The Hamiltonian properties of supergrid graphs

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Abstract
In this paper, we first introduce a novel class of graphs, namely supergrid. Supergrid graphs include grid graphs and triangular grid graphs as their subgraphs. The Hamiltonian cycle and path problems for grid graphs and triangular grid graphs were known to be NP-complete. However, they are unknown for supergrid graphs. The Hamiltonian cycle (path) problem on supergrid graphs can be applied to control the stitching traces of computerized sewing machines. In this paper, we will prove that the Hamiltonian cycle problem for supergrid graphs is NP-complete. It is easily derived from the Hamiltonian cycle result that the Hamiltonian path problem on supergrid graphs is also NP-complete. We then show that two subclasses of supergrid graphs, including rectangular (parallelism) and alphabet, always contain Hamiltonian cycles.

Keywords: Hamiltonian properties, supergrid graph, rectangular supergrid graph, alphabet supergrid graph, grid graph, triangular grid graph, computerized sewing machine

1. Introduction

A Hamiltonian cycle in a graph is a simple cycle in which each vertex of the graph appears exactly once. A Hamiltonian path in a graph is a simple path with the same property. The Hamiltonian cycle (resp., path) problem involves testing whether or not a graph contains a Hamiltonian cycle (resp., path). A graph is said to be Hamiltonian if it contains a Hamiltonian cycle. The Hamiltonian problems include Hamiltonian cycle and Hamiltonian path problems.

They have numerous applications in different areas, including establishing transport routes, production launching, the on-line optimization of flexible manufacturing systems [1], computing the perceptual boundaries of dot patterns [30], pattern recognition [2, 31, 34], and DNA physical mapping [14]. It is well known that the Hamiltonian problems are NP-complete for general graphs [10, 20]. The same holds true for bipartite graphs [23], split graphs [11], circle graphs [8], undirected path graphs [3], grid graphs [19], and triangular grid graphs [12]. In this paper, we will study the Hamiltonian problems on supergrid graphs which contain grid graphs and triangular grid graphs as subgraphs.

The two-dimensional integer grid $G^\infty$ is an infinite graph whose vertex set consists of all points of the Euclidean plane with integer coordinates and in which two vertices are adjacent if and only if the (Euclidean) distance between them is equal to 1. A grid graph is a finite, vertex-induced subgraph of $G^\infty$. For a node $v$ in the plane with integer coordinates, let $v_x$ and $v_y$ represent the $x$ and $y$ coordinates of node $v$, respectively, denoted by $v = (v_x, v_y)$. If $v$ is a vertex in a grid graph, then its possible adjacent vertices include $(v_x, v_y - 1), (v_x - 1, v_y), (v_x + 1, v_y),$ and $(v_x, v_y + 1)$. For example, Fig. 1(a) depicts a fragment of graph $G^\infty$ and Fig. 2(a) shows a grid graph. Recently, the properties of triangular grid graphs, which contain grid graphs as subgraphs, have received much attention. The two-dimensional triangular grid $T^\infty$ is an infinite graph obtained from $G^\infty$ by adding all edges on the lines traced from up-left to down-right. A triangular grid graph is a finite, vertex-induced subgraph of $T^\infty$. The possible adjacent vertices of a vertex $v = (v_x, v_y)$ in a triangular grid graph contain $(v_x, v_y - 1), (v_x - 1, v_y), (v_x + 1, v_y), (v_x, v_y + 1), (v_x - 1, v_y - 1),$ and $(v_x + 1, v_y + 1)$. For instance, Fig. 1(b) depicts a fragment of graph $T^\infty$ and Fig. 2(b) shows a triangular grid.

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Fig. 1: A fragment of infinite graph (a) \(G^\infty\), (b) \(T^\infty\), and (c) \(S^\infty\).

Fig. 2: (a) A grid graph, (b) a triangular grid graph, and (c) a supergrid graph, where solid lines indicate the edges of the graphs.

graph. Note that \(T^\infty\) is isomorphic to the original infinite triangular grid graph in the literature [12] but these graphs are different when considered as geometric graphs. By the same construction of triangular grid graphs from grid graphs, we propose a new class of graphs, namely supergrid graphs, as follows. The two-dimensional supergrid \(S^\infty\) is an infinite graph obtained from \(T^\infty\) by adding all edges on the lines traced from up-right to down-left. A supergrid graph is a finite, vertex-induced subgraph of \(S^\infty\). The possible adjacent vertices of a vertex \(v = (v_x, v_y)\) in a supergrid graph contain \((v_x, v_y - 1), (v_x - 1, v_y), (v_x + 1, v_y), (v_x, v_y - 1), (v_x + 1, v_y + 1), (v_x + 1, v_y - 1),\) and \((v_x - 1, v_y + 1)\). Thus, supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. For example, Fig. 1(c) depicts a fragment of graph \(S^\infty\) and Fig. 2(c) shows a supergrid graph. Obviously, all grid graphs are bipartite [19] but triangular grid graphs and supergrid graphs are not bipartite. The Hamiltonian cycle and path problems on grid graphs and triangular grid graphs have been shown to be NP-complete [12, 19]. However, they are unknown for supergrid graphs. In this paper, we will prove that the Hamiltonian cycle and path problems for supergrid graphs are NP-complete.

The possible application for the Hamiltonian cycle (path) problem on supergrid graphs is presented as follows. Consider a computerized sewing machine given an image. The computerized sewing software is used to compute the sewing traces of a computerized sewing machine. There may be two parts in a computerized sewing software. The first part is to do image processing for the input image, e.g. reduce order of colors and image thinning. It then produces some sets of lattices in which every set of lattices represents a color in the input image for sewing. The second part is given by a set of lattices and then computes a cycle (path) to visit the lattices of the set such that each lattice is visited exactly once. Finally, the software transmits the stitching trace of the computed cycle (path) to the computerized sewing machine, and the machine then performs the sewing work along the trace on the object, e.g. clothes. For example, given an image in Fig. 3(a), the software first analyzes the image and then produces seven colors of regions in which each region is filled with the same color and may consist of some disconnected blocks, as shown in Fig. 3(b). It then produces seven sets of lattices in which every set of lattices represents a region, where each region is filled by a sewing trace with the same color and it may be partitioned into many non-contiguous blocks. Fig.
Fig. 3: (a) An input image for the computerized sewing software, (b) seven colors of regions produced by image processing, (c) a set of lattices for one region of color, (d) a possible sewing trace for the set of lattices in (c), and (e) an overview after computing sewing traces of all regions of colors.

3(c) shows a set of lattices for one region of color, and the software then computes a sewing trace for the set of lattices, as depicted in Fig. 3(d). Since each stitch position of a sewing machine can be moved to its eight neighbor positions (left, right, up, down, up-left, up-right, down-left, and down-right), one set of lattices forms a supergrid graph which may be disconnected. Note that each lattice will be represented by a vertex of a supergrid graph, each region may be separated into many blocks in which each block represented a connected supergrid graph. The desired sewing trace of each set of adjacent lattices is the Hamiltonian cycle (path) of the corresponding connected supergrid graph when it is Hamiltonian. Note that if the corresponding supergrid graph is not Hamiltonian, then the sewing trace contains more than one paths and these paths must be concatenated. After computing the sewing traces of all regions of colors, the software then transmits the computed stitching trace to the computerized sewing machine. Fig. 3(e) depicts the possible sewing result for the image in Fig. 3(a). In addition, the structure of supergrid graphs can be used to design the network topology, and its network diameter is smaller than that of grid graphs.

Related areas of investigation are summarized as follows. Itai et al. [19] showed that the Hamiltonian cycle and Hamiltonian path problems for grid graphs are NP-complete. They also gave the necessary and sufficient conditions for a rectangular grid graph having a Hamiltonian path between two given vertices. Zamfirescu et al. [35] gave the
sufficient conditions for a grid graph having a Hamiltonian cycle, and proved that all grid graphs of positive width have Hamiltonian line graphs. Later, Chen et al. [6] improved the Hamiltonian path algorithm of [19] on rectangular grid graphs and presented a parallel algorithm for the Hamiltonian path problem with two given endpoints in rectangular grid graph (mesh). Also there is a polynomial-time algorithm for finding Hamiltonian cycles in solid grid graphs [25]. In [33], Salman introduced alphabet grid graphs and determined classes of alphabet grid graphs which contain Hamiltonian cycles. Keshavarz-Kohjerdi and Bagheri [21] gave the necessary and sufficient conditions for the existence of Hamiltonian paths in alphabet grid graphs, and presented linear-time algorithms for finding Hamiltonian paths with two given endpoints in these graphs. Recently, Keshavarz-Kohjerdi et al. [22] presented a linear-time algorithm for computing the longest path between two given vertices in rectangular grid graphs. The Hamiltonian cycle (path) on triangular grid graphs has been shown to be NP-complete [12]. Recently, Reay and Zamfirescu [32] proved that all 2-connected, linear-convex triangular grid graphs except one special case contain Hamiltonian cycles. They also proved that all connected, locally connected triangular grid graphs (with one exception) contain Hamiltonian cycles.

In addition, the Hamiltonian cycle problem on hexagonal grid graphs has been shown to be NP-complete [12]. Recently, Reay and Zamfirescu [32] proved that all connected, locally connected triangular grid graphs (with one exception) contain Hamiltonian cycles. They also proved that all 2-connected, linear-convex triangular grid graphs except one special case contain Hamiltonian cycles.

The rest of the paper is organized as follows. In Section 2, some notations and basic terminologies are introduced. Section 3 shows that the Hamiltonian cycle and Hamiltonian path problems for supergrid graphs are NP-complete. In Section 4, we show that rectangular (parallelism) and alphabet supergrid graphs are Hamiltonian. Finally, we make some concluding remarks in Section 5.

2. Notations and terminologies

In this section, we will introduce fundamental terminologies and symbols used in the paper. For graph-theoretic terminology not defined in this paper, the reader is referred to [4]. Let \( G = (V, E) \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). Let \( S \) be a subset of vertices in \( G \), and let \( u, v \) be two vertices in \( G \). We write \( G[S] \) for the subgraph of \( G \) induced by \( S \), \( G - S \) for the subgraph \( G[V - S] \), i.e., the subgraph induced by \( V - S \). In general, we write \( G - v \) instead of \( G - \{v\} \). If \( (u, v) \) is an edge in \( G \), we say that \( u \) is adjacent to \( v \) and \( u, v \) are incident to edge \( (u, v) \). The notation \( u \sim v \) (resp., \( u \sim v \)) means that vertices \( u \) and \( v \) are adjacent (resp., non-adjacent). A neighbor of \( v \) in \( G \) is any vertex that is adjacent to \( v \). We use \( N_G(v) \) to denote the set of neighbors of \( v \) in \( G \). The subscript ‘‘\( G \)’’ of \( N_G(v) \) can be removed from the notation if it has no ambiguity. The degree of vertex \( v \) is the number of vertices adjacent to the vertex \( v \). The distance between \( u \) and \( v \) is the length of the shortest path between these two vertices. A path \( P \) of length \( |P| - 1 \) in a graph \( G \), denoted by \( v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{|P| - 1} \rightarrow v_{|P|} \), is a sequence \( (v_1, v_2, \cdots, v_{|P| - 1}, v_{|P|}) \) of vertices such that \( (v_i, v_{i + 1}) \in E(G) \) for \( 1 \leq i < |P| \). The first and last vertices visited by path \( P \) are denoted by \( \text{start}(P) \) and \( \text{end}(P) \), respectively. We will use \( v_i \in P \) to denote “\( P \) visits vertex \( v_i \)” and use \( (v_i, v_{i+1}) \in P \) to denote “\( P \) visits edge \((v_i, v_{i+1})\)”. A path from vertex \( v_1 \) to vertex \( v_k \) is denoted by \( (v_1, v_2)\)-path. In addition, we use \( P \) to refer to the set of vertices visited by path \( P \) if it is understood without ambiguity. On the other hand, a path is called the reversed path, denoted by \( \text{rev}(P) \), of path \( P \) if it visits the vertices of \( P \) from \( \text{end}(P) \) to \( \text{start}(P) \) in proper sequence; that is, the reversed path \( \text{rev}(P) \) of \( P = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{|P| - 1} \rightarrow v_{|P|} \) is \( v_{|P|} \rightarrow v_{|P| - 1} \rightarrow \cdots \rightarrow v_2 \rightarrow v_1 \). A cycle is a path \( C \) such that \( |V(C)| \geq 3 \) and \( \text{start}(C) \sim \text{end}(C) \).

Let \( S^\infty \) be the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if and only if the difference of their \( x \) or \( y \) coordinates is not larger than 1. A supergrid graph is a finite, vertex-induced subgraph of \( S^\infty \). For a vertex \( v \) in a supergrid graph, let \( v_x \) and \( v_y \) denote respectively \( x \) and \( y \) coordinates of its corresponding point. We color vertex \( v \) to be white if \( v_x + v_y \equiv 0 \) (mod 2); otherwise, \( v \) is colored to be black. Then there are eight possible neighbors of vertex \( v \) including four white vertices and four black vertices. Obviously, all grid graphs are bipartite [19] but supergrid graphs are not bipartite.

Rectangular grid graphs first appeared in [26], where Luccio and Munigia tried to solve the Hamiltonian path problem on them. Itai et al. [19] gave necessary and sufficient conditions for the existence of Hamiltonian \((s,t)\)-path in rectangular grid graphs, where \( s, t \) are two given vertices. In this paper, we expand them to a subclass of supergrid graphs, namely rectangular supergrid graphs. Let \( R(m,n) \) be the supergrid graph whose vertex set \( V(R(m,n)) = \{ v = (v_x, v_y) | 1 \leq v_x \leq m \text{ and } 1 \leq v_y \leq n \} \). Then, \( R(m,n) \) contains \( m \) columns and \( n \) rows of vertices in \( S^\infty \). A rectangular supergrid graph is a supergrid graph that is isomorphic to \( R(m,n) \) for \( m, n \geq 1 \). Thus \( m \) and \( n \), the dimensions, specify a rectangular supergrid graph up to isomorphism. The size of \( R(m,n) \) is defined to be \( mn \), and \( R(m,n) \) is called \( n \)-rectangle. The edge in the boundary of \( R(m,n) \) is called boundary edge. For example, Fig. 4(a)
In the figures, we assume that \( A \) in the paper, we extend them to form a subclass of supergrid graphs, namely alphabet supergrid graphs. An supergrid graph is a finite vertex-induced subgraph of the rectangular super grid graph of a certain type, as follows. For every condition, there are two types of parallelism supergrid graphs. We can see that they are isomorphic although they are different when considered as geometric graphs. Note that the boundary edges of supergrid graphs. For example, each alphabet supergrid graph is isomorphic to \( P(m,n) \). For example, Fig. 4(b) depicts two parallelism supergrid graphs \( P(5, 6) \) and \( P(6, 5) \). In the above definition, there are two types of parallelism supergrid graphs. We can see that they are isomorphic although they are different when considered as geometric graphs. Note that the boundary edges of \( R(m,n) \) and \( P(m,n) \) form a rectangle and a parallelogram, respectively.

In [33], Salman first introduced alphabet grid graphs, which form a subclass of grid graphs, and studied some properties of these graphs. Recently, Keshavarz-Kohjerdi and Bagheri [21] determined the necessary and sufficient conditions for the existence of Hamiltonian \((s, t)\)-path in alphabet grid graphs, where \( s, t \) are two given vertices. In this paper, we extend them to form a subclass of supergrid graphs, namely alphabet supergrid graphs. An alphabet supergrid graph is a finite vertex-induced subgraph of the rectangular supergrid graph of a certain type, as follows. For \( m, n \geq 3 \), an \( L \)-alphabet supergrid graph \( L(m,n) \), \( C \)-alphabet supergrid graph \( C(m,n) \), \( F \)-alphabet supergrid graph \( F(m,n) \), and \( E \)-alphabet supergrid graph \( E(m,n) \) are subgraphs of \( R(3m-2, 5n-4) \). These alphabet supergrid graphs are defined as shown in Fig. 4(c) – (f), where \( m = 4 \) and \( n = 3 \).

3. NP-completeness

In this section, we will prove that the Hamiltonian cycle (path) problem for supergrid graphs is NP-complete. In [19] and [12], the authors showed the Hamiltonian cycle problem for grid graphs and triangular grid graphs to be
NP-complete. We apply the idea of these proofs to show that the Hamiltonian cycle problem remains NP-complete for supergrid graphs. By using similar arguments, we can prove that the Hamiltonian path problem on supergrid graphs is still NP-complete. Notice that grid graphs and triangular grid graphs are not subclasses of supergrid graphs; these classes of graphs have common elements (vertices) but in general they are distinct.

To prove the Hamiltonian cycle problem on supergrid graphs to be NP-complete, we establish a polynomial-time reduction from the Hamiltonian cycle problem for planar bipartite graphs with maximum degree 3. The following theorem was given in [19].

**Theorem 3.1.** (See [19].) The Hamiltonian cycle problem for planar bipartite graphs with maximum degree 3 is NP-complete.

Let $B = (V_0 \cup V_1, E)$ be a planar bipartite graph with maximum degree 3 and let $G_1$ be a rectangular supergrid graph, where $V_0$ and $V_1$ are the bipartition sets of $B$ in which every edge joins a vertex in $V_0$ and a vertex in $V_1$. Similarly to the parity-preserving embedding [19] of a bipartite graph into a rectangular grid graph, let us introduce the following parity-preserving embedding $emb$ of $B$ into $G_1$ (a one-to-one function from $V_0 \cup V_1$ to the vertices of $G_1$ and from $E$ to paths in $G_1$):

1. The vertices of $V_0$ are mapped to white vertices of $G_1$, i.e., if $v \in V_0$, then $emb(v)$ is colored by white.
2. The vertices of $V_1$ are mapped to black vertices of $G_1$, i.e., if $v \in V_1$, then $emb(v)$ is colored by black.
3. The edges of $B$ are mapped to vertex-disjoint paths of $G_1$, i.e., if $e = (u, v) \in E$, then $emb(e)$ is a path $P$ from $emb(u)$ to $emb(v)$, and the intermediate vertices of $P$ do not belong to any other path.

For example, Fig. 5(a) shows a planar bipartite graph $B$ with maximum degree 3, and a parity-preserving embedding $emb(B)$ of $B$ is depicted in Fig. 5(b). The following lemma is given in [19] and shows that the above parity-preserving embedding can be done in polynomial time.

**Lemma 3.2.** (See [19].) Let $B$ be a planar bipartite graph with $n$ vertices and maximum degree 3. Then, a parity-preserving embedding $emb(B)$ of $B$ into a rectangular grid (supergrid) graph $R(kn, kn)$ can be done in polynomial time, where $k$ is a constant.

Now given a planar bipartite graph $B$ with $n$ vertices and maximum degree 3, we shall construct a supergrid graph $G_s$ such that $B$ has a Hamiltonian cycle if and only if $G_s$ contains a Hamiltonian cycle. Let $B = (V_0 \cup V_1, E)$. The construction of $G_s$ from $B$ is sketched as follows. First, we embed graph $B$ into a rectangular supergrid graph $R(kn, kn)$ for some constant $k$, as described in Lemma 3.2. Let the embedding supergrid graph be $G_1$. In the second step, we enlarge the supergrid graph $G_1$ such that each edge in $G_1$ is transformed into a path with 9 edges. Let the enlarged supergrid graph be $G_2$. For example, Fig. 6 shows the supergrid graph $G_2$ enlarged from the supergrid graph $G_1$ in Fig. 5(b) for the planar bipartite graph $B$ in Fig. 5(a). In the third step, each vertex of graph $B$ is transformed into a cluster which is a small supergrid graph. The vertex of $B$ is called critical vertex in $G_2$. Finally, each path in $G_2$ is simulated by a tentacle which is a series of 2-rectangles, and the resultant graph is a supergrid graph $G_s$.

Now, we introduce clusters and tentacles as follows. We transform each vertex of $V_0$ into a white cluster and each vertex of $V_1$ into a black cluster. A white cluster is a supergrid graph with 17 vertices and a black cluster is also a supergrid graph with 13 vertices. Fig. 7 shows the white and black clusters. The center of a white cluster (resp., black}

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**Fig. 5.** (a) A planar bipartite graph $B$ with maximum degree 3, and (b) a parity-preserving embedding $emb(B)$ of $B$, where solid lines indicate the edges in the embedding paths.
cluster) is a white critical vertex (resp., black critical vertex). For example, critical vertex $x$ (resp., $y$) is the center of a white cluster (resp., a black cluster) in Fig. 7(a) (resp., Fig. 7(b)). The distance between any vertex and center in a cluster is at most 2. The vertices $a_1, a_2, a_3, a_4$ (resp., $b_1, b_2, b_3, b_4$) in Fig. 7(a) (resp., Fig. 7(b)) are called the corner vertices of cluster, and the edges $e_1, e_2, e_3, e_4$ in Fig. 7 are called critical edges. In our construction, the corner vertices together with critical edges of a cluster are used to connect to the other cluster. The following two propositions show the properties of clusters.

**Proposition 3.3.** Let $C_{17}$ be a white cluster, $a_1, a_2, a_3, a_4$ be its corner vertices, and let $e_1, e_2, e_3, e_4$ be its critical edges, as in Fig. 7(a). Then for $1 \leq i < j \leq 4$, there exists a Hamiltonian $(a_i, a_j)$-path of $C_{17}$ which contains all four critical edges $\{e_1, e_2, e_3, e_4\}$.

**Proof.** By inspection, the lemma can be verified. For example, Fig. 8(a) and Fig. 8(b) depict a Hamiltonian $(a_1, a_3)$-path and a Hamiltonian $(a_1, a_4)$-path of $C_{17}$, respectively, which contain all critical edges $\{e_1, e_2, e_3, e_4\}$. □

**Proposition 3.4.** Let $C_{13}$ be a black cluster, $b_1, b_2, b_3, b_4$ be its corner vertices, and let $e_1, e_2, e_3, e_4$ be its critical edges, as in Fig. 7(b). Then for $1 \leq i < j \leq 4$, there exists a Hamiltonian $(b_i, b_j)$-path of $C_{13}$ which contains all four critical edges $\{e_1, e_2, e_3, e_4\}$. 7
Fig. 7: (a) A white cluster $C_{17}$, and (b) a black cluster $C_{13}$, where the centers of $C_{17}$ and $C_{13}$ are critical vertices $x$ and $y$, respectively, the critical edges include $e_1, e_2, e_3, e_4$, and the corner vertices include $a_1, a_2, a_3, a_4$ or $b_1, b_2, b_3, b_4$.

Fig. 8: (a) A Hamiltonian $(a_1, a_3)$-path of a white cluster $C_{17}$, (b) a Hamiltonian $(a_1, a_4)$-path of a white cluster, (c) a Hamiltonian $(b_1, b_3)$-path of a black cluster $C_{13}$, and (d) a Hamiltonian $(b_1, b_4)$-path of a black cluster, where arrow lines indicate the edges in such a path and each critical edge is contained in the Hamiltonian path.
Fig. 9: (a) A strip $S(a, b; c, d)$ with corners $a, b, c, d$, and (b) a square tentacle $T'(a_1, b_1; c_5, d_5)$ consisting of 5 strips.

**Proof.** By inspection. For example, For example, Fig. 8(c) and Fig. 8(d) depict a Hamiltonian $(b_1, b_3)$-path and a Hamiltonian $(b_1, b_2)$-path of $C_{13}$, respectively, which contain all critical edges $e_1, e_2, e_3, e_4$. \(\square\)

In our construction, the path in $G_2$ is simulated by a series of 2-rectangles, called tentacle. We will use the similar technique in [19] to make it. A strip is a rectangular supergrid graph with at least 2 squares (see Fig. 9(a)), i.e., it is isomorphic to $R(m, 2)$ with $m \geq 3$. The strip with corners $a, b, c, d$ (the degree of every corner is 3), as in Fig. 9(a), is denoted by $S(a, b; c, d)$. A square in a strip is called terminal if it contains corners. A square tentacle $T'$ is a supergrid graph which is either a strip or a union of a series of strips stuck together by the edges of terminal squares. Let $T' = S(a_1, b_1; c_1, d_1) \cup S(a_2, b_2; c_2, d_2) \cup \cdots \cup S(a_k, b_k; c_k, d_k)$. We define $T'$ to satisfy the following conditions:

1. both $c_i, d_i \in V(S(a_{i-1}, b_{i+1}; c_{i+1}, d_{i+1}))$ for $1 \leq i \leq k - 1$,
2. one of $a_{i+1}, b_{i+1} \in V(S(a_i, b_i; c_i, d_i))$ for $1 \leq i \leq k - 1$, and
3. there is no other intersection between the vertex sets of the strips.

The vertices $a_1, b_1, c_k, d_k$ are called the corners of $T'$, and denote this square tentacle by $T'(a_1, b_1; c_k, d_k)$. For example, Fig. 9(b) shows a square tentacle with 5 strips and corners $a_1, b_1, c_5, d_5$.

In the construction of an enlarged supergrid graph $G_2$ (see Fig. 6) from a planar bipartite graph $B$ (see Fig. 5(a)), the path in $G_2$ is a combination of four possible types of subpaths, as shown in Fig. 10(a). The corresponding square tentacles for these types of subpaths are depicted in Fig. 10(b). A tentacle is then constructed from a square tentacle by attaching a triangle to its terminal square. For instance, Fig. 10(c) depicts the possible tentacles for the square tentacle of type I in Fig. 10(b). We call the attached triangle of a tentacle to be the tail of the tentacle. Let $u, v$ be two vertices of the first square in a tentacle $T$ such that their degrees are 3, and let $w$ be a vertex of the tail in a tentacle $T$ such that its degree 2. Since each strip of a tentacle is isomorphic to a 2-rectangle $R(m, 2)$ with $m \geq 3$, the vertices $u, v, w$ exist. Then we denote the tentacle by $T(u, v; w)$. The vertices $u, v$ are called the twin corners of $T(u, v; w)$, and the vertex $w$ is said to be the tail corner of $T(u, v; w)$. Fig. 10(c) also depicts the corners of tentacles. The following lemma shows the Hamiltonian property of a tentacle.

**Lemma 3.5.** Let $T(u, v; w)$ be a tentacle. Then for any two corners $s, t \in \{u, v, w\}$, there exists a Hamiltonian $(s, t)$-path of $T(u, v; w)$.

**Proof.** Let $T'(a_1, b_1; c_5, d_5) = S(a_1, b_1; c_1, d_1) \cup S(a_2, b_2; c_2, d_2) \cup \cdots \cup S(a_5, b_5; c_5, d_5)$ be the corresponding square tentacle of $T(u, v; w)$. That is, $T(u, v; w)$ is constructed from $T'(a_1, b_1; c_5, d_5)$ by attaching a triangle with vertices $c_5, d_5, w$. Then, $u, v \in \{a_1, b_1\}$ and $w$ is adjacent to both of $c_5$ and $d_5$. We claim that $T'(a_1, b_1; c_5, d_5)$ contains a Hamiltonian $(s', t')$-path for $s' \in \{a_1, b_1\}$ such that edge $(c_5, d_5)$ is in the Hamiltonian path, and has a Hamiltonian $(s', t')$-path for $s' \in \{a_1, b_1\}$ and $t' \in \{c_5, d_5\}$. Since $u, v \in \{a_1, b_1\}$ and $w$ is adjacent to both of $c_5$ and $d_5$, the lemma hence holds true.

We prove the above claim by induction on $\eta$, the number of strips in square tentacle $T'(a_1, b_1; c_5, d_5)$. Initially, let $\eta = 1$. Then, $T'(a_1, b_1; c_1, d_1) = S(a_1, b_1; c_1, d_1)$. By inspection, it is easy to verify that there exists a Hamiltonian $(a_1, b_1)$-path of strip $S(a_1, b_1; c_1, d_1)$ such that edge $(c_1, d_1)$ is in it. That is, the Hamiltonian path contains
(a) A schematic diagram for the relative location of vertices of $I$. Note that if $b$ the vertices of

(b) the corresponding square tentacles for these subpaths in (a), and (c) the possible tentacles for the square tentacle of type $I$ in (b), where the solid lines indicate the edges in paths.

Fig. 10: (a) The possible types of subpaths in an enlarged supergrid graph $G_s$, (b) the Hamiltonian $(a_1, b_1)$-path of $T(a_1, b_1; c_{k+1}, d_{k+1})$, and (c) the Hamiltonian $(a_1, c_{k+1})$-path of $T(a_1, b_1; c_{k+1}, d_{k+1})$, where arrow lines indicate the edges in the Hamiltonian paths.

its all boundary edges except edge $(a_1, b_1)$. In addition, for $s' \in \{a_1, b_1\}$ and $t' \in \{c_1, d_1\}$ we can easily construct a Hamiltonian $(s', t')$-path of $S(a_1, b_1; c_1, d_1)$. Now, assume that the claim holds true when $\eta = k \geq 1$. Then, there exists a Hamiltonian $(a_1, b_1)$-path $P_1$ of $T'(a_1, b_1; c_1, d_1)$ such that edge $(c_1, d_1)$ is in $P_1$, and there exists a Hamiltonian $(s', t')$-path $P_2$ of $T'(a_1, b_1; c_1, d_1)$ for $s' \in \{a_1, b_1\}$ and $t' \in \{c_1, d_1\}$. Consider that $\eta = k + 1$. Then, $T'(a_1, b_1; c_{k+1}, d_{k+1}) = T'(a_1, b_1; c_1, d_1) \cup S(a_{k+1}, b_{k+1}; c_{k+1}, d_{k+1})$. By definition of square tentacle, one of $a_{k+1}, b_{k+1} \in \{c_1, d_1\}$ and both $c_k, d_k \in V(S(a_{k+1}, b_{k+1}; c_{k+1}, d_{k+1}))$. Without loss of generality, suppose that $b_{k+1} = c_k$ and the first square of $S(a_{k+1}, b_{k+1}; c_{k+1}, d_{k+1})$ contains vertices $a_{k+1}, b_{k+1} = c_k, d_k, r$. Let $S' = S(p, q; c, d)$ be a strip obtained from $S(a_{k+1}, b_{k+1}; c_{k+1}, d_{k+1})$ by removing the vertices of the first square. Since each strip contains at least two squares, we have that $S' \neq \emptyset$, and $c_r = c_{k+1}$, and $d_r = d_{k+1}$. Note that if $S(a_{k+1}, b_{k+1}; c_{k+1}, d_{k+1})$ contains only two squares, then $p = c_{k+1}$ and $q = d_{k+1}$. The relative location of $a_{k+1}, b_{k+1}, d_{k+1}, r, p, q, c_{k+1}, d_{k+1}$ is depicted in Fig. 11(a). Then, $d_k, r, p, q$ forms the second square of $S(a_{k+1}, b_{k+1}; c_{k+1}, d_{k+1})$. Note that each square of a strip is a clique.

Consider the following two cases:

Case 1: $s', t' \in \{a_1, b_1\}$. By the induction hypothesis, there exists a Hamiltonian $(a_1, b_1)$-path $P_1$ of $T'(a_1, b_1; c_k, d_k)$ such that edge $(c_k, d_k)$ is in $P_1$. Without loss of generality, suppose that $P_1 = P'_1 \rightarrow c_k \rightarrow d_k \rightarrow P''_1$, i.e., $c_k$ appears before $d_k$ in $P_1$, where $\text{start}(P'_1), \text{end}(P''_1) \in \{a_1, b_1\}$. The case that $c_k$ appears after $d_k$ in $P_1$ can be verified similarly. By inspection, we can construct a Hamiltonian $(p, q)$-path $Q_1$ of $S'(p, q; c_{k+1}, d_{k+1})$ such that edge $(c_{k+1}, d_{k+1})$ is in $Q_1$. Let $P'' = P'_1 \rightarrow c_{k+1} \rightarrow a_{k+1} \rightarrow r \rightarrow Q_1 \rightarrow d_k \rightarrow P''_1$. Then, $P''$ is a Hamiltonian $(a_1, b_1)$-path of $T'(a_1, b_1; c_{k+1}, d_{k+1})$ such that edge $(c_{k+1}, d_{k+1})$ is in $P''$. The construction of such a Hamiltonian $(a_1, b_1)$-path $P''$ is depicted in Fig. 11(b).

Case 2: $s', t' \in \{a_1, b_1\}$ and $t' \in \{c_1, d_1\}$. By the induction hypothesis, there exists a Hamiltonian $(s', d_k)$-path $P_2$
of $T'(a_1, b_1; c_k, d_1)$. By inspection, we can construct a Hamiltonian $(p, t')$-path $Q_2$ of $S' = S(p, q; c_k + 1, d_k + 1)$, where $t' \in [c_k + 1, d_k + 1]$. Let $P' = P_2 \rightarrow a_{k+1} \rightarrow r \rightarrow Q_2$. Then, $P'$ is a Hamiltonian $(s', t')$-path of $T'(a_1, b_1; c_k + 1, d_k + 1)$, where $s' \in [a_1, b_1]$ and $t' \in [c_k + 1, d_k + 1]$. The construction of such a Hamiltonian $(a_1, c_k+1)$-path $P'$ is shown in Fig. 11(c).

It immediately follows from the above cases that the claim holds true. This completes the proof of the lemma. □

Let $B = (V_0 \cup V_1, E)$ be a planar bipartite graph with $n$ vertices and maximum degree 3, $G_1$ be the embedding supergrid graph from $B$, and let $G_2$ be the enlarged supergrid graph by multiplying the scale of $G_1$ by 9. We have simulated the critical vertices of $G_2$ by clusters and the paths of $G_2$ by tentacles. The remaining care is taken as to how the tentacle is connected to the clusters corresponding to two critical vertices of $G_2$. Let $x \in V_0$ and $y \in V_1$ with $(x, y) \in E(B)$. Then, we simulate $x$ and $y$ by a white cluster $C_{17}$ and a black cluster $C_{13}$, respectively, such that $C_{17}$ contains the white vertex $x$ as center and four critical edges $e_1, e_2, e_3, e_4$, and $C_{13}$ contains the black vertex $y$ as center and four corner vertices $b_1, b_2, b_3, b_4$, as shown in Fig. 7. The path between $x$ and $y$ in $G_2$ is then simulated by a tentacle $T(u, v; w)$. Tentacle $T(u, v; w)$ is used to connect clusters $C_{17}$ and $C_{13}$ in the following way. The twin corners, $u$ and $v$, of $T(u, v; w)$ are adjacent to the vertices of one critical edge in $C_{17}$, and the other corner, $w$, of $T(u, v; w)$ is adjacent to one corner vertex of $C_{13}$. For example, Fig. 12 depicts such a connection between white cluster $C_{17}$ and black cluster $C_{13}$ via a tentacle $T(u, v; w)$. Since the maximum degree of the original planar bipartite graph is 3, the number of tentacles connecting to each cluster is at most 3. For a white cluster (resp., black cluster), there are four critical edges (resp., corner vertices) (see Fig. 7) and the number of critical edges (resp., corner vertices) used to connect tentacles is at most 3 since the maximum degree of the original planar bipartite graph is 3. Thus, it is enough to make such a connection. On the other hand, the paths in embedding supergrid graph $G_1$ are vertex disjoint and hence they are vertex disjoint paths in the enlarged supergrid graph $G_2$. Since we enlarge the scale of $G_1$ by 9, i.e., each edge in $G_1$ is transformed into a path with 9 edges, it is easy to construct tentacles from the paths of $G_2$ such that tentacles are disjoint.

Let $T(u, v; w)$ be a tentacle with twin corners $u, v$ and tail corner $w$. By Lemma 3.5, there exists a Hamiltonian $(s, t)$-path of $T(u, v; w)$ for $s, t \in [u, v, w]$. By the definition of tentacle, $u, v$ are adjacent and $w$ is adjacent to neither $u$ nor $v$. We can easily observe that there are only two types of Hamiltonian $(s, t)$-paths in $T(u, v; w)$. The path can be either a return path if $s, t$ are twin corners or a cross path if one of $s, t$ is tail corner. For example, Fig. 13 depicts these two types of Hamiltonian paths in $T(u, v; w)$ shown in Fig. 12. Note that there are many return paths and cross paths.

We have introduced how to construct a supergrid graph $G_2$ from a planar bipartite graph $B$ with maximum degree 3. The construction algorithm is formally presented as follows.

Algorithm SupergridConstruction
Input: $B = (V_0 \cup V_1, E)$, a planar bipartite graph with maximum degree 3.
Output: $G_s$, a supergrid graph constructed from $B$.
Method:

1. Embed graph $B$ into a rectangular supergrid graph $R(kn, kn)$ for some constant $k$ [19], and let the embedding supergrid graph be $G_1$;
2. Enlarge $G_1$ to a supergrid graph $G_2$ such that each edge in $G_1$ is transformed into a path with 9 edges;
3. For each white critical vertex $x$ of $G_2 (x \in V_0)$, $x$ is transformed into a white cluster $C_{17}$ with center $x$;
4. For each black critical vertex $y$ of $G_2 (y \in V_1)$, $y$ is transformed into a black cluster $C_{13}$ with center $y$;
5. Consider a Hamiltonian $(p, t)$-path $Q$ of $S = S(p, q, c_k, d_k)$, where $t \in [c_k + 1, d_k + 1]$. Let $P = P_2 \rightarrow a_{k+1} \rightarrow r \rightarrow Q$. Then, $P$ is a Hamiltonian $(s, t)$-path of $T'(a_1, b_1; c_k, d_k)$, where $s \in [a_1, b_1]$, $t \in [c_k + 1, d_k + 1]$. The construction of such a Hamiltonian $(a_1, c_k)$-path $P'$ is shown in Fig. 11(c).

It immediately follows from the above cases that the claim holds true. This completes the proof of the lemma. □

Let $B = (V_0 \cup V_1, E)$ be a planar bipartite graph with $n$ vertices and maximum degree 3, $G_1$ be the embedding supergrid graph from $B$, and let $G_2$ be the enlarged supergrid graph by multiplying the scale of $G_1$ by 9. We have simulated the critical vertices of $G_2$ by clusters and the paths of $G_2$ by tentacles. The remaining care is taken as to how the tentacle is connected to the clusters corresponding to two critical vertices of $G_2$. Let $x \in V_0$ and $y \in V_1$ with $(x, y) \in E(B)$. Then, we simulate $x$ and $y$ by a white cluster $C_{17}$ and a black cluster $C_{13}$, respectively, such that $C_{17}$ contains the white vertex $x$ as center and four critical edges $e_1, e_2, e_3, e_4$, and $C_{13}$ contains the black vertex $y$ as center and four corner vertices $b_1, b_2, b_3, b_4$, as shown in Fig. 7. The path between $x$ and $y$ in $G_2$ is then simulated by a tentacle $T(u, v; w)$. Tentacle $T(u, v; w)$ is used to connect clusters $C_{17}$ and $C_{13}$ in the following way. The twin corners, $u$ and $v$, of $T(u, v; w)$ are adjacent to the vertices of one critical edge in $C_{17}$, and the other corner, $w$, of $T(u, v; w)$ is adjacent to one corner vertex of $C_{13}$. For example, Fig. 12 depicts such a connection between white cluster $C_{17}$ and black cluster $C_{13}$ via a tentacle $T(u, v; w)$. Since the maximum degree of the original planar bipartite graph is 3, the number of tentacles connecting to each cluster is at most 3. For a white cluster (resp., black cluster), there are four critical edges (resp., corner vertices) (see Fig. 7) and the number of critical edges (resp., corner vertices) used to connect tentacles is at most 3 since the maximum degree of the original planar bipartite graph is 3. Thus, it is enough to make such a connection. On the other hand, the paths in embedding supergrid graph $G_1$ are vertex disjoint and hence they are vertex disjoint paths in the enlarged supergrid graph $G_2$. Since we enlarge the scale of $G_1$ by 9, i.e., each edge in $G_1$ is transformed into a path with 9 edges, it is easy to construct tentacles from the paths of $G_2$ such that tentacles are disjoint.

Let $T(u, v; w)$ be a tentacle with twin corners $u, v$ and tail corner $w$. By Lemma 3.5, there exists a Hamiltonian $(s, t)$-path of $T(u, v; w)$ for $s, t \in [u, v, w]$. By the definition of tentacle, $u, v$ are adjacent and $w$ is adjacent to neither $u$ nor $v$. We can easily observe that there are only two types of Hamiltonian $(s, t)$-paths in $T(u, v; w)$. The path can be either a return path if $s, t$ are twin corners or a cross path if one of $s, t$ is tail corner. For example, Fig. 13 depicts these two types of Hamiltonian paths in $T(u, v; w)$ shown in Fig. 12. Note that there are many return paths and cross paths.

We have introduced how to construct a supergrid graph $G_2$ from a planar bipartite graph $B$ with maximum degree 3. The construction algorithm is formally presented as follows.

Algorithm SupergridConstruction
Input: $B = (V_0 \cup V_1, E)$, a planar bipartite graph with maximum degree 3.
Output: $G_s$, a supergrid graph constructed from $B$.
Method:

1. Embed graph $B$ into a rectangular supergrid graph $R(kn, kn)$ for some constant $k$ [19], and let the embedding supergrid graph be $G_1$;
2. Enlarge $G_1$ to a supergrid graph $G_2$ such that each edge in $G_1$ is transformed into a path with 9 edges;
3. For each white critical vertex $x$ of $G_2 (x \in V_0)$, $x$ is transformed into a white cluster $C_{17}$ with center $x$;
4. For each black critical vertex $y$ of $G_2 (y \in V_1)$, $y$ is transformed into a black cluster $C_{13}$ with center $y$;
5. Consider a Hamiltonian $(p, t)$-path $Q$ of $S = S(p, q, c_k, d_k)$, where $t \in [c_k + 1, d_k + 1]$. Let $P = P_2 \rightarrow a_{k+1} \rightarrow r \rightarrow Q$. Then, $P$ is a Hamiltonian $(s, t)$-path of $T'(a_1, b_1; c_k, d_k)$, where $s \in [a_1, b_1]$, $t \in [c_k + 1, d_k + 1]$. The construction of such a Hamiltonian $(a_1, c_k)$-path $P'$ is shown in Fig. 11(c).

It immediately follows from the above cases that the claim holds true. This completes the proof of the lemma. □
5. for each path between white critical vertex $x$ and black critical vertex $y$, construct a tentacle $T(u, v; w)$ to connect the corresponding clusters of $x$ and $y$ such that $u, v$ are adjacent to two vertices of one critical edges in white cluster and $w$ is adjacent to one corner vertex of black cluster;

6. let the constructed supergrid graph be $G_s$ and output $G_s$.

For example, given a planar bipartite graph $B = (V_0 \cup V_1, E)$ with maximum degree 3 shown in Fig. 5(a), the embedding supergrid graph $G_1$ is shown in Fig. 5(b). The enlarged supergrid graph $G_2$ by multiplying the scale of $G_1$ by 9 is shown in Fig. 6. The constructed supergrid graph $G_s$ is depicted in Fig. 14. By Lemma 3.2, line 1 of Algorithm SupergridConstruction can be done in polynomial time. Clearly, lines 2–6 of the algorithm can be done in polynomial time. Then, Algorithm SupergridConstruction runs in polynomial time and hence the following lemma holds true.

Lemma 3.6. Given a planar bipartite graph $B = (V_0 \cup V_1, E)$ with maximum degree 3, Algorithm SupergridConstruction constructs a supergrid graph $G_s$ in polynomial time.

Next, we will prove that supergrid graph $G_s$ has a Hamiltonian cycle if and only if there exists a Hamiltonian cycle in the planar bipartite graph $B$. Before proving the above property, we first give the relation between tentacle and Hamiltonian cycle of $G_s$. For a cycle $C$ of a graph $G$ and a subgraph $H$ of $G$, we denote the restriction of $C$ to $H$ by $C_H$. Then, $C_H$ is a set of subpaths of $C$. We then have the following lemma.

Lemma 3.7. Let $G_s$ be the supergrid graph constructed from a planar bipartite graph $B$ by Algorithm SupergridConstruction, and let $T = T(u, v; w)$ be a tentacle of $G_s$. If $G_s$ has a Hamiltonian cycle $HC$, then there exists a Hamiltonian cycle $HC^*$ in $G_s$ such that $HC^*|_T$ is a Hamiltonian $(s, t)$-path of $T$ for $s, t \in \{u, v, w\}$.

Proof. By the construction of $G_s$, tentacle $T$ satisfies the following properties:

1. only three vertices $u, v, w$ of $T$ are adjacent to vertices of $G_s - T$, i.e., no vertex of $T - \{u, v, w\}$ is adjacent to vertices of $G_s - T$;
2. each of twin corners $u, v$ of $T$ is adjacent to only two vertices $p, q$ of $G_s - T$, i.e., $N(u) \cap (G_s - T) = N(v) \cap (G_s - T) = \{p, q\}$ and $u, v, p, q$ forms a clique, and
3. the tail corner $w$ of $T$ is adjacent to only one vertex $r$ of $G_s - T$, i.e., $N(w) \cap (G_s - T) = \{r\}$.

Since only three vertices $u, v, w$ of $T$ are adjacent to vertices of $G_s - T$, $G_s - T$ and $T - \{u, v, w\}$ are disjoint. Then, the number of paths in $HC^*|_T$ is not larger than 2, i.e., $|HC^*|_T| \leq 2$. If $|HC^*|_T| = 1$, i.e., $HC^*|_T$ is either a return path or a cross path of $T$, then the lemma is clearly true. Assume that $|HC^*|_T| = 2$ below. Let $u_1, u_2, u_3 \in \{u, v, w\}$ and let $HC = P_1 \rightarrow u_1 \rightarrow Q_1 \rightarrow u_2 \rightarrow P_2 \rightarrow u_3 \rightarrow Q_2$, where $HC|_T = \{u_1 \rightarrow Q_1 \rightarrow u_2, u_3 \rightarrow Q_2\}$ and $P_2 \neq \emptyset$. Suppose that $P_1 = \emptyset$. Since $HC$ is a Hamiltonian cycle of $G_s$, $u_1 \sim \text{end}(Q_2)$ and hence $HC|_T = \{u_2 \rightarrow \text{rev}(Q_1) \rightarrow
Fig. 14: A supergrid graph $G_s$ constructed from a planar bipartite graph $B = (V_0 \cup V_1, E)$ with maximum degree 3 shown in Fig. 5(a), where solid lines indicate the edges in the constructed supergrid graph.
Lemma 3.8. Let \( B = (V_0 \cup V_1, E) \) be a planar bipartite graph with maximum degree 3 and let \( G_s \) be the supergrid graph constructed from \( B \) by Algorithm SupergridConstruction. Then, \( G_s \) has a Hamiltonian cycle if and only if there exists a Hamiltonian cycle in graph \( B \).

Proof. If part: In this part, we will prove that if graph \( B \) has a Hamiltonian cycle, then graph \( G_s \) contains a Hamiltonian cycle. Assume that the planar bipartite graph \( B = (V_0 \cup V_1, E) \) has a Hamiltonian cycle \( HC_B \). We will construct the corresponding Hamiltonian cycle \( HC_s \) of \( G_s \) as follows. Let \((x, y)\) be an edge of graph \( B \) such that \( x \in V_0 \) and \( y \in V_1 \), and let the edge be simulated by a tentacle \( T_{xy} = T(u, v; w) \) in \( G_s \). Starting to form \( HC_s \), we will cover \( T_{xy} \) by a cross path if \((x, y) \in HC_B \), and by a return path otherwise. By Lemma 3.5, there exists a cross path or return path of tentacle \( T_{xy} \). The clusters themselves are covered as in Propositions 3.3 and 3.4. The partial paths can be connected to constitute a Hamiltonian cycle. Note that some critical edges of \( e' / s \) for connecting to a return path in Fig. 7(a) must be deleted in the constructed Hamiltonian cycle. For example, for the planar bipartite graph \( B = (V_0 \cup V_1, E) \) shown in Fig. 5(a), \( HC_B = x_1 \rightarrow y_1 \rightarrow x_2 \rightarrow y_2 \rightarrow x_3 \rightarrow y_3 \) is a Hamiltonian cycle of \( B \). Then, Fig. 15 depicts its corresponding Hamiltonian cycle \( HC_s \) in the constructed supergrid graph \( G_s \) as shown in Fig. 14.

Only If part: In this part, we will prove that if graph \( G_s \) has a Hamiltonian cycle, then graph \( B \) contains a Hamiltonian cycle. Assume now that supergrid graph \( G_s \) has a Hamiltonian cycle \( HC_s \). By Lemma 3.7, we may assume that for any tentacle \( T = T(u, v; w) \) in \( G_s \), the restriction \( HC_T \) of \( HC_s \) to \( T \) is a Hamiltonian \((s, t)\)-path for \( s, t \in \{u, v, w\} \). Then, by our construction of \( G_s \), each tentacle is covered by either a cross path or a return path. In our construction of \( G_s \), each tentacle is connected to only one critical edge of a white cluster, and each white cluster is attached to at most three tentacles. On the other hand, each tentacle is connected to only one corner vertex of a black cluster, and each black cluster is connected to at most three tentacles. We can see that in \( HC_s \) each white or black cluster is incident upon exactly two cross paths. Note that each vertex of \( B \) is transformed into a cluster, and every edge of \( B \) is simulated by a tentacle. To construct a Hamiltonian cycle \( HC_B \) of graph \( B \), we include in \( HC_B \) all edges corresponding to tentacles covered by cross paths. And each vertex of \( HC_B \) is a center of one cluster in \( G_s \). Then, \( HC_B \) is a Hamiltonian cycle of graph \( B \) because each cluster (white or black) of \( G_s \) can not be covered by \( HC_s \) unless it is incident upon exactly two cross paths.

Clearly, the Hamiltonian cycle problem for supergrid graphs is in NP. By Lemmas 3.6 and 3.8, we conclude the following theorem.

Theorem 3.9. The Hamiltonian cycle problem for supergrid graphs is NP-complete.

By similar arguments in proving the above theorem, we can prove the Hamiltonian path problem on supergrid graphs to be also NP-complete, as in the following theorem.

Theorem 3.10. The Hamiltonian path problem for supergrid graphs is NP-complete.

Proof. We give a reduction from the Hamiltonian cycle problem on planar bipartite graphs with maximum degree 3. Given a planar bipartite graph \( B \) with maximum degree 3, we construct a supergrid graph \( G_s' \) as follows:

1. construct a supergrid graph \( G_s' \) by Algorithm SupergridConstruction;
2. let \( C_b \) be a black cluster of \( G_s' \) such that its one critical edge \((s', t')\) is not attached to any tentacle;
3. two new vertices \( s \) and \( t \) are added to be adjacent to \( s' \) and \( t' \), respectively, such that the degree of each new vertex is 1;
Fig. 15: A Hamiltonian cycle $HC$ of a supergrid graph $G_s$ constructed from a Hamiltonian cycle $HC_B = x_1 \rightarrow y_1 \rightarrow x_2 \rightarrow y_2 \rightarrow x_3 \rightarrow y_3$ of a planar bipartite graph $B$ shown in Fig. 5(a), where solid lines indicate the edges in the cycle and $\otimes$ represents the destruction of an edge while constructing such a cycle.
4. let the resultant supergrid graph be $G'_s$.

Since the degree of each vertex in $B$ is at most 3, each tentacle is connected to one critical edges, and there are four critical edges in each black cluster, black cluster $C_{b}$ of $G_{t}$ does exist. The construction for attaching two new vertices is depicted in Fig. 16. By similar arguments in proving Lemma 3.8, we can verify that graph $G'_s$ has a Hamiltonian $(s, t)$-path if and only if there exists a Hamiltonian cycle in graph $B$.

Then, Lemma 3.8 and NP-completeness of the Hamiltonian cycle problem for planar bipartite graphs with maximum degree 3 complete the proof of this theorem. \hfill \Box

4. The Hamiltonian cycle problem on rectangular and alphabet supergrid graphs

In this section, we will study the Hamiltonian cycle property of rectangular and alphabet supergrid graphs. We show that these two subclasses of supergrid graphs always contain Hamiltonian cycles. In the literature, Chen et al. [6] and Salman [33] showed the Hamiltonian properties of rectangular and alphabet grid graphs, respectively, as shown in the following two lemmas.

Lemma 4.1. (See [6].) Let $R'(m, n)$ with $m, n \geq 2$ be a rectangular grid graph which is a subgraph of $R(m, n)$, where $V(R'(m, n)) = \{v = (v_x, v_y) | 1 \leq v_x \leq m \text{ and } 1 \leq v_y \leq n\}$. Then, $R'(m, n)$ contains a Hamiltonian cycle if and only if $mn$ is even.

Lemma 4.2. (See [33].) Let $L'(m, n)$ with $m, n \geq 3$ be an L-alphabet grid graph which is a subgraph of $L(m, n)$, where $L'(m, n)$ is defined similar to $L(m, n)$. Then, $L'(m, n)$ has a Hamiltonian cycle if and only if $mn$ is even.

Since any rectangular grid graph $R'(m, n)$ is a subgraph of a rectangular supergrid graph $R(m, n)$, $R(m, n)$ contains a Hamiltonian cycle if $mn$ is even. However, we will show that $R(m, n)$ contains a Hamiltonian cycle even if $mn$ is odd. Obviously, 1-rectangle contains no Hamiltonian cycle. Without loss of generality, assume that $m \geq n$ for $R(m, n)$. For a 2-rectangle $R(m, 2)$, we can construct a Hamiltonian cycle by visiting all boundary edges of $R(m, 2)$. In the following, consider $R(m, n)$ to satisfy that $m \geq n \geq 3$. By definition, $R(m, n)$ consists of $m$ columns and $n$ rows of vertices. Let $a_{ij}$ be the vertex locating at $i$-th row and $j$-th column of $R(m, n)$. That is, $(i, j)$ is the coordinates of $a_{ij}$ when $a_{11}$ is coordinated as $(1, 1)$. Let $C$ be a cycle of $R(m, n)$ and let $H$ be a boundary of $R(m, n)$, where $H$ is a subgraph of $R(m, n)$. Recall that the restriction of $C$ to $H$ is denoted by $C_{H}$. If $|C_{H}| = 1$, i.e. $C_{H}$ visits all boundary edges of $H$, then $C_{H}$ is called flat face on $H$. If $|C_{H}| > 1$ and $C_{H}$ contains at least one boundary edge of $H$, then $C_{H}$ is called concave face on $H$.

We first consider 3-rectangle $R(m, 3)$. A Hamiltonian cycle of $R(m, 3)$ is called canonical if it contains three flat faces on two shorter boundaries and one longer boundary, and it contains one concave face on the other boundary,
The canonical Hamiltonian cycle of rectangular supergrid graph $R(m, n)$ for (a) $n$ is even, and (b) $n$ is odd, where $m \geq n \geq 4$, solid arrow lines indicate the edges in the cycle, and dashed arrow lines indicate the flat faces in the cycle.

where the shorter boundary contains three vertices. The following lemma shows that $R(m, 3)$ contains a canonical Hamiltonian cycle.

Lemma 4.3. Let $R(m, 3)$ be a 3-rectangle with $m \geq 3$. Then, $R(m, 3)$ contains a canonical Hamiltonian cycle.

Proof. We prove this lemma by constructing a Hamiltonian cycle of $R(m, 3)$ such that it contains all boundary edges of two shorter boundaries and one longer boundary, and it contains at least one boundary edge in the other longer boundary. By inspection, the lemma can be easily verified for $4 \geq m \geq 3$. In the following, assume that $m \geq 5$. Let $P_i = a_{1i} \rightarrow a_{12} \rightarrow \cdots \rightarrow a_{(m-1)i} \rightarrow a_{1m}$, and let $P_i = a_{2i} \rightarrow a_{3i}$ for $m \geq i \geq 1$. Depending on whether $m$ is even or not, we consider the following two cases:

Case 1: $m$ is even. Let $P_i = \text{rev}(P_{i+1}) \rightarrow P_{i+1} \rightarrow \cdots \rightarrow \text{rev}(P_{m-2}) \rightarrow P_{m-1}$, where $j$ is even and $2 \leq j \leq m-2$. Then, $P_i \rightarrow P_m \rightarrow \text{rev}(P_{i+1}) \rightarrow \text{rev}(P_{i})$ is a canonical Hamiltonian cycle of $R(m, 3)$.

Case 2: $m$ is odd. Let $P_i = \text{rev}(P_{i+1}) \rightarrow P_{i+1} \rightarrow \cdots \rightarrow \text{rev}(P_{m-3}) \rightarrow P_{m-2} \rightarrow \text{rev}(P_{m-1})$, where $j$ is even and $2 \leq j \leq m-3$. Then, $P_i \rightarrow P_m \rightarrow \text{rev}(P_{i+1}) \rightarrow \text{rev}(P_{i})$ is a canonical Hamiltonian cycle of $R(m, 3)$.

It immediately follows from the above cases that the lemma holds true.

We have constructed Hamiltonian cycles for 2-rectangles and 3-rectangles. In the following, let $R(m, n)$ satisfy $m \geq n \geq 4$. A Hamiltonian cycle of $R(m, n)$ with $m \geq n \geq 4$ is called canonical if it contains three flat faces on three boundaries, and it contains one concave face on the other boundary. The following lemma shows the Hamiltonian property of $R(m, n)$ with $m \geq n \geq 4$.

Lemma 4.4. Let $R(m, n)$ be a rectangular supergrid graph with $m \geq n \geq 4$. Then, $R(m, n)$ contains four canonical Hamiltonian cycles with concave faces being on different boundaries.

Proof. Depending on whether $n$ is even or not, we consider the following two cases to construct a canonical Hamiltonian cycle of $R(m, n)$:

Case 1: $n$ is even. Let $P_i = a_{11} \rightarrow a_{12} \rightarrow \cdots \rightarrow a_{1m}$, $P_i = a_{2} \rightarrow a_{3} \rightarrow \cdots \rightarrow a_{n}$ for $n \geq i \geq 2$, and let $P_{n+1} = a_{2} \rightarrow a_{3} \rightarrow \cdots \rightarrow a_{(n-1)} \rightarrow a_{1}$. Let $P^* = \text{rev}(P_{n+1}) \rightarrow P_{n+1} \rightarrow \text{rev}(P_{n}) \rightarrow P_{n} \rightarrow \cdots \rightarrow \text{rev}(P_{j}) \rightarrow P_{j}$, where $j$ is even and $2 \leq j \leq n-2$. Let $C_{\text{even}} = P_i \rightarrow P^* \rightarrow \text{rev}(P_{n}) \rightarrow \text{rev}(P_{n-1})$. Then, $C_{\text{even}}$ is a canonical Hamiltonian cycle of $R(m, n)$. The construction of such a canonical Hamiltonian cycle is depicted in Fig. 17(a).

Case 2: $n$ is odd. In this case, $n \geq 5$. Let $P_i = a_{11} \rightarrow a_{12} \rightarrow \cdots \rightarrow a_{1m}$, $P_i = a_{2} \rightarrow a_{3} \rightarrow \cdots \rightarrow a_{n}$ for $n \geq i \geq 2$, and let $P_{n+1} = a_{2} \rightarrow a_{3} \rightarrow \cdots \rightarrow a_{(n-1)} \rightarrow a_{1}$. Let $P^* = \text{rev}(P_{n+1}) \rightarrow P_{n+1} \rightarrow \text{rev}(P_{n}) \rightarrow P_{n} \rightarrow \cdots \rightarrow \text{rev}(P_{j}) \rightarrow P_{j}$, where $j$ is
By Proposition 4.5, let $L(m, n)$ be an L-alphabet supergrid graph with $m, n \geq 3$. Then, $L(m, n)$ contains a Hamiltonian cycle.

By similar partition, we can separate the other types of alphabet supergrid graphs below.
Lemma 4.7. The following lemma can be easily verified.

(1) A partition of an alphabet supergrid graph \(C(m, n)\) is a separation of \(C(m, n)\) into an \(L\)-alphabet supergrid subgraph \(C_1 = L(m, n)\) and a rectangular supergrid subgraph \(C_2 = R(2m - 2, n)\).

(2) A partition of an \(F\)-alphabet supergrid graph \(F(m, n)\) is a separation of \(F(m, n)\) into an \(L\)-alphabet supergrid subgraph \(F_1 = L(m, n)\) and a rectangular supergrid subgraph \(F_2 = R(m, n)\).

(3) A partition of an \(E\)-alphabet supergrid graph \(E(m, n)\) is a separation of \(E(m, n)\) into an \(F\)-alphabet supergrid subgraph \(E_1 = F(m, n)\) and a rectangular supergrid subgraph \(E_2 = R(2m - 2, n)\).

Fig. 20 depicts the partitions of the above alphabet supergrid graphs. By similar arguments in proving Lemma 4.6, the following lemma can be easily verified.

Lemma 4.7. Let \(C(m, n)\), \(F(m, n)\), and \(E(m, n)\) be an \(C\)-alphabet, an \(F\)-alphabet, and an \(E\)-alphabet supergrid graph, respectively, for \(m, n \geq 3\). Then, \(C(m, n)\), \(F(m, n)\), and \(E(m, n)\) contain Hamiltonian cycles.

We have proved that rectangular, parallelism, and alphabet supergrid graphs are Hamiltonian. The following theorem concludes these results.

Theorem 4.8. Let \(A(m, n)\) be a rectangular or parallelism supergrid graph with \(m, n \geq 2\), and let \(B(m, n)\) be an \(L\)-alphabet, \(C\)-alphabet, \(F\)-alphabet, or \(E\)-alphabet supergrid graph with \(m, n \geq 3\). Then, \(A(m, n)\) and \(B(m, n)\) contain Hamiltonian cycles.
5. Concluding remarks

In this paper, we first proposed a novel class of graphs, namely supergrid graphs. The supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. We also give an application to the Hamiltonian properties of supergrid graphs. Then, we prove that the Hamiltonian cycle and Hamiltonian path problems for supergrid graphs are NP-complete. Furthermore, we construct Hamiltonian cycles on some subclasses of supergrid graphs, including rectangular, parallelism, and alphabet supergrid graphs. It is interesting to see whether the Hamiltonian problems for the other subclasses of supergrid graphs, including solid and linear convex, are polynomial solvable. We would like to post it as an open problem to interested readers.

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