

A Presentation for Manin and Schechtman's Higher Braid Groups ¹

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Abstract. Manin and Schechtman's higher braid groups $B(n,k)$ are a generalisation of the ordinary pure braid groups $P_n=B(n,1)$, in which the role that the symmetric group S_n plays in P_n is replaced by a weak Bruhat order. In this paper a family of concrete presentations of $B(n,k)$ will be given, which generalise Artin's presentation of P_n .

1: INTRODUCTION

Let π_1^0, \dots, π_n^0 denote n hyperplanes in \mathbf{C}^k , in general position. That is, it is assumed that for any $S \subseteq \{1, 2, \dots, n\}$,

$$\bigcap_{i \in S} \pi_i^0$$

has codimension $|S|$ in \mathbf{C}^k if $|S| \leq k$, and is empty if $|S| \geq k+1$. Let $U(n,k)$ denote the family of all sets $\{\pi_1, \dots, \pi_n\}$ of n hyperplanes in \mathbf{C}^k , in general position and such that π_i is parallel to π_i^0 for all $1 \leq i \leq n$. Denote by $B(n,k)$, the fundamental group of $U(n,k)$. This is the *higher braid group* in the sense of Manin and Schechtman [MS 2]. Note that although $U(n,k)$ depends on the initial choice of $\{\pi_i^0\}$, the structure of the group $B(n,k)$ and the topology of $U(n,k)$ depend only upon n and k .

In this discussion, n and k are arbitrary positive integers with $n \geq k$. When $k = 1$, $U(n,1)$ reduces to X_n , the configuration space of n distinct, ordered points in \mathbf{C} , whose fundamental group is the pure braid group $B(n,1) = P_n$ on n strings. When $k = n$, $U(n,n)$ is just \mathbf{C} , so that $B(n,n)$ is trivial, while $B(k+1,k) \cong \mathbf{Z}$. Choose hyperplanes specified by,

$$\mathbf{r} \cdot \mathbf{n}_i = \alpha_i, \tag{1.1}$$

for suitable $\mathbf{n}_i \in \mathbf{C}^k$ and $\alpha_i \in \mathbf{C}$, so as to put $\{\pi_i^0\}$ in general position. Then elements of $U(n,k)$ may be specified by points $\mathbf{x} = (x_i) \in \mathbf{C}^n$ with,

$$\pi_i: \mathbf{r} \cdot \mathbf{n}_i = \alpha_i - x_i. \tag{1.2}$$

For every $(k+1)$ -set $J \subseteq I = \{1, 2, \dots, n\}$, a hyperplane π_J may be defined in \mathbf{C}^n , by the condition that $\{\pi_i \mid i \in J\}$ have a common point of intersection. The set of hyperplanes π_i will be in general position so long as \mathbf{x} lies in the complement of the $\binom{n}{k+1}$ hyperplanes $\{\pi_J\}$ in \mathbf{C}^n . Thus $B(n,k)$ is the fundamental group of a complement of hyperplanes. It is therefore possible to obtain a presentation of $B(n,k)$ along the lines of [R].

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In this paper we will explicitly construct such presentations of $B(n, k)$. Using a suitable set of initial hyperplanes π_i^0 , generators for $B(n, k)$ are defined in §2. In §3 the associated relations are obtained by investigating the structure of the codimension-two subsets of \mathbf{C} associated with $\{\pi_i\}$. Some special cases are discussed in §4, while some connections with the combinatorial structures of [MS 1] and those of the ordinary braid groups are considered in §§5 and 6.

It is well known that the complement of a set of complex hyperplanes possesses a strong combinatorial structure (see [OS]). It is therefore not surprising to discover that the formulae obtained for the presentation of $B(n, k)$ are complex, involving inequalities.

2: CONSTRUCTION OF GENERATORS

The hyperplanes (1.1), will be in general position for suitable α_i , so long as, for all $J \subseteq I$ with $|J| = k$,

$$\det(\mathbf{n}_i)_{i \in J} \neq 0,$$

where $(\mathbf{n}_i)_{i \in J}$ denotes the $k \times k$ matrix with columns \mathbf{n}_i indexed by $i \in J$. A suitable choice for \mathbf{n}_i is thus,

$$\mathbf{n}_i = (1, a_i, \dots, a_i^{k-1}), \quad (2.1)$$

where $a_i \in \mathbf{R}^+$ are distinct, say $0 < a_1 < a_2 < \dots < a_n$. Throughout §§2 and 3, this choice of \mathbf{n}_i will be assumed, with a_i fixed. Suppose $J \subset I$ with $|J| = k+1$. Let $j = \max(J)$. The condition for $\{\mathbf{r} \cdot \mathbf{n}_i = \alpha_i \mid i \in J\}$ to not have a common point of intersection is that,

$$\det \mathbf{A}_J \neq 0,$$

where \mathbf{A}_J is a $(k+1) \times (k+1)$ matrix with rows (α_i, \mathbf{n}_i) indexed by $i \in J$. That is,

$$\sum_{i \in J} \left\{ \alpha_i (-1)^{(\#J < i)} \cdot \prod_{\substack{j < k \\ j, k \in J \setminus \{i\}}} (a_k - a_j) \right\} \neq 0. \quad (2.2)$$

The ratio between the coefficients of α_i and α_l ($i, l \in J$) on the l.h.s. here has modulus $|\Pi_{J \setminus \{i, l\}}^{l, i}|$ where,

$$\Pi_K^{i, j} \equiv \prod_{\mu \in K} \left(\frac{a_\mu - a_i}{a_\mu - a_j} \right). \quad (2.3)$$

This shows that if α_j is chosen sufficiently large in comparison with $\{\alpha_i \mid i \in J \setminus \{j\}\}$, then (2.2) will automatically be satisfied.

To be more precise, let π denote $\max\{|\Pi_K^{i, j}|\}$ over all $(k-1)$ sets $K \subseteq I$ and $i, j \notin K$. Consider also the set of all sums of all triples of expressions of the form $\Pi_{K_1}^{i_1, j_1} \Pi_{K_2}^{i_2, j_2}$. Let π' denote the minimum difference between distinct elements of this set. Put $M = 6n\pi^2/\pi'$.

Theorem 1 Suppose that \mathbf{n}_i are defined by (2.1) and $\alpha_1, \dots, \alpha_n$ are chosen to be real numbers with $\alpha_i/\alpha_{i-1} > M$ for $1 \leq i \leq n$ where $\alpha_0 \equiv 1$. Then the hyperplanes π_i^0 of (1.1) are in general position.

The precise value of M is immaterial. It is only important to note that once $\{a_i\}$ are fixed, it is possible to choose M sufficiently large so that when $\{\alpha_i\}$ satisfies the conditions in Theorem 1, the sign of any of the expressions $\sum f_r \alpha_r$ we will wish to compute will be determined by that of f_m , where m is the largest r for which $f_r \neq 0$. Here f_r will be functions of the a 's only, given in terms of $\{\Pi_{i,j}^K\}$. From now on, $\alpha_1, \alpha_2, \dots, \alpha_n$ will be thought of as defining increasing orders of magnitude, in the sense just described.

Consider the hyperplane π_J in \mathbf{C}^n , given by the condition that $\{\pi_i \mid i \in J\}$ have a common point of intersection, where $J \subset I$ with $|J| = k+1$. This hyperplane will intersect the x_j -axis $\iff j \in J$, and the point of intesection will then be at,

$$x_j = x_j^J \equiv \alpha_j - \sum_{r \in J \setminus j} (\alpha_r \Pi_{J \setminus \{j, r\}}^{j, r}). \quad (2.4)$$

Assume $\{a_i\}$ are chosen such that $\{\Pi_K^{i,j} \mid K \subset I \setminus \{i, j\}\}$ are distinct $\forall i > j$. Let $j = \max(J)$. Define the generators β_J of $\pi_1(U_{n,k})$ to be given by a loop in $U_{n,k}$, based at $\mathbf{0}$ and lying in the copy of \mathbf{C} on which $x_i = 0$, $\forall i \neq j$. The loop is given by x_j following a path around x_j^J based at 0, in a clockwise direction, defined with $\Im(x_j) > 0$ along the whole loop except that part 'close' to x_j^J (see Fig 1). Note that in this situation, the values of x_j^J in (2.4) are all positive real numbers. The $\binom{n}{k+1}$ generators β_J of $\pi_1(U_{n,k})$ have now been defined.

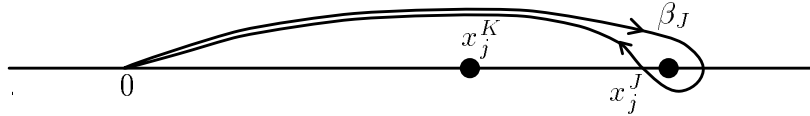


Figure 1

3: ANALYSIS OF RELATIONS

As discussed in [R], the relations in $B_{n,k}$ arise from the consideration of all codimension two subsets of \mathbf{C}^n associated with the arrangement $\mathcal{C} = \{\pi_J \mid J \subset I, |J| = k+1\}$. Such subsets, Δ , come in two main types,

- (i) $\bigcap \{\pi_K \mid K \subset J, |K| = k+1\}$ for $J \subset I$ with $|J| = k+2$;
- (ii) $J \cap L$ for $J, L \subset I$ with $|J| = |L| = k+1$ and $|J \cap L| < k$.

For later convenience, we choose to subdivide case (ii) into two parts,

- (ii)' $\max(J) > \max(L)$;
- (ii)'' $\max(J) = \max(L)$.

Clearly, since J and L may be interchanged, $\max(J) < \max(L)$ need not be considered. In each case, define $j, l \in I$ with $j > l$, as follows,

- (i) $j = \max(J), l = \max(J \setminus \{j\});$
- (ii)' $j = \max(J), l = \max(L);$
- (ii)'' $j = \max(J), l = \max((J \cup L) \setminus \{j\}).$

Without loss of generality, it may be assumed in (ii)'' that $l \in J$.

The relations R_Δ , associated with Δ , may clearly be discussed by only considering the arrangement $\mathcal{C}_{j,l}$ induced by \mathcal{C} upon the two-dimensional subspace of \mathbf{C}^n on which $x_i = 0$ for all $i \neq j, l$. The arrangement $\mathcal{C}_{j,l}$ on \mathbf{C}^2 (axes x_j and x_l) consists of lines,

$$\pi_{M \cup \{j\}}: x_j = \alpha_j - \sum_{r \in M} \alpha_r \Pi_{M \setminus \{r\}}^{j,r} \quad (3.1)$$

$$\pi_{M \cup \{l\}}: x_l = \alpha_l - \sum_{r \in M} \alpha_r \Pi_{M \setminus \{r\}}^{l,r} \quad (3.2)$$

for $M \subset I \setminus \{j, l\}$, $|M| = k$, together with lines $\pi_{M \cup \{j, l\}}$ for $|M| = k-1$, $M \subset I \setminus \{j, l\}$ whose slopes are $\Pi_M^{l,j}$,

$$\pi_{M \cup \{j, l\}}: x_j - \Pi_M^{j,l} x_l = \alpha_j - \sum_{r \in M \cup \{l\}} \alpha_r \Pi_{\{l\} \cup M \setminus \{r\}}^{j,r} = x_j^{M \cup \{j, l\}}. \quad (3.3)$$

The relations associated with Δ are now seen to be associated with a point of intesection $x_\Delta = (x_j^0, x_l^0)$ of n_Δ lines in $\mathcal{C}_{j,l}$ where,

$$n_\Delta = \begin{cases} k+2 & \text{if } \Delta \text{ is of type (i),} \\ 2 & \text{if } \Delta \text{ is of type (ii)' or (ii)''.} \end{cases}$$

These n_Δ lines are said to be the lines in $\mathcal{C}_{j,l}$ associated with Δ .

The generators β_J associated with lines π_J in $\mathcal{C}_{j,l}$, were defined in §2 to be given by loops based at $\mathbf{0}$ in the (complex) x_j -axis, unless $j \notin J$, in which case they are loops in the (complex) x_l -axis. Choose $x_\Delta^\varepsilon \equiv (x_j^0 - \varepsilon, x_l^0 - \varepsilon) = (x_j^\varepsilon, x_l^\varepsilon)$ with $\varepsilon > 0$ sufficiently small, so that the only members of $\mathcal{C}_{j,l}$ crossing the square $[x_j^0 - \varepsilon, x_j^0] \times [x_l^0 - \varepsilon, x_l^0]$ are the lines associated with Δ , while no line of $\mathcal{C}_{j,l}$ cuts the x_l -axis in the interval (x_l^ε, x_l^0) . Let β_J^ε be the generators of,

$$\pi_1(\mathbf{C}^2 \setminus \mathcal{C}_{j,l}, x_\Delta^\varepsilon)$$

defined in the same way as β_J with basepoint x_Δ^ε replacing $\mathbf{0}$.

Definition Suppose g_1, \dots, g_r are elements of a group G . It is said that the relation $\mathcal{R}\{g_1, \dots, g_r\}$ holds if, and only if, the product g_1, \dots, g_r is unchanged by cyclic rotations of the g_i 's.

Lemma 1 [R] *If $\pi_{J_1}, \dots, \pi_{J_{n_\Delta}}$ are the n_Δ lines associated with Δ in an anti-clockwise order, then the relation in $B_{n,k}$ associated with Δ is,*

$$R_\Delta: \mathcal{R}\{\beta_{J_1}^\varepsilon, \dots, \beta_{J_{n_\Delta}}^\varepsilon\}.$$

To complete the evaluation of R_Δ it is only necessary to move the basepoint from x_Δ^ε to $\mathbf{0}$ and determine the transformed β_j^ε in terms of $\{\beta_j\}$. Let us note first that under the conditions of Theorem 1, in all cases $x_j^0, x_l^0 \in \mathbf{R}^+$ while all lines in $\mathcal{C}_{j,l}$ associated with Δ are either parallel to the x_j -axis or cut it at a positive real point.

We shall define the curve followed by the basepoint in two parts. First, move the basepoint from x_{Delta}^ε to $(0, x_l^\varepsilon)$ along a path with x_l fixed, and $\Im(x_j) > 0$ at all intermediate positions. This transforms all $\beta_{J_i}^\varepsilon$ associated with Δ to similarly defined curves, based at $(0, x_l^\varepsilon)$. The fact that no line in $\mathcal{C}_{j,l}$ cuts the x_l -axis between x_l^ε and x_l^0 has been used here. Finally, move the base point from $(0, x_l^\varepsilon)$ to $\mathbf{0}$ along a path in the (complex) x_l -axis with $\Im(x_l) > 0$ at all intermediate points. Any generator $\beta_{J_i}^\varepsilon$ associated with Δ , for which the lines π_{J_i} in $\mathcal{C}_{j,l}$ is parallel to the x_j -axis, will transform to β_{J_i} . The remaining generators transform according to the following lemma, see Fig 2.

Lemma 2 *Under the shift of basepoint described above, a generator β_M^ε ($j \in M$) associated with Δ , transforms to,*

$$\xi^{-1} \eta \beta_M \eta^{-1} \xi$$

where ξ and η are the products of the generators β_K associated with those lines π_K in $\mathcal{C}_{j,l}$ satisfying,

$$\begin{aligned} \xi: x_j^K < 0, \quad 0 < x^K < x_j^0 \\ \eta: 0 < x_j^K < x_j^M, \quad x^K > x_j^0 \end{aligned}$$

respectively, in order of increasing x_j^K . Here x^K denotes the value of x_j at which π_K crosses $x_l = x_l^0$.

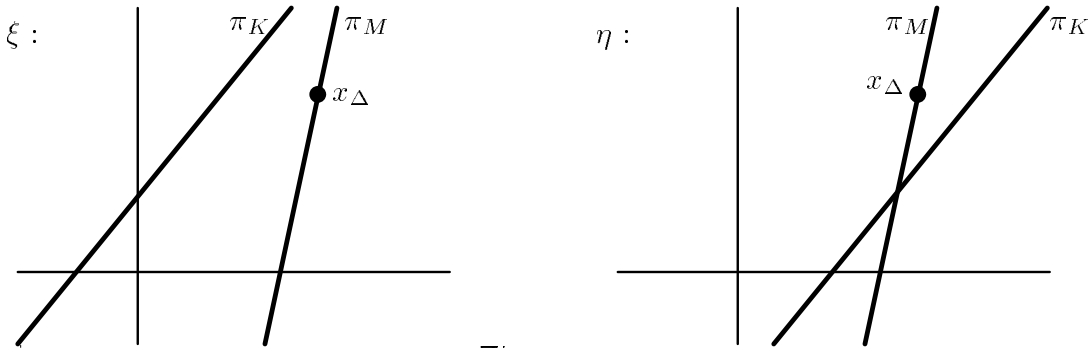


Figure 2

By (2.4), x_j^K can only be negative if $\max(K) > j$. Since x_l^0 has order at most that of α_j , it is impossible for x^K to be positive at the same time as $x_j^K < 0$ (see (3.3)). Hence $\xi = 1$ in Lemma 2, reducing the computation to that of determining the sets K satisfying the condition for η . Note that if K_m denotes $\max(K)$, then the sign of $x_j^{K \cup \{j\}}$ is, by (2.4), determined by that of,

$$\alpha_j - \alpha_{K_m} \prod_{K \setminus \{K_m\}}^{j, K_m}.$$

This is positive if $K_m < j$, while if $K_m > j$ it is only positive if,

$$\prod_{K \setminus \{K_m\}}^{j, K_m} < 0.$$

That is, $x_j^{K \cup \{j\}} > 0$ if, and only if, the number of elements of K greater than j , denoted $\#(K > j)$, is even. Using similar arguments it may be seen that the following lemma holds. Note that in all of cases (i), (ii)' and (ii)'', all the generators β_M associated with Δ , for which $j \in M$, have $\max(M) = j$.

Lemma 3 *The condition $0 < x_j^K < x_j^M$ where $\max(M) = j$ is equivalent to,*

$$(a) \#(K > j) = 0,$$

and (b) $m \equiv \max((K \cup M) \setminus \{j\}) \in K$,

and (c) $\prod_{K \setminus \{j, m\}}^{j, m} > \delta_{m \in M} \prod_{M \setminus \{j, m\}}^{j, m}$, where $\delta_{m \in M}$ is 0 or 1 as $m \notin M$ or $m \in M$.

The consideration of the remaining part of the condition for β_K to be a term in η of Lemma 2, namely $x^K > x_j^0$, must be broken up into three cases.

Case (i): In this case the lines in $\mathcal{C}_{j, l}$ associated with Δ are $\pi_{J \setminus \{j\}}$ and $\pi_{J \setminus \{l\}}$ (parallel to the x_j and x_l axes, respectively) and $\{\pi_{J \setminus \{p\}} \mid p \in J \setminus \{j, l\}\}$ which have positive slopes. The condition η in Lemma 2 thus requires both j and l to be elements of K . Also $x_j^0 = x_j^{J \setminus \{l\}}$ and so $x^K > x_j^0$ requires,

$$\prod_{K \setminus \{j, l\}}^{j, l} x_l^{J \setminus \{j\}} > x_j^{J \setminus \{l\}} - x_j^K$$

by (3.3). Applying (2.4) reduces this to,

$$-\delta_{m \in J} \prod_{K \setminus \{j, l\}}^{j, l} \prod_{J \setminus \{j, l, m\}}^{l, m} > \delta_{m \in K} \prod_{K \setminus \{j, m\}}^{j, m} - \delta_{m \in J} \prod_{J \setminus \{j, l, m\}}^{j, m},$$

where $m = \max((J \cup K) \setminus \{j, l\})$. If $m \notin J$, this condition reduces to $\prod_{K \setminus \{j, m\}}^{j, m} < 0$ which reduces to $m < l$, since by Lemma 3(a), $m < j$. In all cases, $m < l$, and condition η of Lemma 2 applied to $M = J \setminus \{p\}$ ($p \neq j$) gives K of the form $\overline{K} \cup \{j, l\}$ with $\max \overline{K} < l$ and,

$$\left. \begin{aligned} & \prod_{K \setminus \{j, l, m\}}^{j, l} > \delta_{l \neq p} \prod_{J \setminus \{j, l, p\}}^{j, l} \\ & \delta_{m \in J} \prod_{J \setminus \{j, l, m\}}^{l, m} (\prod_{\overline{K}}^{j, l} - \prod_{J \setminus \{j, l, m\}}^{j, l}) < \delta_{m \in K} \prod_{K \setminus \{j, m\}}^{j, m} \end{aligned} \right\} \quad (3.4)$$

As was mentioned above, the second condition is automatically satisfied if $m \notin J$.

Case (ii)^y: In this case, the two lines associated with Δ are π_L (parallel to the x_j -axis) and π_J . Hence $x_l^0 = x_l^L$ and $x^K > x_j^0$ requires,

$$(\delta_{l \in K} \Pi_{K \setminus \{j, l\}}^{j, l} - \delta_{l \in J} \Pi_{J \setminus \{j, l\}}^{j, l}) x_l^L > x_j^J - x_j^K .$$

This reduces to,

$$(\Pi_{K \setminus \{j, l\}}^{j, l} \delta_{l \in K} - \Pi_{J \setminus \{j, l\}}^{j, l} \delta_{l \in J}) \Pi_{L \setminus \{l, m\}}^{l, m} \delta_{m \in L \setminus \{l\}} < (\Pi_{J \setminus \{j, m\}}^{j, m} \delta_{m \in J} - \Pi_{K \setminus \{j, m\}}^{j, m} \delta_{m \in K}), \quad (3.5)$$

where $m = \max\left(\left((J \cup K) \setminus \{j, l\}\right) \cup (L \setminus \{l\})\right)$. Thus condition η of Lemma 2 applied to $M = J$ gives K of the form $\overline{K} \cup \{j, l\}$ with $\max(\overline{K}) < l$, so long as $\max(J \setminus \{j\}) \leq l$ and (3.5) together with,

$$\Pi_{K \setminus \{j, l\}}^{j, l} > \delta_{l \in J} \Pi_{J \setminus \{j, l\}}^{j, l} . \quad (3.6)$$

Indeed, if $\max(J \setminus \{j\}) > l$, no K satisfies condition η of Lemma 2 for $M = J$.

Case (ii)^{y'}: In this case the two lines associated with Δ are π_J and π_L . Using a similar argument to those in the last two cases, it can be seen that condition η of Lemma 2 is satisfied by those K of the form $\overline{K} \cup \{j, l\}$ for which $\max(\overline{K}) < l$ and,

$$\begin{aligned} \Pi_{K \setminus \{j, m\}}^{j, m} \delta_{m \in K} < & \left[(\Pi_{K \setminus \{j, l\}}^{j, l} - \Pi_{L \setminus \{j, l\}}^{j, l} \delta_{l \in L}) \Pi_{J \setminus \{j, m\}}^{j, m} \delta_{m \in J} \right. \\ & \left. + (\Pi_{J \setminus \{j, l\}}^{j, l} - \Pi_{K \setminus \{j, l\}}^{j, l}) \Pi_{L \setminus \{j, m\}}^{j, m} \delta_{m \in L} \right] (\Pi_{J \setminus \{j, l\}}^{j, l} - \delta_{l \in L} \Pi_{L \setminus \{j, l\}}^{j, l})^{-1} , \end{aligned} \quad (3.7)$$

where $m = \max\left((J \cup K \cup L) \setminus \{j, l\}\right)$. It is also required that,

$$\Pi_{K \setminus \{j, l\}}^{j, l} > \Pi_{M \setminus \{j, l\}}^{j, l} \delta_{l \in M} . \quad (3.8)$$

These give the possible K 's corresponding to both $M = L$ and $M = J$.

Theorem 2 $B(n, k)$ has a presentation with generators $\{\beta_J \mid J \subset I, \quad |J| = k+1\}$ and relations,

- (i) $\mathcal{R}\{\beta_{J \setminus \{p\}}^*, \dots, \beta_{J \setminus \{l\}}^*, \beta_{J \setminus \{j\}}^*\}$ for $|J| = k+2$, $j = \max(J)$, $l = \max(J \setminus \{j\})$, where p ranges over $J \setminus \{j\}$ in increasing order;
- (ii)^y $\mathcal{R}\{\beta_J^*, \beta_L\}$ for $|J| = |L| = k+1$, $|J \cap L| < k$, $\max(J) = j > l = \max(L)$;
- (ii)^{y'} $\mathcal{R}\{\beta_J^*, \beta_L^*\}$ for $|J| = |L| = k+1$, $|J \cap L| < k$, $\max(J) = \max(L) = j$ with $\max(J \setminus \{j\}) = l \geq \max(L \setminus \{j\})$.

Here $\beta_M^* = \eta \beta_M \eta^{-1}$ where η is the product of generators β_K over K whose maximal elements are j, l and which satisfy (3.4) in (i); (3.5), (3.6) in (ii)^y; (3.7), (3.8) in (ii)^{y'}. The product is taken in order of decreasing $\Pi_{K \setminus \{j, l\}}^{j, l}$. In case (ii)^y, if $\max(J \setminus \{j\}) > l$ then $\beta_J^* \equiv \beta_J$.

This presentation depends on the choice of positive real numbers $a_1 < a_2 < \dots < a_n$ such that,

$$\Pi_J^{i,j} = \prod_{k \in J} \frac{a_k - a_i}{a_k - a_j}$$

are distinct over all $J \subset I \setminus \{i, j\}$ of order $(k-1)$.

4: SPECIAL CASES

In this section we shall illustrate the use of Theorem 2 in some special cases. If $n = k+1$, there is only one generator, so that $B(k+1, k) \cong \mathbf{Z}$. If $n = k+2$, there are n generators labelled by the $k+1$ -sets $I \setminus \{j\}$. There is just one relation, which is of type (i),

$$\{\beta_{I \setminus \{1\}}, \dots, \beta_{I \setminus \{n\}}\}.$$

In other words, the only relation states that $\beta_{I \setminus \{1\}}, \dots, \beta_{I \setminus \{n\}}$ is central.

The generators for $B(n, 1)$ are labelled by pairs $\{i, j\} \subset I$. There are no relations of type (ii)'. The relations of type (i) give,

$$\{\beta_{jk}, \beta_{ik}, \beta_{ij}\} \quad i < j < k.$$

The relations of type (ii)' give,

$$\begin{aligned} \{\beta_{ij}, \beta_{kl}\} & \quad \text{for } j > i > l > k \text{ or } j > l > k > i \\ \{\beta_{jl}\beta_{ij}\beta_{jl}^{-1}, \beta_{kl}\} & \quad \text{for } j > l > i > k. \end{aligned}$$

These are the standard relations existing between generators of the pure braid group P_n . Here all Π 's are 1.

In the case of $B(n, 2)$, the generators are labelled by triples $\{i, j, k\} \subset I$. The relations of type (i) are,

$$\{\beta_{jkl}, \beta_{ikl}, \beta_{ijl}^*, \beta_{ijk}\} \quad \text{for } i < j < k < l,$$

where $\beta_{ijl}^* = \eta\beta_{ijl}\eta^{-1}$ and η is the product of generators β_{qkl} with $q < i$, in decreasing order. The relations of type (ii)' give,

$$\begin{aligned} \{\beta_{ijk}, \beta_{lms}\} & \quad \text{for } l < m < s < j < k, \quad i < j; \\ \{\beta_{ijk}^*, \beta_{lms}\} & \quad \text{for } i < j \leq s < k, \quad l < m < s. \end{aligned}$$

In the latter case, $\beta_{ijk}^* = \eta\beta_{ijk}\eta^{-1}$, where η is a product of generators β_{qsk} with q decreasing and such that $\max\{i, m\} < q < j$, for $j = s$, while for $j < s$, q must satisfy,

$$\Pi_q^{ks} \Pi_l^{sm} \delta_{\alpha m} < \Pi_i^{kj} \delta_{\alpha j} - \Pi_s^{kq} \delta_{\alpha q} \quad (4.1)$$

with $\alpha = \max\{m, q, j\}$. Finally the relations of type (ii)'' are given by,

$$\{\beta_{ilj}^*, \beta_{kmj}^*\} \quad \text{for } k < m < l < j, \quad k \neq i < l$$

and $\beta_{ilj}^* = \xi\beta_{ilj}\xi^{-1}$, $\beta_{kmj}^* = \eta\beta_{kmj}\eta^{-1}$ in which ξ and η are suitable products of generators β_{plj} , in decreasing order of p . The p 's involved in ξ are those with $i < p < m$ (equality only if $k < i$), while the condition required for η is that,

$$\Pi_i^{jl}\Pi_l^{ip}\delta_{\alpha p} > \Pi_l^{ip}\Pi_p^{jl}\delta_{\alpha i} - \Pi_k^{jm}\Pi_p^{il}\delta_{\alpha m},$$

where $\alpha = \max\{i, m, p\}$. This latter condition reduces to,

$$\left. \begin{array}{l} l > p > i \text{ and } p \geq m \text{ (equality only if } k < i) \\ \text{or } p < i < m \end{array} \right\}$$

In a similar way, (4.1) (type (ii)') may be reduced to,

$$\left. \begin{array}{l} m < q < s \\ \text{or } m < j, \quad q < s \\ \text{or } m = j > q \text{ and } \Pi_q^{ks}\Pi_l^{sj} < \Pi_i^{kj} \\ \text{or } m = j = q \text{ and } l < i \end{array} \right\} \quad (4.2)$$

It is apparent that the values of $\{a_i\}$ only enter into the presentation given above, in those relations associated with case (ii)', for which $i, l < j = m < s < k$ ($i \neq l$) and then the allowable values of q are such that $q < j$ with $\Pi_q^{ks}\Pi_l^{sj} < \Pi_i^{kj}$.

5: RELATION TO ORDINARY BRAID GROUP

Let \mathcal{C} be an arrangement of $n+1$ hyperplanes in \mathbf{C}^k , constructed as in as in §2, with $n+1$ replacing n . Then, by construction, $\{\pi_{J \cup \{n+1\}} \mid J \subset I, |J| = k\}$ cut the x_{n+1} -axis in $\binom{n}{k}$ distinct points. Any element of $B(n, k)$ is associated with a loop in \mathbf{C}^n , based at $\mathbf{0}$, and as $\mathbf{x} = (x_1, \dots, x_n)$ moves, the $\binom{n}{k}$ values $a_J(\mathbf{x})$ of x_{n+1} , for which π_{n+1} passes through the intersection of k other hyperplanes $\{\pi_i \mid i \in J\}$, will follow paths in \mathbf{C} . It is possible that some of the $a_J(\mathbf{x})$ may coincide, for some values of \mathbf{x} .

For any $J, K \subset I$, $|J| = |K| = k$, $|J \cap K| < k-1$, a path in $U(n, k)$ exists in which $a_J(\mathbf{x}) = a_K(\mathbf{x})$ at some point \mathbf{x} on the path, while all other $a_L(\mathbf{x})$ remain distinct along the entire path. For every such pair, J, K , pick such a path, and deform it slightly so that $a_J(\mathbf{x})$ and $a_K(\mathbf{x})$ never coincide, but do wind around each other. Denote the corresponding element of $P_{\binom{n}{k}}$ by $h_{J,K}$. This will be conjugate to $\beta_{J,K}$ in $P_{\binom{n}{k}}$, using the standard notation for generators of $B(*, 1)$ (see §4). For every $J \subset I$ of order $k+1$, choose an element $g_J \in P_{\binom{n}{k}}$, corresponding to a path in $U(n, k)$ based at $\mathbf{0}$, associated with the generator β_J of $B(n, k)$. Such a path may need to be slightly deformed so as to ensure $\{a_J(\mathbf{x})\}$ distinct throughout the path.

Let G be the group generated by $\{g_J \mid |J| = k+1\}$ and $\{h_{J,K} \mid |J| = |K| = k, |J \cap K| < k-1\}$. Let H be the subgroup of G generated by $\{h_{J,K}\}$, and K be its normal closure in G . Then the choice of g_J is arbitrary up to composition with elements of K .

The analysis above shows that,

$$B(n, k) \cong G/K, \quad (5.1)$$

a quotient of a subgroup of $P_{\binom{n}{k}}$. The generators g_J and $h_{J,K}$ used to construct G and K will be conjugates of,

$$\prod_{j \in J} \left(\prod_{\substack{i \in J \\ i < j}} \beta_{J \setminus \{i\}, J \setminus \{j\}} \right) \text{ and } \beta_{J,K},$$

respectively, in $P_{\binom{n}{k}}$.

The construction described here is analogous to viewing $B(n, 1)$ as P_n . That is, consider $X_{n+1,1}$ fibred over $X_{n,1}$ with projection map given by forgetting the last point. The fibre will be \mathbf{C} with n points removed, and has fundamental group F_n . In the case of general k , one can still consider the map from $X_{n+1,k}$ to $X_{n,k}$ obtained by forgetting the last hyperplane. However, this map is not a fibration; this corresponds to the non-triviality of K .

It should be observed that when $k = 1$, the higher braid group reduces to P_n , the pure braid group, and not to B_n , the full braid group. A natural action of S_n exists on X_n , by permuting the points, and B_n appears as the fundamental group of X_n/S_n . There is no obvious action of S_n on $U(n, k)$ for $k > 1$. Let $V(n, k)$ denote the space of all (\mathbf{a}, \mathbf{x}) with $\mathbf{a} \in X_n$ and $\mathbf{x} \in \mathbf{C}$, for which the hyperplanes $\{\mathbf{r} \cdot \mathbf{n}_i = x_i\}$ with \mathbf{n}_i defined by (2.1), are in general position. Clearly, the projection to X_n , given by $(\mathbf{a}, \mathbf{x}) \mapsto \mathbf{a}$ defines a fibration with fibre $U(n, k)$. A free action of S_n exists on $V(n, k)$, given by,

$$\sigma(\mathbf{a}, \mathbf{x}) = (\sigma(\mathbf{a}), \sigma(\mathbf{x}))$$

for $\sigma \in S_n$. The resulting quotient fibration,

$$\begin{array}{c} V(n, k)/S_n \\ \downarrow \\ X_n/S_n \end{array}$$

gives rise to a homomorphism,

$$B_n \longrightarrow \text{Aut}(B(n, k))$$

in much the same way that a homomorphism $B_n \rightarrow \text{Aut}(F_n)$ exists (see for example [B]). This provides yet another construction for braid group representations, distinct from, but similar to [L].

6: FURTHER REMARKS

In [MS 1], it was shown how the space of orders on k -sets satisfying a certain condition, generalises, in some sense, the structure of the symmetric group S_n . These orders may be viewed as the order of $\{a_J(\mathbf{0}) \mid |J| = k\}$, which clearly depends on the choice of a_1, \dots, a_{n+1} .

It is clear from the analysis of this paper, that $B(n, k)$, for $k > 1$, has no unique natural presentation, but rather possesses a family of such, indexed by possible real $\{a_i\}$. In other words, a different presentation may be defined for each component of the space of allowed $\{a_i\}$, a complement of hypersurfaces in \mathbf{R}^n . This corresponds to the fact that for $k > 1$, $B(n, k)$ is associated with only a *weak* Bruhat order. The results of §5 (see (5.1)) may also be seen as the formulation in terms of groups, analogous to the constructions of partially ordered sets in [MS 1].

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