Control Design for Time-Delay Linear Systems: a Rational Transfer Function Based Approach

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Abstract

The aim of this paper is to present new results on $\mathcal{H}_\infty$ control synthesis for time-delay linear systems. We extend the use of a finite order LTI system, called comparison system to $\mathcal{H}_\infty$ analysis and design. Differently from what can be viewed as a common feature of other control design methods available in the literature to date, the one presented here treats time-delay systems control design with classical numeric routines based on Riccati equations arisen from $\mathcal{H}_\infty$ theory. The proposed algorithm is simple, efficient and easy to implement. Some examples illustrating state and output feedback design are solved and discussed in order to put in evidence the most relevant characteristic of the theoretical results. Moreover, a practical application involving a 3-DOF networked control system is presented.

Keywords: Time-delay systems, linear systems, $\mathcal{H}_\infty$ control design.

1 Introduction

Time delays in dynamic systems generally imply on poor performance, or even instability. For this reason, in the past decades, there has been a great effort for the development of efficient control design techniques to cope with time-delay. See the books [10] and [15], and the survey paper [19] among many others. The control synthesis becomes more difficult in the presence of exogenous disturbances, which can degrade the system performance, justifying efforts in developing new approaches to eliminate this undesired effect. In this context, the $\mathcal{H}_\infty$ controller plays the central role of maintaining the $\mathcal{H}_\infty$ norm of the transfer function between the external disturbance and the controlled output below a pre-specified level $\gamma > 0$ for a given value of time delay, keeping the closed-loop system stable [1].
There are two main classes of synthesis problems for time-delay systems, which provide delay-dependent and delay-independent controllers. In the literature, many works have dealt with state feedback design in the Riccati equation framework, as for example [13] and [21] for the delay-independent case, whereas [7], [8] and [14] address the same design problem by means of Lyapunov-Krasovskii functionals, obtaining delay-dependent controllers. For the output feedback problem, [3] and [9] have proposed delay-independent controllers obtained from the solution of Riccati equations, whereas [5] solved the same problem using Lyapunov-Krasovskii functionals. In the context of delay-dependent output feedback design problem, to the best of our knowledge, much less attention has been given, see [8].

Our goal is to present delay-dependent design procedures for state and output feedback control design. Towards this end, we apply the Rekasius substitution [18] to replace the delay operator by a rational first order transfer function. The paper [16] has proposed an useful technique for stability analysis of time-delay system applying the Rekasius substitution and the Routh-Hurwitz criterion. However, this method imposes the need to handle the characteristic equation of the time-delay system, a difficulty that has been circumvented in [12], by addressing the same problem in the frequency domain, based on the celebrated Nyquist criterion. An important consequence of the Rekasius substitution in [12] is the definition of a finite order linear time invariant system, called comparison system, which provides, for the example solved, a tight lower bound to the \( \mathcal{H}_\infty \) norm of the time-delay system, allowing a simple and efficient filter synthesis algorithm. A comparison system for time-delay systems was first introduced in [23], yielding one of the most important results for stability analysis and \( \mathcal{H}_\infty \) norm calculation. Indeed, adopting such a comparison system approach, the well known Padé approximation is used to determine linear time invariant systems of increasing but finite order, allowing the direct determination of stability margin and bounds for the \( \mathcal{H}_\infty \) norm performance of the time-delay system. It is shown that the quality of the result is better whenever the order of the Padé approximation increases. However, high order approximations are not suitable for control synthesis.

This paper follows the same stream as that proposed in [12] and [23]. A linear time invariant comparison system of order twice the number of state variables of the time-delay system, built from the Rekasius substitution, is defined and the relationship between the comparison and the time-delay systems stability, highlighted by the results from [12], is discussed. It is shown that the comparison system \( \mathcal{H}_\infty \) norm provides an useful lower bound to the time-delay system \( \mathcal{H}_\infty \) norm, which is used for delay-dependent linear control design to impose lower and upper bounds on the closed-loop transfer function \( \mathcal{H}_\infty \) norm. The procedure is compared with methods from the literature by means of several illustrative examples. Finally, it is important to stress that comparing to [12], the present paper innovates in the
following directions:

- The output feedback controller design needs a new parameterization from the stabilizing solution of a Riccati equation (filter) and any feasible solution of a Riccati inequality (control). It assures the existence of a starting feasible solution that is essential for the development of our design method. Moreover, the design procedure, when compared to the ones already cited, is simpler to be implemented and provides more accurate results.

- A practical application concerning the control of a 3-DOF system of sixth order and four control inputs is presented. It puts in evidence that the proposed method is well adapted to deal with control-delayed signals arising in networked control systems.

The paper is organized as follows. Section 2 is devoted to the definition and discussion of the comparison system: in Section 2.1 and 2.2 we introduce the relationship between stability and $\mathcal{H}_\infty$ norm, respectively, of the comparison and the time-delay systems. Section 3 addresses the problem of state feedback design and Section 4 deals with output feedback synthesis. Section 5 brings examples to put in evidence the more relevant aspects of the theoretical results and in Section 6 we discuss the output feedback design for a real world plant. The paper ends with a conclusion in Section 7 where the main contributions are briefly commented. The notation used throughout is standard. Capital letters denote matrices and small letters denote vectors. For scalars, small Greek letters are used. For real matrices or vectors (') indicates transpose. Exclusively, rational transfer functions of LTI systems are denoted as

\[
C(sI - A)^{-1}B + D = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

where the matrices $A, B, C, D$ are real and of compatible dimensions. The maximum singular value of a complex matrix $Q$ is a nonnegative real number denoted by $\sigma(Q)$. The nonnegative real number $r_s(Q)$ denotes the spectral radius of a square matrix $Q$. The identity matrix of any dimension is denoted by $I$.

2 Rational Comparison System

The LTI comparison system, introduced in [12], associated to the following time-delay system

\[
\begin{align*}
\dot{x}(t) &= A_0x(t) + A_1x(t - \tau) + E_0w(t) \\
z(t) &= C_0x(t) + C_1x(t - \tau)
\end{align*}
\]
is presented for both stability analysis and $H_\infty$ norm calculation. The system state vector is $x \in \mathbb{R}^n$, $w \in \mathbb{R}^r$ is the exogenous input and $z \in \mathbb{R}^q$ is the output. It is assumed throughout that $x(t) = 0 \forall t \in [-\tau, 0]$, and the delay $\tau \geq 0$ is constant with respect to time. The basic idea stems from the fact, already denominated by several authors as Rekasius’ substitution, that for $s = j\omega$ with $\omega \in \mathbb{R}$, the equality

$$e^{-s\tau} = \frac{1 - \lambda^{-1}s}{1 + \lambda^{-1}s}$$

is not a mere approximation, instead it holds for some $\lambda \in \mathbb{R}$ such that $\omega/\lambda = \tan(\omega\tau/2)$. It is important to notice that for a given pair $(\lambda, \omega)$ there exist many $\tau \geq 0$ satisfying this relation. Based on this relationship we introduce what is called a rational comparison system associated to (2)-(3) as follows:

$$H(\lambda, s) = \begin{bmatrix} A\lambda & E \\ Cz & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \lambda I \\ A_o + A_1 & A_o - A_1 - \lambda I \end{bmatrix} \begin{bmatrix} E \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} C_o + C_z1 & C_o - C_z1 \end{bmatrix}$$

(5)

Denoting by $T(\tau, s)$ the non-rational transfer function of the time-delay system from the input $w$ to the output $z$, this LTI system has been determined in such a way that the equality $H(\lambda, j\omega) = T(\tau, j\omega)$ holds whenever $\lambda \in \mathbb{R}$, $\tau \geq 0$ and the frequency $\omega \in \mathbb{R}$ are related by the nonlinear expression $\omega/\lambda = \tan(\omega\tau/2)$ that emerges from equality (4), see [12].

At this point, we are interested in verifying if it is possible to design a controller based on $H(\lambda, s)$ that assures some desired property to the time-delay system under consideration defined by the non-rational transfer function $T(\tau, s)$. To this end, we need to make clear the relationship between the comparison and the time-delay systems as far as stability and $H_\infty$ norm calculation are concerned.

2.1 Stability Analysis

A standard problem in the framework of time-delay systems consists in the determination of a time delay $\tau^* > 0$ such that the system (2)-(3) remains asymptotically stable for all $\tau \in [0, \tau^*)$. Clearly, the determination of $\tau^*$ depends on the ability to calculate the poles of the transfer function $T(\tau, s)$, that are the roots of the characteristic equation

$$\Delta_T(\tau, s) = \det(sI - A_0 - A_1e^{-s\tau})$$

(6)
which is transcendental (whenever $\tau > 0$) and admits, generally, infinitely many roots being thus hard to solve. Many methods have been developed in order to circumvent this difficulty. Most of the existing procedures are based on the detection of the crossings of poles through the imaginary axis. This fact comes from one important property of time-delay systems, the root continuity argument. It means that for any positive value of the delay, the position of the poles varies continuously with respect to delay, so that any root crossing from the left to the right half-plane will need to pass through the imaginary axis. An interesting comparison between some algorithms proposed in order to find the position where the roots cross the imaginary axis is provided by [22]. All of those share some similarities since they are based on a particular feature of the roots of some polynomial with the degree higher than the one of the original system. Another strategy to the solution of this problem is presented in [12]. Based on the celebrated Nyquist criterion, a simple graphical test can be performed in order to decide if the system is asymptotic stable for a given value of the delay. Hence, by increasing $\tau \geq 0$, it is possible to determine the value $\tau^*$ corresponding to the first occurrence of an unstable pole, which defines the so-called stability margin of the time-delay system.

A property exploited in [16] is the cluster simple nature of the roots of (6). Indeed, putting aside that $s = 0$ is a solution for $\Delta_T(\tau, s) = 0$, since otherwise the system would be unstable for every $\tau$, one can show two important properties. Firstly, all zeros crossing the imaginary axis will happen in complex conjugate pairs, i.e., if $\Delta_T(\tau, j\omega) = 0$ then $\Delta_T(\tau, -j\omega) = 0$. Secondly, there is a $2\pi/\omega$ periodicity for the existence of zeros in specific locations of the imaginary axis, which means that if $\Delta_T(\tau, j\omega) = 0$ then $\Delta_T(\tau + 2k\pi/\omega, j\omega) = 0$, for all $k \in \mathbb{N}$. With these results at hand, [16] has shown that a set $\Omega$ with a finite and manageable number of elements $(\omega_i, \tau_i)$, with $\omega_0 > 0$ and $0 \leq \tau_i < 2\pi/\omega_i$ completely characterizes the stability of the time-delay system (2)-(3), in the sense that for any zero of (6) through the imaginary axis, i.e., $\Delta_T(\tau^*, j\omega^*) = 0$, there exists an element $(\omega_i, \tau_i) \in \Omega$ and $k \in \{0, 1, \ldots\}$ such that $|\omega^*| = \omega_i$ and $\tau^* = \tau_i + 2k\pi/\omega_i$. The importance of this discussion in the present context comes from the fact that any $\lambda$ given by (4) calculated by $\lambda = \omega/\tan(\omega \tau/2)$ is invariant inside the set of solutions of $\Delta_T(\tau^*, j\omega^*) = 0$ with $(\omega^*, \tau^*)$ derived from a single element $(\omega_i, \tau_i) \in \Omega$. In other words, for each element $(\omega_i, \tau_i) \in \Omega$, there exists $\lambda_i = \omega_i/\tan(\omega_i \tau_i/2) \in (-\infty, \infty)$ such that the characteristic equation for the comparison system

$$\Delta_H(\lambda, s) = \det(sI - A_\lambda)$$

(7)

satisfies $H(\lambda_i, j\omega_i) = 0$.

Regarding the use of the Rekasius substitution (4) for the problem of stability of the time-delay
system (2)-(3), [12] presents an extensive analysis of the subject, including its limitations. The main problem is the strong dependence of $\lambda_i$ with the location of the crossing $\omega_i$, which hinders us in the direct use of this result to provide the delay stability margin. Recently, [6] deals with the same problem but with the following substitution

$$e^{-s\tau} = \left(\frac{1 - \lambda_s^{-1} s}{1 + \lambda_s^{-1} s}\right)^\kappa$$

which, as before, is not a mere approximation, but also holds for $s = j\omega$ and some $\lambda \in \mathbb{R}$ such that $\omega/\lambda = \tan(\omega \tau/2\kappa)$.

The first clear advantage of this approach is the restriction of the range of values of $\lambda$. Since we can restrict our analysis to the first occurrence of the delay for each cluster in $\Omega$, $\tau_i < 2\pi/\omega_i$, and therefore, for any $\kappa \geq 2$, we have $\lambda_i \geq 0$. Larger values of $\kappa$ would further restrict the upper bounds for $\lambda$. A second important property is that the higher the value of $\kappa$, the lower is the dependence of $\lambda_i$ with $\omega_i$, in the sense that there exists a $\kappa^* \geq 1$ such that if we order the elements of $\Omega$ by the values of the delay, i.e., $\tau_1 < \tau_2 < \ldots < \tau_n$, then their respective values of $\lambda$ are also in order $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. In this case, the system with delay is stable for $\tau = [0, \tau_1)$ if and only if the comparison system is stable for $\lambda = (\lambda_1, \infty)$.

Although the appealing benefit of using a higher order Rekasius substitution for analysis, the same cannot be told for synthesis. As an example, by regarding $A_\lambda$ for $\kappa = 2$

$$A_\lambda = \begin{bmatrix} 0 & \lambda I & 0 \\ 0 & 0 & \lambda I \\ A_0 + A_1 & 2A_0 - 2A_1 - \lambda I & A_0 + A_1 - 2\lambda I \end{bmatrix}$$

it becomes clear the difficulties one have to face when trying to deal with the linear dependence of the elements of the last row. For that reason we restrain ourselves for the case $\kappa = 1$ and develop a strategy exploiting the relations of the norm $H_{\infty}$ of both the time-delay and the comparison systems.

### 2.2 $H_{\infty}$ Norm Calculation

In this section our purpose is to discuss how to address the problem

$$\|T(\tau, s)\|_{\infty} = \sup_{\omega \in \mathbb{R}} \sigma(T(\tau, j\omega))$$

(10)
for a given $\tau \in [0, \tau^\ast)$. As it was discussed on [12], we define the positive scalar

$$\lambda_o = \inf \{ \lambda \mid A_\xi \text{ is Hurwitz } \forall \xi \in (\lambda, \infty) \}$$

assuring that $\|H(\lambda,s)\|_{\infty} = \sup_{\omega \in \mathbb{R}} \sigma(H(\lambda,j\omega))$ holds and the norm $\|H(\lambda,s)\|_{\infty}$ is bounded for $\lambda \in (\lambda_o, \infty)$. In this framework, the next theorem relates the $H_\infty$ norm of the time delay and the comparison systems, see [12].

**Theorem 1** Assume that $A_0 + A_1$ is Hurwitz. For each $\lambda \in (\lambda_o, \infty)$ define $\alpha \geq 0$ such that

$$\alpha = \arg \sup_{\omega \in \mathbb{R}} \sigma(H(\lambda,j\omega))$$

(11)

and determine $\tau(\lambda)$ from $\alpha/\lambda = \tan(\alpha\tau/2)$. If $\tau(\lambda) \in [0, \tau^\ast)$ then $\|H(\lambda,s)\|_{\infty} \leq \|T(\tau(s),s)\|_{\infty}$ holds.

Under the assumption of Theorem 1, we can conclude that the transfer functions $H(\infty,s)$ and $T(0,s)$ are asymptotically stable, equal and, consequently, their norms satisfy $\|H(\infty,s)\|_{\infty} = \|T(0,s)\|_{\infty} < \infty$. Moreover, notice that we can not discard the possibility that for some $\lambda \in (\lambda_o, \infty)$ the value of the time delay calculated from (11) be such that $\tau(\lambda) \notin [0, \tau^\ast)$. In this case, the lower bound provided by Theorem 1 remains valid but only in a subset of $(\lambda_o, \infty)$. This aspect is treated in the next corollary of Theorem 1, see [12].

**Corollary 1** Assume that $A_0 + A_1$ is Hurwitz. For any given positive parameter $\gamma > \|H(\infty,s)\|_{\infty}$ there exist $\lambda_\gamma \geq \lambda_o > 0$ and $0 \leq \tau_\gamma \leq \tau^\ast$ such that

$$\|H(\lambda,s)\|_{\infty} \leq \|T(\tau(s),s)\|_{\infty} < \gamma$$

(12)

holds $\forall \lambda \in (\lambda_\gamma, \infty)$ whenever the time-delay function $\tau(\lambda)$ given by Theorem 1 is continuous in the same interval.

Theorem 1 makes clear that it may not be possible to generate through $\lambda \in (\lambda_o, \infty)$ a lower bound valid for all $\tau \in [0, \tau^\ast)$. However, from Corollary 1 we know that it is possible to determine a sub-interval of $\lambda > 0$ such that the lower and upper bounds $\|H(\lambda,s)\|_{\infty} \leq \|T(\tau(s),s)\|_{\infty} < \gamma$ hold. It is important to keep in mind that the determination of the sub-interval defined by $\lambda_\gamma$ must be done with care due to the eventual occurrence of multiple solutions $\alpha(\lambda)$ to problem (11), which may cause discontinuities on the associated value of the time-delay $\tau(\lambda)$ extracted from the nonlinear relationship provided in Theorem 1. From Corollary 1, a possible numeric procedure to calculate these bounds is as follows:
for each element of a strictly decreasing sequence \( \lambda_k = \{\infty, \cdots, \lambda_0\} \) the time-delay value \( \tau_k = \tau(\lambda_k) \) is computed. The index \( k \) is increased whenever \(-2/\lambda^2 < d\tau(\lambda)/d\lambda < 0 \) at \( \lambda = \lambda_k \) and \( \|T(\tau_k, s)\|_\infty < \gamma \).

When this procedure stops we get \( \lambda_\gamma = \lambda_{k-1} \) and \( \tau_\gamma = \tau_{k-1} \). An important note about this algorithm is that the calculation of \( \|T(\tau_k, s)\|_\infty \) is essential to impose the desired upper bound and to compute the limits \( \lambda_\gamma \) and \( \tau_\gamma \). Moreover, the existence of the derivative \( d\tau(\lambda)/d\lambda < 0 \) implies the continuity and monotonicity of \( \tau(\lambda) \) and avoids its sudden variation with respect to the variation of \( \lambda \). This allows us to identify any unboundedness tendency of \( \|T(\tau(\lambda), s)\|_\infty \) and also the stability margin \( \tau^* \) since this norm is continuous within the entire interval \( (\lambda_\gamma, \infty) \). The constraint \(-2/\lambda^2 < d\tau(\lambda)/d\lambda < 0 \) at \( \lambda = \lambda_k \), inspired by inequality 0 \( \leq \tau(\lambda) < 2/\lambda \), is numerically implemented through the simple test \( 0 < \tau_k - \tau_{k-1} < 2(\lambda_{k-1} - \lambda_k)/\lambda_k^2 \) whose accuracy is controlled by taking \( |\lambda_{k+1} - \lambda_k| \) sufficiently small.

Remark 1 From the above discussion, the first element of the sequence \( \{\lambda_k\} \) can be chosen as \( 2/\varepsilon \), where \( \varepsilon > 0 \) is such that each norm of the finite order systems, namely \( \|H(2/\varepsilon, s)\|_\infty \) and \( \|T(0, s)\|_\infty \), are close to each other, which means that their distance is within some precision defined by the designer. Such an \( \varepsilon > 0 \) satisfying this condition always exists.

3 State Feedback Design

Considering the above discussion we now move our attention to the time delay system

\[
\dot{x}(t) = A_0x(t) + A_1x(t - \tau) + B_0u(t) + E_0w(t) \tag{13}
\]

\[
z(t) = C_0x(t) + C_1x(t - \tau) + D_{zu}u(t) \tag{14}
\]

whose output \( z \in \mathbb{R}^r \) must be controlled by means of a state feedback control law \( u(t) = K_0x(t) + K_1x(t - \tau) \in \mathbb{R}^m \) to be designed. Connecting it to (13)-(14) we get a closed-loop system state space realization of the form of (2)-(3), with transfer function \( T_K(\tau, s) \). Furthermore, adopting the same reasoning as before, the associated rational transfer function \( H_K(\lambda, s) \) is given by

\[
H_K(\lambda, s) = \begin{bmatrix}
A_\lambda + BK & E \\
Cz + D_{zu}K & 0
\end{bmatrix} \tag{15}
\]

where \( B' = [0 \ B'_0] \) and

\[
K = \begin{bmatrix}
K_0 + K_1 & K_0 - K_1
\end{bmatrix} \tag{16}
\]
It is important to observe that the transfer function $H_K(\lambda, s)$ has a closed-loop structure and once the matrix gain $K \in \mathbb{R}^{m \times 2n}$ is determined, it immediately provides the time delay system closed-loop gains $K_0 \in \mathbb{R}^{m \times n}$ and $K_1 \in \mathbb{R}^{m \times n}$. Hence, we turn now our attention to the $\mathcal{H}_\infty$ control design problem consisting on the determination, if any, of a state feedback gain $K \in \mathbb{R}^{m \times 2n}$ such that $\|T_K(\tau, s)\|_\infty < \gamma$ which, from the results of the previous section, is replaced by the determination of $K \in \mathbb{R}^{m \times 2n}$ such that

$$\|H_K(\lambda, s)\|_\infty < \gamma$$

(17)

For a given $\gamma > 0$, under the usual assumptions $D'_{cu}C_z = 0$ and $D'_{cu}D_{cu} = I$, imposed just for simplicity, the state feedback gain is readily determined. Indeed, the so called central solution of (17) is given by (see, for instance, [4] for details) $K = -B'P$ where $P > 0$ satisfies the Riccati inequality

$$A'_\lambda P + PA_\lambda + C'_zC_z - P(BB' - \gamma^{-2}EE')P < 0$$

(18)

Moreover, the observability of $(A_\lambda, C_z)$ and the controllability of $(A_\lambda, B)$ assure that the Riccati equation obtained by replacing the inequality in (18) by equality admits a positive definite stabilizing solution as well.

**Theorem 2** Consider $\gamma > \min_K \|H_K(0, s)\|_\infty$. For $\lambda > 0$ large enough, the central solution of (18) defined by the pair $(K, P)$ is such that $P^{-1}$ has the particular structure

$$P^{-1} = \begin{bmatrix} Y + R & -R \\ -R & R \end{bmatrix}$$

(19)

where matrices $Y \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{n \times n}$ are positive definite. Furthermore, the relations $\|H_K(\infty, s)\|_\infty = \|T_K(0, s)\|_\infty < \gamma$ hold.

**Proof:** Under the assumption $\gamma > \min_K \|H_K(\infty, s)\|_\infty$ (18) admits a positive definite solution $P \in \mathbb{R}^{2n \times 2n}$. Hence, partitioning $P$ in four $n \times n$ matrix blocks such as

$$P^{-1} = \begin{bmatrix} X & U \\ U' & \hat{X} \end{bmatrix} > 0$$

(20)

it is a matter of simple verification that the Riccati inequality (18) multiplied from both sides by $P^{-1}$ can
be rewritten as $\Psi + \lambda \Gamma < 0$ where

$$
\Gamma = \begin{bmatrix}
U + U' & \hat{X} - U \\
\hat{X} - U' & -2\hat{X}
\end{bmatrix}
$$

(21)

and $\Psi \in \mathbb{R}^{2n \times 2n}$ depends on the four blocks of $P$ but not on $\lambda$. In addition, considering the nonsingular matrix

$$
\Sigma = \begin{bmatrix}
I & (1/2)(U - \hat{X})\hat{X}^{-1} \\
0 & I
\end{bmatrix}
$$

(22)

it is verified that the factorisation

$$
\Gamma = \Sigma \begin{bmatrix}
(1/2)(U + \hat{X})\hat{X}^{-1}(U + \hat{X})' & 0 \\
0 & -2\hat{X}
\end{bmatrix} \Sigma'
$$

(23)

holds, and the fact that the matrix placed in the first diagonal block in (23) is semidefinite positive puts in evidence that for the Riccati inequality $\Psi + \lambda \Gamma < 0$ to be satisfied for $\lambda > 0$ large enough two conditions are required. The first condition is that $\hat{X} = -U = R > 0$. Consequently, defining $Y = X - R > 0$ the structure (19) follows. The second one is that the first diagonal block of the matrix $\Sigma^{-1}\Psi\Sigma^{-1} \in \mathbb{R}^{2n \times 2n}$ must be negative definite, that is

$$
(A_0 + A_1)Y + Y(A_0 + A_1)' + Y(C_{z0} + C_{z1})' \times
\times (C_{z0} + C_{z1})Y - (B_0B_0' - \gamma^{-2}E_0E_0') < 0
$$

(24)

Furthermore, determining the state feedback gain $K = -B'P$ with $P^{-1}$ given in (19), from the state feedback gain formula (16) we obtain $K_0 + K_1 = -B_0'Y^{-1}$. Consequently, multiplying both sides of inequality (24) by $Y^{-1}$, we conclude that such a state feedback gain imposes $\|T_k(0,s)\|_\infty < \gamma$. Finally, adopting the same reasoning to get (12) for $H_K(\lambda,s)$, instead of the open loop transfer function $H(\lambda,s)$, the equality $\|H_K(\infty,s)\|_\infty = \|T_k(0,s)\|_\infty$ follows.

Starting from $\lambda > 0$ large enough, the previous results enable us to search for a continuous function $\tau(\lambda)$ and a matrix gain $K$ (depending on $\lambda$) such that $\|H_K(\lambda,s)\|_\infty \leq \|T_k(\tau(\lambda),s)\|_\infty < \gamma$. Moreover, for $\lambda \to \infty$ (with makes $\tau(\lambda) \to 0$) such state feedback gain always exists. These points are common to the $H_\infty$ filter design, since the state feedback synthesis is the dual problem of state estimation treated in [12]. However, it is worth to mention that the control problem does not need a similarity transformation,
as in [12] for the filter design, in order to obtain the controller gains \( K_0 \) and \( K_1 \).

4 Output Feedback Design

In this section we address the main problem of this work, by considering the following time-delay system with minimal realization

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + B_0 u(t) + E_0 w(t) \tag{25}
\]
\[
y(t) = C_{y0} x(t) + C_{y1} x(t - \tau) + D_{yu} w(t) \tag{26}
\]
\[
z(t) = C_{z0} x(t) + C_{z1} x(t - \tau) + D_{zu} u(t) \tag{27}
\]

where, in addition to the assumptions and the variables defined in previous sections, \( y(t) \in \mathbb{R}^p \) is the measured signal. The aim at this point is to design a full order dynamic output feedback controller with the following structure

\[
\dot{\hat{x}}(t) = \hat{A}_0 \hat{x}(t) + \hat{A}_1 \hat{x}(t - \tau) + \hat{B}_0 y(t) \tag{28}
\]
\[
u(t) = \hat{C}_0 \hat{x}(t) + \hat{C}_1 \hat{x}(t - \tau) \tag{29}
\]

where \( \hat{x}(t) \in \mathbb{R}^n \). When connected to (25)-(27) this controller yields the regulated output \( z(t) \) as

\[
\dot{\xi}(t) = F_0 \xi(t) + F_1 \xi(t - \tau) + G_0 w(t) \tag{30}
\]
\[
z(t) = J_0 \xi(t) + J_1 \xi(t - \tau) \tag{31}
\]

where \( \xi(t) = [x(t) \quad \dot{x}(t)]' \in \mathbb{R}^{2n} \) is the state and the indicated matrices stand for

\[
F_0 = \begin{bmatrix} A_0 & B_0 \hat{C}_0 \\ B_0 C_{y0} & \hat{A}_0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} A_1 & B_0 \hat{C}_1 \\ B_0 C_{y1} & \hat{A}_1 \end{bmatrix} \tag{32}
\]

\[
G_0 = [E_0', D_{yu}\hat{B}_0], \quad J_0 = [C_{z0}, D_{zu}\hat{C}_0] \quad \text{and} \quad J_1 = [C_{z1}, D_{zu}\hat{C}_1].\] The transfer function from the external input \( w(t) \) to the controlled output \( z(t) \) is similar to the estimation error transfer function calculated in [12] and is given by

\[
T_C(\tau, s) = \left( J_0 + J_1 e^{-s\tau} \right) \left( sI - F_0 - F_1 e^{-s\tau} \right)^{-1} G_0 \tag{33}
\]
where the subindex “C” indicates its dependence on a given controller of the form (28)-(29). Hence, the goal is to design a controller such that \( \|T_C(\tau, s)\|_\infty < \gamma \) for a given \( \gamma > 0 \), which is accomplished by the definition of the following 4n-th order rational comparison system (see [12] for details)

\[
H_C(\lambda, s) = \begin{bmatrix} F_\lambda & G \\ J & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda I & 0 \\ F_0 + F_1 & F_0 - F_1 - \lambda I & G_0 \\ J_0 + J_1 & J_0 - J_1 & 0 \end{bmatrix}.
\]  

Then, solve the corresponding \( \mathcal{H}_\infty \) output feedback design problem for each \( \lambda > 0 \) and extract the corresponding time-delay \( \tau(\lambda) \) as indicated in Corollary 1. Notice that, even though the matrices of the state space realization of \( H_C(\lambda, s) \) depend on an intricate manner of the control state space realization matrices, applying the similarity transformation used in [12]

\[
S = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}
\]  

one can rewrite (34) in the equivalent form

\[
H_C(\lambda, s) = \begin{bmatrix} SF_\lambda S^{-1} & SG \\ JS^{-1} & 0 \end{bmatrix} = \begin{bmatrix} A_\lambda & B\hat{C} \\ \hat{B}C_y & \hat{A}_\lambda \end{bmatrix} \begin{bmatrix} E \\ \hat{B}D_{yw} \end{bmatrix} \begin{bmatrix} C_z \\ D_{zw}\hat{C} \end{bmatrix} 0
\]  

where the system matrices \( (A_\lambda, E, C_z) \) have been defined in (5), \( B' = [0, B'_0], C_y = [C_{y0} + C_{y1}, C_{y0} - C_{y1}] \) and the controller matrices are given by

\[
\hat{A}_\lambda = \begin{bmatrix} 0 & \lambda I \\ \hat{A}_0 + \hat{A}_1 & \hat{A}_0 - \hat{A}_1 - \lambda I \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 \\ \hat{B}_0 \end{bmatrix}
\]
and \( \hat{C} = [\hat{C}_0 + \hat{C}_1, \hat{C}_0 - \hat{C}_1] \), indicating that they are in the comparison form. Hence, the controller (28)-(29) whenever connected to the time-delay system (25)-(27) produces an LTI comparison system associated to the regulated output (30)-(31) whose transfer function can be alternatively determined from the connection of the LTI comparison system of the system (25)-(27) and the LTI comparison system of the controller (28)-(29).

Once matrices given in (37) have a particular structure, as it occurred in the filter design [12], we adopt the same strategy proposed in [12] such that controller matrices \((\hat{A}_\lambda, \hat{B}, \hat{C})\) will be replaced by general matrix variables \((A_C, B_C, C_C)\). These two realizations are coupled by a nonsingular matrix \(V \in \mathbb{R}^{2n \times 2n}\) which defines the similarity transformation \((\hat{A}_\lambda, \hat{B}, \hat{C}) = (VA_CV^{-1}, VB_C, C_CV^{-1})\). Once we have the controller matrices at hand, it is a simple matter of computation to determine whether \(\|T_C(\tau(\lambda), s)\|_\infty < \gamma\) holds.

For a given \(\gamma > 0\), under the usual assumptions \(D_y'C_yz = 0, ED_y'D_yw = I\) and \(D_w'D_wz = I\), imposed just for simplicity, there exists a full order controller with realization \((A_C, B_C, C_C)\) such that \(\|H_C(\lambda, s)\|_\infty < \gamma\) if and only if there exist stabilizing matrices \(P = P' > 0\) and \(\Pi = \Pi' > 0\) satisfying

\[
A_\lambda \Pi + \Pi A_\lambda' + EE' - \Pi (C'_yC_y - \gamma^{-2} C'_zC_z) \Pi = 0
\]  
(38)

\[
A_\lambda' P + PA_\lambda + C'_zC_z - P (BB' - \gamma^{-2} EE') P < 0
\]  
(39)

and the spectral radius constraint \(r_s(\Pi \Pi) < \gamma^2\). In the affirmative case, the desired controller has the state space realization defined by matrices in (57) and (58) (see Appendix A and references [4] and [26] for details). Notice that the observer-based controller for the comparison system is obtained by solving a Riccati equality (38) and a Riccati inequality (39).

**Lemma 1** For \(\lambda > 0\) large enough, the stabilizing positive definite solution of (38) and any positive definite feasible solution of (39) exhibit the structures

\[
\Pi = \begin{bmatrix}
Z & \lambda^{-1} Q \\
\lambda^{-1} Q' & \lambda^{-1} W
\end{bmatrix}, \quad P^{-1} = \begin{bmatrix}
Y + R & -R \\
-R & R
\end{bmatrix}
\]  
(40)

where \(Z > 0, W > 0, Q, Y > 0\) and \(R > 0\) are \(n \times n\) matrices.

**Proof:** See Appendix B.

This lemma is used in the proof of the next theorem where, similarly to the state feedback case, a central result valid for \(\lambda \to \infty\) (the behavior at infinity) is established. It states that at infinity the
close-loop delay system and the associated comparison system have transfer functions with equal $\mathcal{H}_\infty$ norms.

**Theorem 3** Consider $\gamma > \min \|H_C(\infty, s)\|_\infty$. For $\lambda > 0$ large enough the relations $\|H_C(\infty, s)\|_\infty = \|T_C(0, s)\|_\infty < \gamma$ hold.

**Proof:** For matrices $\Pi > 0$ and $P > 0$ with the structures (40) and satisfying $r_s(\Pi P) < \gamma^2$ we determine the gains $B_C$ and $C_C$ from (57). The first gain can be partitioned as $B_C = [B'_C, 0]'$ where $B'_C = -Z(C_{y0} + C_{y1})'$, while the second one can be written as $C_C = [C_{c0}, C_{c0} + B'_C R^{-1}]$ where $C_{c0} = B'_C Y^{-1}(I - \gamma^{-2} Z Y^{-1})^{-1}$. For $\lambda \to \infty$ and $|s|$ finite, the central controller transfer function can be written as $C_\infty(s) = C_C(sI - A_C)^{-1}B_C = C_C(I - \Xi(B_C C_y + \gamma^{-2} \Pi C(z)))^{-1} \Xi B_C$, where

$$\Xi \approx \begin{bmatrix} (sI - (A_0 + A_1) + B_0 C_{c0})^{-1} & I \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda^{-1} N A_C \end{bmatrix} (41)$$

Defining $A_C = A_0 + A_1 - B_0 C_{c0} + B_0 (C_{y0} + C_{y1})' + \gamma^{-2} Z(C_{y0} + C_{z1})' (C_{z0} + C_{z1})$, we conclude that $C_\infty(s) = C_{c0}(sI - A_C)^{-1}B_{c0}$, which guarantees $\|T_C(0, s)\|_\infty < \gamma$ since, from Lemma 1, the spectral radius condition $r_s(\Pi P) < \gamma^2$ reduces to $r_s(Z Y^{-1}) < \gamma^2$, for $\lambda > 0$ large enough. Indeed, $C_\infty(s)$ is the central controller associated to the time-delay system when $\tau = 0$. As a consequence $\|H_C(\infty, s)\|_\infty = \|T_C(0, s)\|_\infty < \gamma$. \qed

As in the filter design problem, an important point is how to obtain a suitable similarity transformation $V \in \mathbb{R}^{2n \times 2n}$ that guarantees closed-loop stability and $\mathcal{H}_\infty$ performance. As discussed in [12], $V$ does not define a similarity transformation for the time-delay system and thus must be computed with care.

**Lemma 2** Assume that $\dim(y) = p \leq n = \dim(x)$, $\lambda > 0$ and the non-singular matrix

$$V = \begin{bmatrix} N \\ \lambda^{-1} N A_C \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \quad (42)$$

where $N' \in \mathbb{R}^{2n \times n}$ belongs to the null space of $B'_C$. Under these conditions the equality $(\hat{A}_\lambda, \hat{B}, \hat{C}) = (V A_C V^{-1}, V B_C, C_C V^{-1})$ holds.

**Proof:** See [12]. \qed

Partitioning matrix $N = [N_1 \ N_2]$, where $N_1$ is assumed to be non-singular, we have that the central controller obtained from Theorem 3 when $\lambda \to \infty$, after applying the similarity transformation defined
\[ C_\infty(s) = \begin{bmatrix} N_1A_{C0}N_1^{-1} & N_1B_{C0} \\ C_{C0}N_1^{-1} & 0 \end{bmatrix} \] \tag{43}

This fact indicates, as expected, that the similarity transformation does not affect the controller transfer function when \( \tau = 0 \), and consequently \( \|H_C(\infty, s)\|_\infty = \|T_C(0, s)\|_\infty \) for any nonsingular \( V \in \mathbb{R}^{2n \times 2n} \).

From the above discussion we are able to extend the algorithm proposed to calculate the \( H_\infty \) norm of a time-delay system to obtain intervals \( \lambda \in (\lambda_\gamma, \infty) \) and \( \tau(\lambda) \in [0, \tau_\gamma) \) assuring the existence of a controller with realization (28)-(29) for each pair \((\lambda, \tau(\lambda))\) that guarantees \( \|H_C(\lambda, s)\|_\infty \leq \|T_C(\tau(\lambda), s)\|_\infty < \gamma \). At each iteration \( k \) we must calculate not only the time delay \( \tau_k = \tau(\lambda_k) \), but also the central controller \((A_{Ck}, B_{Ck}, C_{Ck})\) and the similarity transformation matrix \( V_k \). In order to assure the continuity of \( \|T_C(\tau(\lambda), s)\|_\infty \) in the considered interval, what enables us to detect any unboundedness tendency, we must compute matrix \( N_k \) in (42) with care. As in [12] we have chosen \( N_k' \) as the first \( n \) column vectors provided by the Matlab null space routine applied to \( B_{Ck} \). Furthermore, at \( \lambda_k \) we propose to verify the continuity by evaluating the norm condition \( \|N_k - N_{k-1}\| \leq \varepsilon \), with \( \varepsilon > 0 \) sufficiently small, what enable us to circumvent the difficulty of generating a continuous null space basis for matrix \( B_C \) [2]. A similar reasoning can be adopted to design state feedback gains \( K_0k = K_0(\lambda_k) \) and \( K_{1k} = K_1(\lambda_k) \) for each pair \((\lambda_k, \tau(\lambda_k))\). However, notice that in this case the design procedure is simpler, since it is not necessary to determine a similarity transformation, even though the continuity of \( \tau(\lambda) \) at \( \lambda = \lambda_k \) must still be verified as indicated in Section 2.2.

### 5 Examples

To illustrate the state feedback design we consider a second order example borrowed from [8] where the matrices corresponding to the state space realization (13)-(14) are as follows

---

Figure 1: \( H_\infty \) performance versus time delay for \( \gamma = 1 \).
Figure 2: $\mathcal{H}_\infty$ performance versus time delay for $\gamma = 1$.

\[
\begin{bmatrix}
A_0 & A_1 & E_0 & B_0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & -0.9 & 1 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
C_{y0} & C_{y1} & D_{yw}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.1 \\
\end{bmatrix}
\]

Our main purpose with this simple example is to put in evidence the importance of the gain $K_1 \neq 0$ for performance improvement by comparing our results with those in [8] where a state feedback control of the form $u(t) = K_0x(t)$ is designed. Setting $\gamma = 0.13$, for the minimum value $\lambda_\gamma = 1.0198$ we have calculated $\tau_\gamma = \tau(\lambda_\gamma) = 1.2122$ [s], the feedback gains $K_0 = [-0.0060 \ - 18.1024]$, $K_1 = [0.0053 \ 2.2108]$ and the norms $\|T_K(\tau_\gamma, s)\|_\infty = 0.1218$ and $\|H_K(\lambda_\gamma, s)\|_\infty = 0.1209$. In [8], for approximatively the same value of $\gamma$ and $\tau = 0.999$ the gain $K_0$ given is a high gain (of order $10^6$). Figure 1 gives in dashed line the lower bound $\|H_C(\lambda, s)\|_\infty$ and in solid line the exact value $\|T_C(\tau(\lambda), s)\|_\infty$ for $\gamma = 1$. Both coincides for all $\tau(\lambda) \in [0 \ 1.5673)$. It is interesting to verify that, in [8] for $\gamma > 0$ big enough (since only asymptotic stability is concerned) the maximum value assuring the existence of a state feedback stabilizing controller is $\tau_{\text{max}} = 1.28$. Meanwhile, for $\gamma = 1$ and $\lambda = 1.0023$ our design procedure provides the controller gains $K_0 = [0.6169 \ - 10.5939]$ and $K_1 = [0.3615 \ 1.2740]$ which are stabilizing for $\tau(\lambda) = 1.5672$.

Let us now consider the same second order example in [8] for illustration and comparison of the output feedback control design. The time-delay system (25)-(27) matrices are defined as before\(^1\) and

\[
\begin{bmatrix}
C_{y0} & C_{y1} & D_{yw}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

For this system we have applied the proposed algorithm to generate a sequence of stabilizing controllers.

\(^1\)In this case, a simple change of variables must be performed in order to get a model satisfying $D'_{yw}D_{yw} = I$ and $D_{yw}D'_{zw} = I$. 

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for each pair \((\lambda_k, \tau(\lambda_k))\) such that \(\lambda_k \in (\lambda, \infty)\) and \(\tau(\lambda_k) \in [0, \tau_\gamma)\). For the upper bound \(\gamma = 1\) we have determined \(\lambda_\gamma = 1.0100\) and \(\tau_\gamma = 1.2477\) \([s]\). In Figure 2 we present in dashed line the lower bound \(\|H_C(\lambda, s)\|_\infty\) and in solid line the exact value \(\|T_C(\tau(\lambda), s)\|_\infty\) when imposing \(\gamma = 1\). It is interesting to note that for \(\tau(\lambda) \in [0, 0.5)\) the values of the lower bound and the true value of \(\|T_C(\tau(\lambda), s)\|_\infty\) are identical and we verify that, for the remaining interval, the maximum difference between them is about 3.2%.

For comparison purposes, the point mark in \(\|T_C(\tau(\lambda), s)\|_\infty\) curve, in Figure 2, refers to the time delay \(\tau = 0.9990\) \([s]\), for \(\lambda = 1.40438\), and the corresponding norm \(\|T_C(0.9990, s)\|_\infty = 0.2731\), which is 68% smaller than the \(\mathcal{H}_\infty\) norm obtained by [8]. In the same figure we point out the lower bound \(\|H_C(1.40438, s)\|_\infty = 0.2681\), which is only 1.81% smaller than the true norm value. Also for \(\tau = 0.9990\) \([s]\) we have determined the controller matrices given as follows:

\[
\begin{bmatrix}
\hat{A}_0 & \hat{A}_1 \\
\end{bmatrix} = \begin{bmatrix}
-28.6072 & 1.4110 \\
-76.1020 & 3.8891 \\
3.6807 & -2.4378 \\
11.2365 & -7.4419 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{B}_0 & \hat{C}_0^T \\
\hat{C}_1 \\
\end{bmatrix} = \begin{bmatrix}
15.0420 & -10.5733 & 2.2117 \\
36.8268 & 0.4678 & -0.9181 \\
\end{bmatrix}
\]

As it can be verified, this time-delay controller makes the closed-loop system asymptotically stable with the transfer function \(\mathcal{H}_\infty\) norm previously calculated.

### 6 Practical Application - Networked Control

Networked control systems (NCS) have received a great amount of attention in recent years. The main feature of NCS is that measurement and control actions are supported by a communication network. As a result, the control design has to take into account some phenomena that can deteriorate the final
performance and reduce the stability margin. Among them, it is worth mention bandwidth limitations, packet dropout, and delay. See the survey paper [11] and the papers [25, 24] for interesting and useful discussions on the main relevant aspects of this class of control systems design. Presently, we focus our attention to the effect of network-induced delay in NCS control, exclusively, see [25]. That is, it is further supposed that the network is ideal as far as the other mentioned characteristics are concerned. In addition, since depending on the medium access control (MAC) protocol of the control network, network-induced delay can be constant, time varying, or even random [25], we also make the assumption that the time-delay is deterministic and constant. Hence, using the previous results, our main purpose is to design a dynamic output feedback controller that exhibits both good stability margin and small performance deterioration in terms of maximum delay and $\mathcal{H}_\infty$-norm cost.

In this framework, we consider a system with 4 propellers mounted on a three degrees of freedom (3-DOF) pivot, as depicted in Figure 3, [17]. Each pair of diametrically opposed propellers generate lift forces that control the pitch $p$ and roll $r$ angles, while the total torque causes a yaw $y$ to the body as well, as discussed in [17]. The lift forces indicated in Figure 3 are proportional to the voltages applied to the motors that command the propellers. Also, the angular displacements are measured by encoders placed in the three rotation axis of the body. It is important to mention that to change the measurement frame to the body axis, instead of the encoder axis, it must be performed a basis transformation by means of a nonsingular matrix $T_{\text{meas}}$ defined in [17]. Considering the data in [17], we define the state vector $x = [p \ r \ y \ \dot{p} \ \dot{r} \ \dot{y}]'$ and the control input vector $u = [V_f \ V_b \ V_r \ V_l]'$, defined by the voltages applied to the propellers. Moreover, the vector of external disturbances $w$ belongs to $\mathbb{R}^7$, whose first 4 elements correspond to control signal noise, and its last 3 components are related to the measurement noise. Hence, the state space realization (25)-(27) is characterized by

$$A_0 = \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix}, \quad A_1 = 0, \quad B_0 = \begin{bmatrix} 0 \\ B_{0l} \end{bmatrix}, \quad E_0 = \begin{bmatrix} 0 & 0 \\ B_{0l} & 0 \end{bmatrix}$$

where

$$B_{0l} = \begin{bmatrix} 2.1188 & -2.1188 & 0 & 0 \\ 0 & 0 & 2.1188 & -2.1188 \\ -0.0978 & -0.0978 & 0.0978 & 0.0978 \end{bmatrix}$$

We assume that we measure the angular positions $p$, $r$ and $y$, and inspired in [20] that are sent to the controller through a network which adds a total delay $\tau > 0$ to the signals. The delayed signals are to be processed by the controller (28)-(29) in order to provide the control signal $u$. Under this context, our
goal is to control the angular positions \( p, r \) and \( y \) subjected to a \( \mathcal{H}_\infty \) norm level \( \gamma = 5 \). Thus we define the output matrices

\[
C_{y0} = 0, \quad C_{y1} = \begin{bmatrix} I_3 & 0 \end{bmatrix}, \quad D_{yw} = \begin{bmatrix} 0 & 0.1I_3 \end{bmatrix}
\]

\[
C_{zo} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{z1} = 0, \quad D_{zu} = \begin{bmatrix} 0 \\ I_4 \end{bmatrix}
\]

In order to evaluate the robustness of the controller designed for \( \tau = 0 \) we consider the autonomous system \((w = 0)\) with initial condition \( x(0) = [\pi/6 \, \pi/4 \, \pi/3 \, 0 \, 0 \, 0] \) [rad] measured in the encoder axis frame. Fig. 4 depicts the time simulation for a time delay value \( \tau = 0.34 \) [s]. The continuous line denotes the pitch angle \( p \), the dashed line denotes the raw angle \( r \) and the dot-dashed line denotes the yaw angle \( y \) measured, in degrees, in the body axis frame. The time delay \( \tau = 0.34 \) [s] corresponds to the stability threshold, and it can be verified that although the stability is guaranteed, the \( \mathcal{H}_\infty \) norm level is not preserved. On the other hand, Fig. 5 shows again the time simulation for the position angles behavior based on the same assumptions as before, but for a controller properly designed to cope with
Figure 6: Time response for $\tau = 0.5960$ [s].

the time delay $\tau = 0.34$ [s], assuring the pre-specified norm level. It is clear that the performance of the second controller is much better than in the first one, and this fact agrees with the discussion that in a networked control framework, neglecting the existence of time-delays may lead to poor performance and, even worse, lead the closed-loop system to instability.

Applying the algorithm proposed in Section 4, we have determined $\lambda_\gamma = 3.3229$ and the corresponding $\tau_\gamma = 0.5960$ [s]. For this maximum time delay, we evaluated the controller performance by considering $x(0) = 0$ and exciting the system with a decaying sine wave in the first exogenous input channel $w_1(t) = 0.5e^{-0.2t}\sin(\omega_T t)$ [V], where $\omega_T = 2.4643$ [rad/s] is the frequency associated to the $\mathcal{H}_\infty$ norm of the closed-loop system. As it can be verified, the control action is efficient in reducing the influence of the disturbance in the output variables in an acceptable interval of time.

7 Conclusions

In this paper we have proposed a new procedure for time-delay control design. It is based on what we call comparison system which is an LTI system with order twice the number of state variables of the time-delay system. Moreover, under some weak limitations, it is well adapted not only for stability analysis but also for linear control design of time-delay dynamic systems. The most important feature of the comparison system is that it makes possible the control design by manipulating finite order LTI systems, exclusively. As a consequence, the classical routines for control synthesis can be applied, opening the possibility to handle time-delay systems with high number of state variables. Examples borrowed from the literature illustrate the theory and a practical application in the framework of networked controls systems, discuss the control limitations imposed by delay effects. Some points are still open and deserve future research efforts, as for instance the generalization of the procedure for delay-independent, robust control design and neutral systems.
**A Parameterization of $\mathcal{H}_\infty$ controllers**

In this appendix we discuss the $\mathcal{H}_\infty$ output feedback design when dealing with Riccati inequalities, instead of equalities. To this end, consider the 2nth order LTI system

\[
\dot{x}(t) = A\lambda x(t) + Bu(t) + Ew(t) \tag{44}
\]
\[
y(t) = C_y x(t) + D_{yw} w(t) \tag{45}
\]
\[
z(t) = C_z x(t) + D_{zu} u(t) \tag{46}
\]

and the complete order controller to be designed

\[
\dot{x}_c(t) = A_C x(t) + B_C y(t) \tag{47}
\]
\[
u(t) = C_C x_c(t) \tag{48}
\]

such that the closed-loop system, with state vector $\xi(t) = [x(t)' \ x_c'(t)']' \in \mathbb{R}^{4n}$, has its dynamics governed by

\[
\dot{\xi}(t) = \tilde{A} \xi(t) + \tilde{B} w(t) \tag{49}
\]
\[
z(t) = \tilde{C} \xi(t) \tag{50}
\]

where

\[
\begin{bmatrix}
\tilde{A} & \tilde{B} & \tilde{C}' \\
A\lambda & B C_y & E \\
B_C C_y & A_C & B_C D_{yw} \\
& & C_z'C_{zu}'
\end{bmatrix} \tag{51}
\]

Then, the closed-loop satisfies an $\mathcal{H}_\infty$ performance level $\gamma > 0$ if there exist $\tilde{P} > 0 \in \mathbb{R}^{2n}$ such that

\[
\tilde{A}'\tilde{P} + \tilde{P}\tilde{A} + \tilde{C}'\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}\tilde{B}'\tilde{P} < 0 \tag{52}
\]

Considering the classical partitioning

\[
\tilde{P} = \begin{bmatrix} X & U \\ U' & \tilde{X} \end{bmatrix}, \quad \tilde{P}^{-1} = \begin{bmatrix} Y & V \\ V' & \tilde{Y} \end{bmatrix}, \quad \tilde{\Gamma} = \begin{bmatrix} Y & I \\ V' & 0 \end{bmatrix} \tag{53}
\]
and multiplying (52) by $\Gamma'$ on the left and by its transpose on the right, we obtain the following inequalities

$$A_{\lambda}Y + YA'_{\lambda} + YC'C_Y + \gamma^{-2}EE' +$$

$$+ BL + L'B' + L'L < 0 \quad (54)$$

$$XA_{\lambda} + A'_{\lambda}X + \gamma^{-2}XX'E'X + C'C_Y +$$

$$+ FC_Y + C'Y' + \gamma^{-2}FF' < 0 \quad (55)$$

where $L = C'C_Y$ and $F = UB_C$. Moreover, in order to (54) and (55) be equivalent to (52) the following equality must also hold

$$XA_{\lambda}Y + FC_Y + XBL + UA_{\lambda}V' + A'_{\lambda} +$$

$$+ C'C_Y + \gamma^{-2}XEE' = 0 \quad (56)$$

From (54) and (55) we determine the gains $L = -B'$ and $F = -\gamma^2C'$, corresponding to the central controller parameterization. Considering these values, we multiply (54) and (55) on both sides by $Y^{-1}$ and $\gamma^{-2}X^{-1}$, respectively, and define $P = Y^{-1}$ and $\Pi = \gamma^2X^{-1}$. As an immediate consequence, (39) is obtained from (54). Since the equality $XY + UV' = I$ must hold, we can fix $U = X = \gamma^2\Pi^{-1}$ and calculate

$$B_C = -\Pi C_Y$$

$$C_C = B'P(I - \gamma^{-2}\Pi P)^{-1} \quad (57)$$

On the other hand, with no loss of generality, adopting a slight perturbation consider that $EE' > 0$ and determine a positive definite solution $\Pi = \gamma^2X^{-1}$ lying in the border of inequality (55) by replacing it by the correspondent Riccati equation. This procedure together with the proposed change of variables yield (38) and then, from (56) and (38), we can recover after some tedious calculations the controller matrix

$$A_C = A_{\lambda} + \gamma^{-2}\Pi C_Y' + B_C C_Y - BC_C \quad (58)$$

Moreover, the condition $\hat{P} > 0$ is equivalent to $X > Y^{-1} > 0$, which from the above change of variables reduces to $\Pi > 0$, $P > 0$ and $\gamma^2\Pi^{-1} > P > 0$. 

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B Behavior at infinity

In this appendix we determine the behavior of positive definite matrices $\Pi$ and $P$ stated in Lemma 1, for $\lambda \to \infty$. First, to evaluate the behavior of $\Pi$ with respect to $\lambda$, let us consider the matrix function

$$\Phi_\lambda = A_\lambda \Pi + \Pi A'_\lambda + E E' - \Pi (C'_y C_y - \gamma^{-2} C'_z C_z) \Pi$$

where $\Pi$ is given in (40). Calculating $\lim_{\lambda \to \infty} \Phi_\lambda = \Phi_\infty$ we obtain the following matrix blocks identities:

$$\Phi_{\infty}^{11} = Q + Q' - Z (C_{y0} + C_{y1})' (C_{y0} + C_{y1}) Z +$$
$$+ \gamma^{-2} Z (C_{z0} + C_{z1})' (C_{z0} + C_{z1}) Z$$

$$\Phi_{\infty}^{12} = W + Z (A_0 + A_1)' - Q$$

$$\Phi_{\infty}^{22} = -2W + E_0 E_0'$$

Assuming $E_0 E_0' > 0$, otherwise perturb it slightly, the condition $\Phi_\infty = 0$ is satisfied whenever $Z > 0$ is the stabilizing solution of the Riccati equation

$$(A_0 + A_1) Z + Z (A_0 + A_1)' + E_0 E_0' - Z ((C_{y0} + C_{y1})' \times$$
$$\times (C_{y0} + C_{y1}) - \gamma^{-2} (C_{z0} + C_{z1})' (C_{z0} + C_{z1})) Z = 0$$

Moreover, this solution is possible since the condition $\Pi > 0$ for $\lambda \to \infty$ is equivalent to $Z > 0$ and $W > 0$. Hence, the first part of Lemma 1 follows.

For the second part, considering any $P > 0$ feasible for inequality (39), multiplying it by $P^{-1}$ on both sides we can apply the same procedure as in Theorem 2, which yields the conclusion that (39) is satisfied for $\lambda \to \infty$ if and only if $P^{-1}$ has the structure given in (40) with $Y > 0$ satisfying

$$(A_0 + A_1)' Y^{-1} + Y^{-1} (A_0 + A_1) - Y^{-1} (B_0 B_0') -$$
$$- \gamma^{-2} E_0 E_0' Y^{-1} + (C_{z0} + C_{z1})' (C_{z0} + C_{z1}) < 0$$

and $R > 0$ arbitrary.
References


