**Lemma 5.8.** There is no PTAS for the maximum independent set of triangles in $\mathbb{R}^3$, unless $P = NP$. 

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Figure 5.2: Illustration of the proof of Lemma 5.7: (A) A 3-regular graph with its 3 coloring. (B) Placing the vertices on a circle. (C) Three edges and their associated triangles. (D) All the triangles.

Proof: The problem **Independent Set** is **APX-Hard** even for graphs with maximum degree 3 [ACG+99]. Let \( G = (V, E) \) be a given graph with maximum degree 3, where \( V = \{v_1, \ldots, v_n\} \). We will create a set of triangles, such that their intersection graph is \( G \).

If one spreads \( n \) points \( p_1, \ldots, p_n \) on the positive branch of the moment curve in \( \mathbb{R}^3 \) [Sei91, EK03], their Voronoi diagram is **neighborly**; that is, every Voronoi cell is a convex polytope that shares a non-empty two dimensional boundary face with each of the other cells of the diagram. Let \( C_i \) denote the cell of the point \( p_i \) in this Voronoi diagram, for \( i = 1, \ldots, n \).

Now, for every vertex \( v_i \in V \), we form a set \( P_i \) of (at most) three points, as follows. If \( v_i v_j \in E \), then we place a point \( p_{ij} \) on the common boundary of \( C_i \) and \( C_j \), and we add this point to both \( P_i \) and \( P_j \). After processing all the edges in \( E \), each point set \( P_i \) has at most three points, as the maximum degree in \( G \) is three.

For \( i = 1, \ldots, n \), let \( f_i \) be the triangle formed by the convex-hull of \( P_i \) (if \( P_i \) has fewer than three points then the triangle is degenerate).

Let \( T = \{f_1, \ldots, f_n\} \). Observe that the triangles of \( T \) are disjoint except maybe in their common vertices, as their interior is contained inside the interior of \( C_i \), and the cells \( C_1, \ldots, C_n \) are interior disjoint. Clearly \( f_i \cap f_j \neq \emptyset \) if and only if \( v_i v_j \in E \). Thus, finding an independent set in \( G \) is equivalent to finding an independent set of triangles of the same size in \( T \). We conclude that the problem of finding maximum independent set of triangles is **APX-Hard**, and as such does not have a **PTAS** unless \( P = \text{NP} \).

Implicit in the above proof is that any graph can be realized as the intersection graph of convex bodies in \( \mathbb{R}^3 \) (we were a bit more elaborate for the sake of completeness and since we needed slightly more structure). This is well known and can be traced to a result of Tietze from 1905 [Tie05].

### 5.5. On the hardness implications of our results

#### 5.5.1. A review of complexity terms

The **exponential time hypothesis** (**ETH**), is that 3SAT can not be solved in time better than \( 2^{\Omega(n)} \), where \( n \) is the number of variables. The **strong exponential time hypothesis** (**SETH**), is that the time to solve \( k \text{SAT} \) is at least \( 2^{c_k n} \), where \( c_k \) converges to 1 as \( k \) increases.

Showing that a problem \( X \) is **APX-Hard** implies that one can not do better than a constant approximation. Specifically, if one can get a \((1 + \varepsilon)\)-approximation for such a problem, for any constant
<table>
<thead>
<tr>
<th>Objects</th>
<th>Approx. Alg.</th>
<th>Hardness</th>
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<tbody>
<tr>
<td>Disks/pseudo-disks</td>
<td>QPTAS [MRR14b]</td>
<td>Exact version NPHARD [FG88]</td>
</tr>
<tr>
<td>Fat triangles of same size</td>
<td>O(1) [CV07]</td>
<td>APX-HARD: Lemma 5.3.6.15 I.e., no PTAS possible.</td>
</tr>
<tr>
<td>Fat objects in $\mathbb{R}^2$</td>
<td>$O(\log^*\text{opt})$ [ABES14]</td>
<td>APX-HARD: L5.3</td>
</tr>
<tr>
<td>Objects $\subseteq \mathbb{R}^d$, $O(1)$ density E.g. fat objects, $O(1)$ depth.</td>
<td>PTAS: Corollary 4.9 p11</td>
<td>Exact version NPHARD [FG88]</td>
</tr>
<tr>
<td>Objects with polylog density</td>
<td>QPTAS: C4.9</td>
<td>No PTAS under ETH Observation 5.9 p19</td>
</tr>
<tr>
<td>Objects with density $\rho$ in $\mathbb{R}^d$</td>
<td>PTAS: C4.9</td>
<td>No $(1 + \varepsilon)$-approx with RT $n^{\text{polylog}(\rho)}$ assuming ETH: O5.9</td>
</tr>
</tbody>
</table>

Figure 5.3: Known results about the complexity of geometric set-cover. Specifically, the input is a given set of points, and a set of objects, and the task is to find the smallest subset of objects that covers the points. To see that the hardness proof Feder and Greene [FG88] indeed implies the above, on just need to verify that the input instance their proof generates has bounded depth.

$\varepsilon > 0$, then one can $(1 + \varepsilon)$-approximate 3SAT. By the PCP Theorem, this would imply an exact algorithm for 3SAT. Under ETH, showing that a problem is APX-HARD implies that it does not even have a QPTAS.

5.5.2. Implications
Given a set of objects $\mathcal{F}$ in $\mathbb{R}^d$, and a set of points $P$, we would like to find minimum cardinality subset of the objects in $\mathcal{F}$ that covers the points of $P$. We refer to this problem as the geometric set cover problem.

Observation 5.9. Assuming ETH, an instance of geometric set cover with $n$ fat triangles, and density at least $\Omega(\log^c n)$, can not be $(1 + \varepsilon)$-approximated in polynomial time, where $c$ is a sufficiently large constant.

Indeed, suppose we had such a PTAS, and consider an instance $I$ of 3SAT of size at least $c' \log^2 n$, where $c'$ is a sufficiently large constant. ETH implies that any algorithm solving such an instance must have running time at least $n^{O(\log n)}$. On the other hand, the instance $I$ can be converted to a set of fat triangles of the same size with polylog density. As such, a PTAS in this case, would contradict ETH.

Similarly, for arbitrary density $\rho$ and arbitrarily small constant $\varepsilon > 0$, one can have an $(1 + \varepsilon)$-approximation algorithm with running time $n^{\Omega(\rho)}$, since such an algorithm would imply a QPTAS for 3SAT.

Similar conclusions can be drawn about geometric hitting set.

6. Conclusions
We have defined and studied the family of graphs arising out of low-density scenes. We showed that this family of graphs have many interesting properties. In particular, it has arbitrarily large minors,
<table>
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<td>APX-HARD Lemma 5.7,p17</td>
</tr>
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<td>Objects with $O(1)$ density.</td>
<td>PTAS: Corollary 4.8,p11</td>
<td>Exact ver. NPHARD [FG88]</td>
</tr>
<tr>
<td>Objects polylog density.</td>
<td>QPTAS: C4.8</td>
<td>No PTAS under ETH Observation 5.9 / L5.7</td>
</tr>
<tr>
<td>Objects with density $\rho$ in $\mathbb{R}^d$</td>
<td>PTAS: C4.8, run time $n^{O(\rho^{(d+1)/d}/\epsilon^d)}$</td>
<td>No $(1 + \epsilon)$-approx with RT $n^{\text{polylog}(\rho)}$ assuming ETH: O5.9</td>
</tr>
</tbody>
</table>

Figure 5.4: Known results about the complexity of geometric hitting set. Specifically, the input is a given set of points, and a set of objects, and the task is to find the smallest subset of points such that any object is hit by one of these points.

despite having small separators. We showed that density provides a useful characterization as far as approximability, by providing new approximation algorithms and hardness results.

There are many natural directions for future research, from developing efficient approximation algorithms for other problems (Steiner tree for example), to improving known approximation algorithms for geometric set-cover or geometric hitting-set.

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References


[EK03] J. Erickson and S. Kim. Arbitrarily large neighborly families of congruent symmetric


Stearns. NC-approximation schemes for NP- and PSPACE-hard problems for geometric


A. Properties of low density graphs

A.1. Depth, fatness and density

An object $g \subseteq \mathbb{R}^d$ is $\alpha$-fat if for any ball $b$ with a center inside $g$, that does not contain $g$, we have $\text{vol}(b \cap g) \geq \alpha \text{vol}(b)$ [BKSV02]. A set $\mathcal{F}$ of objects is fat if all its members are $\alpha$-fat for some constant $\alpha$. A collection of objects $\mathcal{F}$ has depth $k$ if any point in the underlying space lies in at most $k$ objects of $\mathcal{F}$. The depth index of a set of objects is a lower bound on the density of the set, as a point can be viewed as a ball of radius zero. The following is well known, and we include a proof for the sake of completeness.

Lemma A.1. A set $\mathcal{F}$ of $\alpha$-fat convex objects in $\mathbb{R}^d$ with depth $k$ has density $\alpha 2^d / k$. In particular, if $\alpha, k$ and $d$ are bounded constants, then $\mathcal{F}$ has a bounded density.

Proof: Let $b = b(p, r)$ be any ball in $\mathbb{R}^d$, and consider an $\alpha$-fat object $g \in \mathcal{F}$ intersecting $b$ with larger diameter. A ball $b(q, r)$ centered at a point $q$ that is in $g \cap b$ does not contain $g$, so $\text{vol}(b(q, r)) \geq \text{vol}(b(q, r) \cap g) \geq \alpha \text{vol}(b(q, r)) = \alpha \text{vol}(b)$. Furthermore, since $b(q, r)$ is contained in the ball $b' = b(p, 2r)$, and as such $\text{vol}(b' \cap g) \geq \text{vol}(b(q, r) \cap g) \geq \alpha \text{vol}(b) = \frac{\alpha}{2^d} \text{vol}(b')$.

Each point in $b'$ can be covered by at most $k$ objects of $\mathcal{F}$, and each large object intersecting $b$ covers a $(\alpha / 2^d)$-fraction of $b'$. Therefore, there at most $k 2^d / \alpha$ objects in $\mathcal{F}$ with intersecting $g$ with radius larger than $r$.

Definition A.2. A metric space $\mathcal{X}$ is a doubling space if there is universal constant $c_{\text{dbl}} > 0$, such that any ball $b$ of radius $r$ can be covered by $c_{\text{dbl}}$ balls of half the radius. Here $c_{\text{dbl}}$ is the doubling constant, and its logarithm is the doubling dimension. In $\mathbb{R}^d$ the doubling constant is $c_d = 2^{O(d)}$, and the doubling dimension is $O(d)$ [Ver05], making the doubling dimension a natural abstraction of the notion of dimension in the Euclidean case.

Lemma A.3. Let $\mathcal{F}$ be a set of objects in $\mathbb{R}^d$ with density $\rho$. Then for any $\alpha \in (0, 1)$, and any ball $b = b(c, r)$ can intersect at most $\rho c_{\text{dbl}}^{\lfloor \log_2 1/\alpha \rfloor}$ objects of $\mathcal{F}$ with diameter $\geq \alpha \cdot 2r$, where $\lfloor \log_2 \rfloor$ and $c_{\text{dbl}}$ is the doubling constant of $\mathbb{R}^d$.

Proof: Cover $b$ by minimum number balls of radius $\leq \alpha r$. By the definition of the doubling constant, the number of balls needed is $c_{\text{dbl}}^{\lfloor \log_2 1/\alpha \rfloor}$. Clearly, each one of these balls, by the density definition, can intersect at most $\rho$ objects of $\mathcal{F}$ of diameter larger than $\alpha \cdot 2r$, which implies the claim.

The density definition can be made to be somewhat more flexible, as follows.

Lemma A.4. Let $\alpha > 0$ be a parameter, and let $\mathcal{F}$ be a collection of objects in $\mathbb{R}^d$ such that, for any $r$, any ball with radius $r$ intersects at most $\rho c_{\text{dbl}}^{\lfloor \log_2 \alpha \rfloor}$ objects of $\mathcal{F}$ with diameter $\geq 2\alpha r$. Then $\mathcal{F}$ has density $c_{\text{dbl}}^{\lfloor \log_2 \alpha \rfloor} \rho$.

Proof: Let $b$ be a ball with radius $r$. We can cover $b$ with $c_{\text{dbl}}^{\lfloor \log_2 \alpha \rfloor}$ balls with radius $r/\alpha$. Each $r/\alpha$-radius ball can intersect at most $\rho$ objects with diameter $2\alpha \cdot (r/\alpha) = 2r$, so $b$ intersects at most $c_{\text{dbl}}^{\lfloor \log_2 \alpha \rfloor} \rho$ balls with radius $r$.

Lemma A.5. Let $\mathcal{F}$ and $\mathcal{G}$ be two collections of objects in $\mathbb{R}^d$. We have the following:

(i) $\text{density}(\mathcal{F} \cup \mathcal{G}) \leq \text{density}(\mathcal{F}) + \text{density}(\mathcal{G})$.

(ii) If $\mathcal{G} \subseteq \mathcal{F}$, then $\text{density}(\mathcal{G}) \leq \text{density}(\mathcal{F})$. 

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A.2. Graph minors

An undirected graph $H$ is a minor of a graph $G = (V, E)$ if $H$ can be formed from $G$ by deleting edges and vertices and contracting edges.

A specific sequence of deletions and contractions taking $G$ to a minor $H$ generates an equivalence relation on $V$ where all the vertices of $G$ that collapse to the same vertex of $H$ are equivalent. The vertices of $G$ that are in an equivalence class form an $H$-cluster. We remind the reader that a subgraph $G'$ of $G$ is a $t$-patch if the diameter of $G'$ is bounded by $t$. As such, the graph $H$ is a $t$-shallow minor of $G$, if for all $H$-clusters are $t$-patches.

We now redefine these notions in terms of the objects underlying the intersection graph. Let $\mathcal{F}$ and $\mathcal{G}$ be two collections of objects in $\mathbb{R}^d$. $\mathcal{G}$ is a minor of $\mathcal{F}$ if it can be obtained by deleting objects and replacing pairs of overlapping objects $f$ and $g$ (i.e., $f \cap g \neq \emptyset$) with their union $f \cup g$. A sequence of unions and deletions taking $\mathcal{F}$ to $\mathcal{G}$ induces an equivalence class on $\mathcal{F}$, and the equivalence classes are called $\mathcal{G}$-clusters. A set objects $\mathcal{G}$ is a $t$-shallow minor of $\mathcal{F}$ if all $\mathcal{G}$-clusters are $t$-patches.

These definitions for object sets are similar to the graph theoretic definitions applied to their intersection graphs. If $\mathcal{G}$ is a minor of $\mathcal{F}$, then the intersection graph $G_\mathcal{G}$ is a graph minor of $G_\mathcal{F}$; an $\mathcal{G}$-cluster is a $G_\mathcal{G}$-cluster; and $\mathcal{G}$ is $t$-shallow if $G_\mathcal{G}$ is. We refrain from defining minors of object sets via their intersection graphs for lack of an intuitive analogue to edge deletion.

There is a simple relationship between the depth of a shallow minor of objects and its density.

Lemma A.6. Let $\mathcal{F}$ be a collection of objects with density $\rho$ in $\mathbb{R}^d$, and let $\mathcal{G}$ be $t$-shallow minor of $\mathcal{F}$. Then $\mathcal{G}$ has density at most $t^{O(1)} \rho$.

Proof: Every object $g \in \mathcal{G}$ has a defining subset $F_g \subseteq \mathcal{F}$. These sets are disjoint, and let $\mathcal{P} = \{ F_g \mid g \in \mathcal{G} \}$ be the induced partition of $\mathcal{F}$ into clusters. Next, consider any ball $B = B(c, r)$, and suppose that $g \in \mathcal{G}$ intersects $B$ and it has diameter at least $2r$, and let $F_g \in \mathcal{P}$ be its defining cluster, and $H = G_{\mathcal{F}_g}$ be its associated intersection graph. By assumption $H$ has (graph) diameter $\leq t$.

Now, let $h$ be any object in $F_g$ that intersect $B$, let $d_H$ denote the shortest path metric of $H$ (under the number of edges), and let $h'$ be the object in $F_g$ closest to $h$ (under $d_H$), such that $\text{diam}(h') \geq 2r/t$ (if there is no such object then the diameter of $\text{diam}(g) < t(2r/t) \leq 2r$, which is a contradiction).

Consider the shortest path $\pi \equiv h_1, \ldots, h_\tau$ between $h = h_1$ and $h' = h_\tau$ in $H$, where $\tau \leq t$. Observe that, for $i = 1, \ldots, \tau - 1$, $\text{diam}(h_i) < 2r/t$, and thus the distance between $B$ and $h'$ is bounded by $\sum_{i=1}^{\tau-1} \text{diam}(h_i) \leq (\tau - 1)2r/\tau < 2r$. We refer to $h'$ as the representative of $g$, denoted by $\text{rep}(g) \in F_g$.

Now, let $\mathcal{H} = \{ \text{rep}(g) \in \mathcal{F} \mid g \in \mathcal{G}, \text{diam}(g) \geq 2r, \text{ and } g \cap B \neq \emptyset \}$. The representatives in $\mathcal{H}$ are all unique, each is of diameter $\geq 2r/t$, all of them intersect $B(c', 3r)$, and they all belong to $\mathcal{F}$, a set of density $\rho$. Lemma A.3 implies that $|\mathcal{H}| \leq \rho c_{\text{dim}}^{\lceil \log t \rceil}$, implying the claim.

Low-density graphs do not in general exclude graph minors, since they are not closed under minors. Figure 2.1p4 constructs a low-density collection of $n$ fat rectangles that contains a $\Theta(\sqrt{n})$-vertex clique minor. Thus implying the following.

Lemma A.7. There are constant density graphs induced by fat objects (specifically, axis parallel rectangles) in the plane that contains arbitrarily large clique minors.

A.3. Nowhere dense graphs

For a graph $G$, and an integer $k$, let $\nu_k(G)$ denote the set of all $k$-shallow minors of $G$ of depth $k$. For example, $\nu_0(G)$ is the set of all subgraphs of $G$. For a class of graphs $C$, let $\nu_k(C) = \bigcup_{G \in C} \nu_k(G)$ denote the set of
all $d$-shallow minors of graphs in $\mathcal{C}$. A graph family $\mathcal{C}$ is **nowhere dense** if $\lim_{k \to \infty} \limsup_{G \in \mathcal{V}_k(\mathcal{C})} \frac{\log |E(G)|}{\log |V(G)|} = 1$.

This definition is taken from Nešetřil and Ossona de Mendez [NO11, Corollary 3.3]. Nowhere dense classes of graphs include families of graphs excluding a fixed minor and classes of bounded degree graphs, but also include constant degree expanders.

**Lemma A.8.** Any graph $G$ with $n$ vertices and density $\rho$, with realization in $\mathbb{R}^d$, has at most $c_{d}\rho n$ edges, where $c_{d}$ is the doubling constant in $\mathbb{R}^d$.

*Proof:* Let $G$ be realized by a set of objects $\mathcal{F}$ in $\mathbb{R}^d$. Consider the smallest diameter object $f \in \mathcal{F}$ and consider the smallest enclosing ball $B$ of $f$. Observe that $B$ intersects at most $c_{d}\rho$ objects of $\mathcal{F}$, and this bounds the degree of $f$ in $G$. Now, remove $f$ from $G$, and argue inductively on the remaining graph.  

**Lemma A.9.** For any fixed $\rho > 0$ and $d$, the class of graphs with density $\leq \rho$ (realized by objects in $\mathbb{R}^d$) is nowhere dense.

*Proof:* Let $\mathcal{C}$ be the family of graphs with density at most $\rho$ for objects in $\mathbb{R}^d$. Fix $k$, and consider the graph family $\mathcal{V}_k(\mathcal{C})$. By Lemma A.6, for every graph $G \in \mathcal{V}_k(\mathcal{C})$ has density $\rho' = k^{c}\rho$ for some constant. As such, by Lemma A.8, for $n = |V(G)|$, we have that

$$\frac{\log |E(G)|}{\log |V(G)|} \leq \frac{\log(c_{d}k^{c}n)}{\log n} = 1 + \log_n(c_{d}k^{c}),$$

and this quantity goes to 1 as $n$ increases. As such, it readily implies that

$$\limsup_{G \in \mathcal{V}_k(\mathcal{C})} \frac{\log |E(G)|}{\log |V(G)|} = 1 \implies \lim_{k \to \infty} \limsup_{G \in \mathcal{V}_k(\mathcal{C})} \frac{\log |E(G)|}{\log |V(G)|} = 1.$$ 

Recently, Grohe, Kreutzer and Siebertz showed that first-order properties are fixed-parameter tractable on nowhere dense graph classes, which applies also to low-density classes of graphs [GKS14].

### B. Some additional details

#### B.1. Proof of Lemma 3.5

*Proof:* Let $\alpha = \frac{1}{(d+2)(c_{d})^2}$.

For every object $f \in \mathcal{F}$, place a representative point $p_{f} \in f$ arbitrarily with weight $w(f)$. Let $P$ be the resulting set of weighted points. We assume that $P$ is in general position; namely, no $d+2$ points in $P$ lie on a sphere. If any point $p$ has weight at least $\alpha \cdot W$, then we simply return the degenerate sphere of radius 0 centered at $p$, observing that at most $\rho$ objects in $\mathcal{F}$ contain $p$.

Next, let $B(c,r)$ be the smallest ball containing weight $\alpha \cdot W$ of points of $P$. At most $d+1$ points of $P$ lies on the boundary of $B(c,r)$, and each point has weight at most $\alpha \cdot W$, so $B(c,r)$ contains weight at most $(d+1) \cdot \alpha \cdot W$, as does any other ball with radius $r$. The objects of $\mathcal{F}$ lying outside $B(c,r)$ have weight at most $(1-\alpha) \cdot W$.

As in the unweighted case, we randomly pick $R$ uniformly in the range $[r,2r]$. The same argument as for the unweighted case shows that the sphere $\mathbb{S} = \mathbb{S}(c,R)$ intersects $O\left(\rho + \rho^{1/d}n^{(d-1)/d} \log^{1/d} n\right)$ in expectation. As in the unweighted case, we can find a two-approximation for the smallest sphere with weight $\alpha \cdot W$ of points of $P$ with [HR13], and expect to succeed in a constant number of iterations.  

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B.2. Proof of Lemma 4.1: Building $\lambda$-divisions

**Proof** By assumption, the given graph $G = (V, E)$ is $\alpha$-breakable. That is, there is a constant $c$, such that for any induced subgraph of $G$ on $m$ vertices, there is a balanced separator of size $f(m) \leq cm^\alpha \ln m$. For the sake of simplicity of exposition we assume every connected component after the removal of the separator is of size at most $(2/3)m$.

Our strategy is to break $G$ into smaller pieces. Specifically, at every step the algorithm takes the largest remaining piece $G_{|U}$, compute a balanced separator $Z \subseteq U$ for it, with $L, R \subseteq U$ being the two separated pieces. Specifically, we have

(I) $Z = L \cap R$,

(II) $L \cup R = U$,

(III) $|L| \leq (2/3)|U|$ and $|R| \leq (2/3)|U|$,

(IV) $L \setminus Z$ is separated from $R \setminus Z$ in $G_{|U}$.

(V) $|Z| \leq f(\lfloor |U| \rfloor)$.

Now, the algorithm replaces $G_{|U}$ by the two “broken” pieces $G_{|L}$ and $G_{|R}$. The algorithm continues in this process until all pieces are of size smaller than $b$ (and by construction, of size at least, say, $b/4$). This generates a natural binary separator tree, where the final pieces of the division are the leafs.

Let $N_i = (2/3)^i n$, for $i = 0, \ldots, h = \lceil \log_{3/2} n \rceil$. A piece $G_{|U}$ is at level $i$ if $N_{i+1} < |U| \leq N_i$. There are at most $2^i$ nodes in the separator tree in the $i$th level. Consider such a subproblem at node $y$, which is at level $i$ with $\nu$ vertices. The total size of the subproblems of its two children is $\leq \nu + 2f(\nu)$ (here, somewhat confusingly, we count the separator vertices as new, in both subproblems – this makes the following argument somewhat easier). As such, the fraction of the new vertices created as subproblems move from the $i$th level to the next is bounded by

$$\leq \nu + 2f(\nu) \leq \nu + 2c\nu^\alpha \ln \nu = \left(1 + \frac{2c \ln \nu}{\nu^{1-\alpha}}\right) \nu \leq \beta_i \nu, \quad \text{for} \quad \beta_i = 1 + 2 \frac{c \ln N_{i+1}}{(N_{i+1})^{1-\alpha}}.$$

In particular, the total number of vertices in the $k$th level is at most $\Delta_k n$, where

$$\Delta_k = \prod_{j=0}^{k-1} \beta_j \leq \prod_{j=0}^{k-1} \exp \left(2 \frac{c \ln N_{j+1}}{(N_{j+1})^{1-\alpha}}\right) = \exp \left(\sum_{j=0}^{k-1} \frac{2c \ln N_{j+1}}{(N_{j+1})^{1-\alpha}}\right) \leq \exp \left(\frac{c' \ln N_k}{(N_k)^{1-\alpha}}\right) \leq 1 + \frac{2c' \ln N_k}{(N_k)^{1-\alpha}},$$

since the summation behaves like an increasing geometric series, and $c'$ is a constant that depends on $c$. The last step follows as $e^x \leq 1 + 2x$, for $0 \leq x \leq 1/2$. In particular, because of the double counting of the separator vertices, the total number of marked vertices in the first $k$ levels is bounded by $n(\Delta_k - 1)$. As such, we need that $\Delta_k - 1 \leq \varepsilon$. This is equivalent to

$$\frac{2c' \ln N_k}{(N_k)^{1-\alpha}} \leq \varepsilon \iff \frac{2c'}{\varepsilon} \leq \frac{(N_k)^{1-\alpha}}{\ln N_k},$$

which holds if $N_k \geq \left(\frac{2c'}{\varepsilon} \ln \varepsilon^{-1} \ln \varepsilon^{-1}\right)^{1/(1-\alpha)}$, where $c''$ is a sufficiently large constant. In particular, setting $b$ to be (say) twice this threshold implies the claim. 

C. On exposed sets of segments and their density

Let $\sigma > 0$ be a fixed parameter. We say that an object $f$ $\sigma$-shadows (or simply shadows) another object $g$ if

$$\max_{q \in g} d(q, f) \leq \sigma \cdot \diam(g),$$

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where \( d(q,f) = \min_{u \in f} ||q - u|| \). Equivalently, \( f \sigma \)-shadows \( g \) if and only if \( g \subseteq f \oplus b(0, \sigma \cdot \text{diam}(g)) \). Here, \( X \oplus Y = \{q + u \mid q \in X, u \in Y\} \) denotes the Minkowski sum of \( X \) and \( Y \). A set of objects \( F \) is \( \sigma \)-exposed if no object in \( F \) \( \sigma \)-shadows another object in \( F \).

Observation C.1. Let \( f \) and \( g \) be two objects and \( \sigma \geq 0 \). If \( f \subseteq g \), then \( g \) \( \sigma \)-shadows \( f \).

C.1. On the density of exposed segments

C.1.1. Intervals in \( \mathbb{R} \)

Following the above, interval \( I = [\ell, r] \) \( \sigma \)-exposes \( I' = [\ell', r'] \) if \( I' \) is not contained in the interval \([\ell - \sigma \| J \|, r + \sigma \| J \|]\), where \( \| I' \| = r' - r' \) denotes the length of \( I' \).

Lemma C.2. Let \( I = [\ell, r] \) and \( I' = [\ell', r'] \) be two overlapping intervals on the real line. If \( I \) and \( I' \) \( \sigma \)-expose each other, then \( |\ell' - \ell| \geq \sigma \| I \| \) and \( |r' - r| \geq \sigma \| I' \| \).

Proof: Without loss of generality, assume that \( \ell \leq \ell' \). Since \( I \) and \( I' \) are overlapping, we have \( \ell' \leq r \). Furthermore, if \( r' \leq r \), then \( I \subseteq I' \) and by Observation C.1 the interval \( I \) \( \sigma \)-shadows \( I' \). So it must be that \( r' > r \). Since the left endpoint \( \ell' \) of \( I' \) is contained in \( I \), \( I \) does not \( \sigma \)-shadow \( I' \) only if \( r' \) extends at least \( \sigma \| I' \| \) past \( r \). Similarly, if \( I' \) does not \( \sigma \)-shadow \( I \), then \( \ell' - \ell \geq \sigma \| I \| \).

Lemma C.3. Let \( \mathcal{I} \) be a set of intervals all covering a common point \( p \). If \( \mathcal{I} \) is a \( \sigma \)-exposed set of intervals, then \( |\mathcal{I}| = O(1/\sigma^2) \).

Proof: Let the \( i \)th interval of \( \mathcal{I} \) be \( I_i = [\ell_i, r_i] \), for \( i = 1, \ldots, n \). Furthermore, assume that \( \ell_1 \leq \ell_2 \leq \cdots \ell_n \). By Dilworth’s theorem, there exists a subsequence \( i_1 < i_2 < \cdots < i_k \) with \( k \geq \sqrt{n} \) such that either \( r_{i_1} \leq r_{i_2} \leq \cdots \leq r_{i_k} \) or \( r_{i_1} \geq r_{i_2} \geq \cdots \geq r_{i_k} \). The later possibility implies that \( I_{i_1} \subseteq I_{i_2} \), which contradicts the assumption that \( \mathcal{I} \) is \( \sigma \)-exposed. Assume, without loss of generality, that for at least half the intervals in this sequence, we have \( |I(I)| \geq r(I) \), and let \( I_1', \ldots, I_{k/2}' \) be the resulting subsequence restricted to these intervals, where \( I_i' = [\ell_i', r_i'] \) for all \( i \). (The other case is handled by symmetric argument.)

We have \( \ell_1' \leq \cdots \ell_{k/2}' \leq 0 \leq r_1' \leq \cdots \leq r_{k/2}' \). By Lemma C.2, we have that for any \( i \), we have \( r_i' - r_{i-1}' \geq \sigma \| I_i' \| \geq \sigma |\ell_i'| \). Summing this inequality for \( i = 2, \ldots, t \), we have

\[
r_t' > r_1' - r_1' \geq \sigma \sum_{i=1}^{t-1} |\ell_i'| \geq \sigma \cdot (t - 1) |\ell_t| > |\ell_t|,
\]

for \( t = \lceil 1/\sigma \rceil + 2 \), which is a contradiction. We conclude that \( \sqrt{n}/2 \leq k/2 \leq \lceil 1/\sigma \rceil + 2 \), which readily implies the claim.

Figure C.1: (A) Objects \( f, g, h \). (B) \( f \) 1/8-shadows \( g \). (C) \( f \) does not 1/8-shadow \( h \).
C.1.2. Line segments through a point

Lemma C.4. Let \( \mathcal{L} \) be a set of segments in \( \mathbb{R}^d \), and \( \sigma > 0, \theta \in (0, \pi/2) \) be parameters. Furthermore, assume that (i) \( \mathcal{L} \) is \( \sigma \)-exposed, (ii) \( \bigcap_{s \in \mathcal{L}} s \neq \emptyset \), (iii) for all pairs \( \ell_1, \ell_2 \in \mathcal{L} \), the angle between \( \ell_1 \) and \( \ell_2 \) is at most \( \theta \), and (iv) \( \sin \theta \leq \frac{4}{\pi} \). Then \( |\mathcal{L}| = O(1/\sigma^2) \).

Proof: Without loss of generality, we assume that the lines intersect at the origin and the angle between any line and the \( x \)-axis is at most \( \theta \). For each \( s \in \mathcal{L} \), let \( \ell(s) \) be the left endpoint of \( s \), let \( x\ell(s) \) be the \( x \)-coordinate of \( \ell(s) \), and let \( y\ell(s) \) the distance from \( \ell(s) \) to the \( x \)-axis. Similarly we define \( r(s), x_r(s), \) and \( y_r(s) \) with respect to the right endpoint. For \( s \in \mathcal{L} \), let \( I_s = [x\ell(s), x_r(s)] \) be the projection of \( s \) onto the \( x \)-axis, and let \( \mathcal{I}_\mathcal{L} = \{I_s \mid s \in \mathcal{L}\} \).

We claim that \( \mathcal{I} \) is \( (\sigma/4) \)-exposed. Indeed, suppose that there are two segments \( s, s' \), such that \( I_s = [\ell, r] \) is \( \sigma/4 \)-shadowing \( I_{s'} = [\ell', r'] \). We define the following sequence of points:

1. \( p \) is any point on \( s' \),
2. \( q \) is the projection of \( p \) into \( I_{s'} \),
3. \( u \) is any point in \( I_s \) that is in distance at most \( (\sigma/4) ||I_{s'}|| \) from \( q \), and it is closer to the origin than \( u \), and
4. \( v \) is the point on \( s \) whose projection on \( I_s \) is \( u \).

See the figure on the right.

We now have that \( ||p - q|| \leq ||s'|| \sin \theta \leq (\sigma/4) ||s'|| \). Similarly, as \( u \) is closer to the origin than \( q \), we have that \( ||u - v|| \leq (\sigma/4) ||s'|| \). Also, since \( ||q - u|| \leq (\sigma/4) ||I_{s'}|| \leq (\sigma/4) ||s'|| \), we have by the triangle inequality that \( ||p - v|| \leq ||p - q|| + ||q - u|| + ||u - v|| \leq \sigma ||s'|| \), which implies that \( s' \) is \( \sigma \)-shadowed by \( s \), a contradiction.

Now, Lemma C.3 implies that \( |\mathcal{L}| = |\mathcal{I}_\mathcal{L}| = O(1/\sigma^2) \). \[ \square \]

Lemma C.5. Let \( \mathcal{L} \) be a set of segments in \( \mathbb{R}^d \) and \( \sigma \in (0, 1) \) a fixed parameter, such that (i) \( \mathcal{L} \) is \( \sigma \)-exposed, and (ii) \( \bigcap_{s \in \mathcal{L}} s \neq \emptyset \). Then \( |\mathcal{L}| = O(1/\sigma^{d+2}) \).

Proof: Partition \( \mathcal{L} \) into \( O(\sigma^{-d}) \) clusters such that any two lines in the same cluster forms an angle \( \leq \sigma/4 \). By Lemma C.4, each cluster contains at most \( O(1/\sigma^2) \) segments, and the claim follows. \[ \square \]

C.1.3. Large segments all intersecting a common ball

Lemma C.6. Let \( \mathfrak{b} \) be a ball of radius \( r \), and let \( \mathcal{L} \) be a set of segments both in \( \mathbb{R}^d \). Furthermore, assume that (i) \( \mathcal{L} \) is \( \sigma \)-exposed, (ii) all the segments of \( \mathcal{L} \) intersect \( \mathfrak{b} \), and (iii) they are all of length \( \geq r \). Then, we have \( |\mathcal{L}| = O(1/\sigma^{2d+2}) \).

Proof: Let \( \mathcal{B} \) be a set of \( O(\sigma^{-d}) \) balls of radius \( \sigma r/4 \), that cover \( \mathfrak{b} \). For each \( s \in \mathcal{L} \), pick a small ball \( \mathfrak{b}_s \in \mathcal{B} \) intersecting \( s \), and translate \( s \) by at most \( \sigma r/4 \) so that it passes through the center of \( \mathfrak{b}_s \). For \( s \in \mathcal{L} \), let \( s' \) denote the translated segment, and let \( \mathcal{L}' = \{s' \mid s \in \mathcal{L}\} \).

Since \( \mathcal{L} \) is \( \sigma \)-exposed, and the length of each segment of \( \mathcal{L} \) is at least \( r \), it follows that \( \mathcal{L}' \) is \( \sigma/2 \)-exposed, as can be easily verified.

Now, for every ball \( \mathfrak{b}_s(c, \sigma r/4) \in \mathcal{B} \), consider the segment of segments \( \mathcal{L}'(c) \) that passes through \( c \). By Lemma C.5, we have \( |\mathcal{L}'(c)| = O(1/\sigma^{d+2}) \). This implies that \( |\mathcal{L}| = |\mathcal{L}'| = O(|\mathcal{B}|/\sigma^{d+2}) = O(1/\sigma^{2d+2}) \). \[ \square \]
C.1.4. Putting things together

Lemma C.7. Let $L$ be a set of segments in $\mathbb{R}^d$ and $\sigma > 0$ a fixed parameter. If $L$ is $\sigma$-exposed, then $L$ has density $O(\sigma^{-2d-2})$.

Proof: Consider any ball $b(c,r)$ in $\mathbb{R}^d$. By Lemma C.6, there could be at most $O(\sigma^{-2d-2})$ segments of length $\geq 2r$ of $L$ intersecting it, and the result follows. ■

C.2. On $(\sigma,k)$-shadowing

A set of objects $F$ in $\mathbb{R}^d$ is $(\sigma,k)$-exposed if each object $f \in F$ is $\sigma$-shadowed by at most $k$ other objects in $F$.

Lemma C.8. Let $\sigma > 0$ be a fixed parameter and $G$ a set of objects, such that for any subset $H \subseteq G$ that is $\sigma$-exposed, we have that density($H$) $\leq \rho$. If $G$ is $(\sigma,k)$-exposed, then density($G$) $\leq (2k+1)\rho$.

Proof: We create a graph $G$ over $G$, with an edge between two objects $g,h \in G$ if one shadows the other. By assumption, the average degree in $G$ is bounded by $2k$, and in particular the graph is $2k$-degenerate and can be partitioned into $2k+1$ independent sets. Every independent set is $\sigma$-exposed, and by assumption has density $\leq \rho$. Since density is subadditive under unions, $G$ has density at most $(2k+1)\rho$. ■

Corollary C.9. Let $L$ be a set of segments in $\mathbb{R}^d$ that $(\sigma,k)$-exposed. Then $L$ has density $O(k\sigma^{-4})$. 