Nonlinear Adaptive Control for Electromagnetic Actuators

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Abstract
We study here the problem of robust ‘soft-landing’ control for electromagnetic actuators. The soft landing requires accurate control of the actuators moving element between two desired positions. We present here two nonlinear adaptive controllers to solve the problem of robust trajectory tracking for the moving element. The first controller is based on classical nonlinear adaptive technique. We show that this controller ensures bounded tracking errors of the reference trajectories and bounded estimation error of the uncertain parameters. Second, we present a controller based on the so-called Input-to-State Stability (ISS), merged with gradient descent estimation filters to estimate the uncertain parameters. We show that it ensures bounded tracking errors for bounded estimation errors, furthermore, due to the ISS results we conclude that the tracking errors bounds decrease as function of the estimation errors. We demonstrate the effectiveness of these controllers on a simulation example.

1 Introduction
In many practical applications such us valves of combustion engines or artificial hearts, electromagnetic actuators are preferred to other type of actuators. In this work we concentrate on a particular control problem of nonlinear electromagnetic actuator called ‘soft landing’ problem. The soft landing requires accurate control of the moving element of the actuator between two desired positions. This ‘soft-landing’ performance has to be guaranteed over long period of time during which the actuator components may age. The main objective is to attain small contact velocity, which in turn ensures low component-wear operation of the actuator. Due to these practical constraints we have developed a robust control algorithm that aims for a zero impact velocity, and adapts to the actuator aging parts. We present here the results of this study.

Many papers have been dedicated to the soft-landing problem for electromagnetic actuators, e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9]. Several controllers have been developed in [1, 4, 5, 9] based on linear models of the system. Linear models allow a relatively easy design of the control but due to their linearity, are not valid for a full operation range of the actuator. To control the system over a larger operating state space, the controller has to be based on more complex nonlinear models of the actuators. Different nonlinear controllers have been used in [2, 3, 6, 8, 10, 11]. For example in [6], the authors proposed a nonlinear controller to solve the problem of armature stabilization for an electromechanical valve actuator. The authors proved a global asymptotic stability result using Sontag’s nonlinear controller. However, this approach did not solve the problem of armature trajectory tracking and did not consider robustness of the controller with respect to system’s uncertainties and changes in parameters over time. In [2], the authors studied the problem of electromagnetic valve actuator control in an internal combustion engine. The solution proposed by the author is based on iteratively solving a constrained nonlinear optimal problem using Nelder-Mead algorithm. The robustness of this feedforward-based approach has neither been proven nor tested. In [11], the authors designed a backstepping based controller for electromagnetic actuators position regulation. However, robustness w.r.t. uncertainties in parameters of the system are not considered in this paper. In [8], a nonlinear sliding mode approach was used to solve the problem of trajectory tracking for an electromagnetic valve actuator. The authors used a nonlinear model to design the sliding mode control. The reported results showed good tracking performances, however, this sliding mode controller does not ensure robustness with respect to model uncertainties. In [3], the authors used a single parameter extremum seeking learning method to solve the problem of soft landing for an electromechanical valve actuator. In [12] a

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Consider the nonlinear time-varying dynamical system
\begin{equation}
\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \geq t_0
\end{equation}
where \( x(t) \in D \subseteq \mathbb{R}^n \) such that \( 0 \in D, f : [t_0, t_1] \times D \to \mathbb{R}^n \) is such that \( f(t, \cdot) \) is jointly continuous in \( t \) and \( x \), and for every \( t \in [t_0, t_1] \), \( f(t, 0) = 0 \) and \( f(t, \cdot) \) is locally Lipschitz in \( x \) uniformly in \( t \) for all \( t \) in compact subsets of \( [0, \infty) \). The above assumptions guarantee the existence and uniqueness of the solution \( x(t) \) over the interval \([t_0, t_1]\). Without loss of generality, we assume \( t_0 = 0 \).

**Definition 1.** (LaSalle-Yoshizawa [14]) Consider the time-varying system (2.1) and assume \([0, \infty) \times D\) is a positively invariant set with respect to (2.1) where \( f(t, \cdot) \) is Lipschitz in \( x \), uniformly in \( t \). Assume there exist a \( C^1 \) function \( V : [0, \infty) \times D \to \mathbb{R} \), continuous positive definite functions \( W_1(\cdot) \) and \( W_2(\cdot) \) and a continuous nonnegative function \( W(\cdot) \), such that for all \((t, x) \in [0, \infty) \times D\),
\begin{equation}
W_1(x) \leq V(t, x) \leq W_2(x),
\end{equation}
\begin{equation}
\dot{V}(t, x) \leq -W(x)
\end{equation}
hold. Then there exists \( D_0 \subseteq D \) such that for all \((t_0, x_0) \in [0, \infty) \times D_0\), \( x(t) \to \mathcal{R} \triangleq \{ x \in D : W(x) = 0 \} \) as \( t \to \infty \). If, in addition, \( D = \mathbb{R}^n \) and \( W_1(\cdot) \) is radially unbounded, then for all \((t_0, x_0) \in [0, \infty) \times \mathbb{R}^n\), \( x(t) \to \mathcal{R} \triangleq \{ x \in \mathbb{R}^n : W(x) = 0 \} \) as \( t \to \infty \).

**Definition 2.** (K Function [15]) A continuous function \( \alpha : [0, a) \to [0, \infty) \) is said to belong to class \( \mathcal{K} \) if it is strictly increasing and \( \alpha(0) = 0 \). It is said to belong to class \( \mathcal{K}_\infty \) if \( a = \infty \) and \( \alpha(r) \to \infty \) as \( r \to \infty \).

**Definition 3.** (KL Function [15]) A continuous function \( \beta : [0, a] \times [0, \infty) \to [0, \infty) \) is said to belong to class \( \mathcal{KL} \) if, for each fixed \( s \), the mapping \( \beta(r, \cdot) \) belongs to class \( \mathcal{K} \) with respect to \( r \) and, for each fixed \( r \), the mapping \( \beta(r, \cdot) \) is decreasing with respect to \( s \) and \( \beta(r, s) \to 0 \) as \( s \to \infty \).

**Definition 4.** (Integral Input-to-State Stability [16]) Consider the system
\begin{equation}
\dot{x} = f(t, x, u),
\end{equation}
where \( x \in \mathcal{D} \subseteq \mathbb{R}^n \) such that \( 0 \in \mathcal{D}, \) and \( f : [0, \infty) \times \mathcal{D} \times \mathcal{D}_u \to \mathbb{R}^n \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \) and \( u \), uniformly in \( t \). The inputs are assumed to be measurable and locally essentially bounded functions \( u : [0, \infty) \times \mathcal{D}_u \to \mathbb{R}^m \). Given any control \( u \in \mathcal{D}_u \) and any \( \xi \in \mathcal{D}_0 \subseteq \mathcal{D} \), there is a unique maximal solution of the initial value problem \( \dot{x} = f(t, x, u) \).
\( x(t_0) = \xi. \) Without loss of generality, assume \( t_0 = 0. \) The unique solution is defined on some maximal open interval, and it is denoted by \( x(t; \xi, u). \) System (2.3) is locally integral input-to-state stable (LiISS) if there exist functions \( \alpha, \gamma \in C_\text{loc} \) and \( \beta \in C_\text{loc} \) such that, for all \( \xi \in D_0 \) and all \( u \in D_u, \) the solution \( x(t; \xi, u) \) is defined for all \( t \geq 0 \) and

\[
\alpha(||x(t; \xi, u)||) \leq \beta(||\xi||, t) + \int_0^t \gamma(||u(s)||)ds
\]

for all \( t \geq 0. \) Equivalently, system (2.3) is LiISS if and only if there exist functions \( \beta \in C_\text{loc} \) and \( \gamma_1, \gamma_2 \in C \) such that

\[
\|x(t; \xi, u)|| \leq \beta(||\xi||, t) + \gamma_1 \left( \int_0^t \gamma_2(||u(s)||)ds \right)
\]

for all \( t \geq 0, \) all \( \xi \in D_0 \) and all \( u \in D_u. \)

**Definition 5. (Weakly Zero-Detectability [17])**

Let an output for the system (2.3) be a continuous map \( h: D \to \mathbb{R}^p, \) with \( h(0) = 0. \) For each initial state \( \xi \in D_0, \) and each input \( u \in D_u, \) let \( y(t; \xi, u) \) be the corresponding output function; i.e., \( y(t; \xi, u) = h(x(t; \xi, u)), \) defined on some maximal interval \( [0, T_{\xi, u}] \). The system (2.3) with output \( h \) is said to be weakly zero-detectable if, for each \( \xi \) such that \( T_{\xi, 0} = \infty \) and \( y(t; \xi, 0) \equiv 0, \) it must be the case that \( x(t; \xi, 0) \to 0 \) as \( t \to \infty. \)

### 3 System modelling

Following [11, 10, 3], we consider the nonlinear electromagnetic actuator model

\[
\begin{align*}
\frac{d^2x}{dt^2} &= k(x_0 - x) + \eta \frac{dx}{dt} - \frac{a i^2}{2(b+x^2)} + f_d, \\
u &= Ri + a \frac{di}{dt} - \frac{a i^2}{2(b+x^2)}.
\end{align*}
\]

where, \( x \) represents the armature position physically constrained between the initial position of the armature \( 0, \) and the maximal position of the armature \( x_f, \) \( \frac{dx}{dt} \) represents the armature velocity, \( m \) is the armature mass, \( k \) the spring constant, \( x_0 \) is the initial length of the spring, \( \eta \) the damping coefficient (assumed to be constant), \( \frac{a i^2}{2(b+x^2)} \) represents the electromagnetic force (EMF) generated by the coil, \( a, b \) being constant parameters of the coil, \( f_d \) a constant term modelling disturbance forces, e.g., static friction, \( R \) the resistance of the coil, \( L = \frac{a}{a + b} \) the inductance (assumed to be armature-position dependent). \( \frac{di}{dt} \) represents the back EMF. Finally, \( i \) denotes the coil current, \( \frac{di}{dt} \) its time derivative and \( u \) represents the control voltage applied to the coil. In this model we do not consider the saturation region of the flux linkage in the magnetic field generated by the coil, since we assume a current and armature motion ranges within the linear region of the flux.

### 4 Adaptive Nonlinear Backstepping Control

#### 4.1 Classical Backstepping Adaptive Controller

Consider the dynamical system (3.6). Defining the state vector \( z := [z_1, z_2, z_3]^T = [x, \dot{x}, \ddot{x}]^T, \) the objective of the control is to make the variables \( (z_1, z_2) \) track a sufficiently smooth (at least \( C^2 \)) time-varying position and velocity trajectories \( z_1^{ref}(t), z_2^{ref}(t) = \frac{dx_1^{ref}(t)}{dt}, \) that satisfy the following constraints:

\[
\begin{align*}
z_1^{ref}(t_0) &= z_{1, in}, \\
z_1^{ref}(t_f) &= z_1, \\
z_2^{ref}(t_0) &= z_2^{ref}(t_f) = 0, \\
z_2^{ref}(t_0) &= z_2^{ref}(t_f) = 0,
\end{align*}
\]

where \( t_0 \) is the starting time of the trajectory, \( t_f \) is the ending time, \( z_{1, in} \) is the initial position and \( z_1 \) is the final position. To start, let us first write the system (3.6) in the following way:

\[
\begin{align*}
\dot{z_1} &= z_2, \\
\dot{z_2} &= \frac{k}{m}(x_0 - z_1) + \eta \frac{dx}{dt} - \frac{a i^2}{2(b+x^2)} + f_d, \\
\dot{z_3} &= - \frac{R}{m} z_3 + \frac{c_1}{m} z_1 + \frac{c_3}{m} (z_1 - z_1^{ref}) + c_1 (z_2 - z_2^{ref}).
\end{align*}
\]

Together with the control input

\[
\begin{align*}
u &= \frac{a}{a + b} \left( \frac{R(x_0 - z_1)}{m} - \frac{2a i^2}{2(b+x^2)} \right) + \frac{a}{a + b} (-c_2 (z_2^{ref} - z_2) - \frac{a}{2m} \frac{a i^2}{2(b+x^2)} (z_2 - z_2^{ref})) \\
&\quad - \frac{m(b+x^2)}{m^2} \left( \frac{c_2}{m} z_2 + \frac{c_3}{m} (x_0 - z_1) + \frac{c_3}{m} (z_1 - z_1^{ref}) + c_1 (z_2 - z_2^{ref}) + \frac{c_2}{m} (x_0 - z_1) + \frac{c_3}{m} (z_1 - z_1^{ref}) + \frac{c_3}{m} (z_1 - z_1^{ref}) + \frac{c_3}{m} (z_1 - z_1^{ref}) \right)
\end{align*}
\]

In addition, we will make use of the following equations for the parameter estimation dynamics:

\[
\begin{align*}
\dot{k} &= \gamma_2 (z_2 - z_2^{ref}) - (\dot{\theta} + mc_1) (\frac{z_2^{ref}}{z_2^{ref} - z_2}) \\
\dot{\theta} &= \gamma_2 \gamma_2 (z_2 - z_2^{ref}) - (\dot{\theta} + mc_1) (\frac{z_2^{ref}}{z_2^{ref} - z_2}) \\
\dot{f_d} &= \gamma_2 \gamma_2 (z_2 - z_2^{ref}) - (\dot{\theta} + mc_1) (\frac{z_2^{ref}}{z_2^{ref} - z_2})
\end{align*}
\]

with \( \gamma_1, \gamma_2, \gamma_3 > 0 \) design parameters. The controller (4.9), (4.10) and (4.11) is obtained by the constructive proof of the next lemma.

**Lemma 4.1.** Consider the closed-loop dynamics given by (4.8), (4.9), (4.10) and the parameter update laws...
given by (4.11). Then, there exist positive gains $c_i, \sigma_i$, $i = 1, 2, 3$ such that $z_1(t), z_2(t), z_3(t), \hat{k}(t), \hat{n}(t), \hat{f}_d(t)$ are globally bounded, and satisfy $\lim_{t \to \infty} z_2(t) = z_2^{ref}(t)$, $\lim_{t \to \infty} z_3^2(t) = \bar{u}_{ref}(t)$.

Proof. Primarily, we consider the mechanical subsystem, described by the dynamics (4.8), with the virtual control input $\tilde{u} := z_3^2$. Consider the Lyapunov function

$$V_{subad} = \frac{1}{2}(z_1 - z_1^{ref})^2 + \frac{1}{2}(z_2 - z_2^{ref})^2 + \frac{1}{2m} (k - k)(k - k) + \frac{1}{2m} (n - n)(n - n) + \frac{1}{2m} (f_d - f_d)^2,$$

where $z_1^{ref}$ and $z_2^{ref}$ are known $C^k$ functions, $c_3, \sigma_1, \sigma_2, \sigma_3 > 0$ are design parameters, and $\hat{k}, \hat{n}, \hat{f}_d$ are estimates of the actual parameters $k, n$ and $f_d$. Taking the derivative of $V_{subad}$ along the first two equations of (4.8), we get

$$\dot{V}_{subad} = -(z_2 - z_2^{ref})^2 + c_{22}(z_2 - z_2^{ref}) + c_{21}(z_2 - z_2^{ref}) - c_{21}(z_2^{ref}) - c_{22}(z_2^{ref}).$$

Substituting (4.19) into (4.12) , and defining $e_k := k - \hat{k}, e_n := n - \hat{n}, e_f := f - \hat{f}_d$ we have

$$\dot{V}_{subad} = -c_{21}(z_2 - z_2^{ref})^2 + c_{22}(z_2^{ref} - z_2^{ref}) - c_{21}(z_2^{ref}) - c_{22}(z_2^{ref}).$$

Next, we define the augmented Lyapunov function for the full system, $V_{augad} = V_{subad} + \frac{e_2^2}{2}$ with $e := z_3^2 - \bar{u}$. Taking the derivative along the trajectories of the whole system and utilizing (4.9), we obtain

$$\dot{V}_{augad} = -(z_2 - z_2^{ref})^2 + c_{21}(z_1^{ref} - z_1^{ref}) - c_{21}(z_2^{ref}) - c_{22}(z_2^{ref}) - c_{21}(z_2^{ref}) - c_{22}(z_2^{ref}).$$

Rewriting (4.14) by grouping the terms involving $e_k, e_n$ and $e_f$, we get the following inequality:

$$\dot{V}_{augad} = -c_{21}(z_2^{ref} - z_2^{ref})^2 + c_{22}(z_2^{ref} - z_2^{ref}) - c_{21}(z_2^{ref}) - c_{22}(z_2^{ref}).$$

Finally, LaSalle-Yoshizawa Theorem implies the regulation of $z_2$ to $z_2^{ref}$ and $z_3^2$ to $\bar{u}$.

Remark 1. Notice that $\hat{k}, \hat{n}$ and $\hat{f}_d$ do not necessarily converge to $k, n$ and $f_d$ when the control scheme discussed in Lemma 4.1 is utilized. This prevents us from provving convergence of $z_2$ to $z_2^{ref}$ by analyzing the zero dynamics of the mechanical subsystem. Hence, by analyzing the zero dynamics of (4.9), (4.10) and (4.11) with the output $(z_2 - z_2^{ref}), z_3^2 - \bar{u}$, we can only conclude about the boundedness of $\|z_2 - z_2^{ref}\|$.

4.2 ISS Adaptive Backstepping Controller

We now address the control problem of the adaptive trajectory tracking with asymptotic convergence of the estimation errors $e_k, e_n$ and $e_f$. First, the backstepping controller is modified as follows:

$$u = \frac{e}{2m} + \frac{c_{21}(z_1^{ref} - z_1^{ref}) + c_{22}(z_2^{ref} - z_2^{ref}) - c_{21}(z_2^{ref}) - c_{22}(z_2^{ref})}{2m} \frac{e_2}{2m} + \frac{c_{22}(z_2^{ref} - z_2^{ref})}{2m} \frac{e_f}{2m} - \frac{c_{22}(z_2^{ref} - z_2^{ref})}{2m} \frac{e_f}{2m} - \frac{c_{22}(z_2^{ref} - z_2^{ref})}{2m} \frac{e_f}{2m}.$$

Rewriting (4.14) by grouping the terms involving $e_k, e_n$ and $e_f$, we get the following inequality:
where the uncertain parameters $k$, $\eta$, $f_d$ have been replaced by their estimated parameters $\hat{k}$, $\hat{\eta}$, $\hat{f}_d$, with $\psi \triangleq \left[ \frac{z_m}{a_{2,1}} z_{m} \frac{1}{b+1} \right]^T$. We can now state the following lemma.

**Lemma 4.2.** Consider the closed-loop dynamics given by (4.8), (4.18) and (4.19), with constant but unknown parameters $k$, $\eta$, $f_d$ and the parameter error vector $\Delta \triangleq \left[ k - \hat{k} \quad \eta - \hat{\eta} \quad f_d - \hat{f}_d \right]^T$. Then, there exist positive gains $c_1$, $c_2$, $c_3$, $\kappa_1$, $\kappa_2$ and $\kappa_3$ such that $(z_1(t), z_2(t))$ are uniformly bounded and the system (4.8) is locally integral-input-to-state stable (LiISS) with respect to $(\Delta, \Delta)$.

**Proof.** Consider the full mechanical subsystem that consists of only the first two equations with the virtual control input $\tilde{u} := \tilde{z}_3^2$:

\begin{equation}
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= \frac{k}{m} (x_0 - z_1) + \frac{\eta}{m} z_2 + \frac{f_d}{m} - \frac{a}{2m(b+1)} \tilde{z}_3^2 \tilde{u}.
\end{aligned}
\end{equation}

Defining the Lyapunov function $V_{sub} = \frac{c_3}{2} (z_1 - z_{ref}^2)^2 + \frac{1}{2} (z_2 - z_{ref}^2)^2$, with $c_3 > 0$, we would like to design $\tilde{u}$ so that $V_{sub} = -c_1 (z_2 - z_{ref}^2)^2$ along the trajectories of (4.20), but since the system parameters $k$, $\eta$ and $f_d$ are unknown, we design the virtual input to be $\tilde{u}$ given by (4.19). Inserting $\tilde{u}$ from (4.19) into $V_{sub}$, we have the following derivation:

\begin{equation}
\begin{aligned}
\dot{V}_{sub} &= c_3 (z_1 - z_{ref}^2) (\dot{z}_1 - z_{ref}) + (z_2 - z_{ref}^2) (\dot{z}_2 - z_{ref}) \\
&= (z_2 - z_{ref}) (c_3 (z_1 - z_{ref}^2) + \frac{\eta}{m} (x_0 - z_1) + \frac{f_d}{m}) \\
&= -\frac{z_{ref}^2}{2m(b+1)} \hat{a} \tilde{z}_3^2 \tilde{u} \\
&= -c_3 (z_2 - z_{ref}^2)^2 - c_1 (z_2 - z_{ref}^2) \|\psi\|^2 + \frac{1}{2} \left| \Delta \| \psi \|_2 \right|^2.
\end{aligned}
\end{equation}

Using the definitions of the vectors $\psi$ and $\Delta$, we have

\begin{equation}
\begin{aligned}
\dot{V}_{sub} \leq c_3 (z_2 - z_{ref}^2)^2 + \frac{1}{2} \left| \Delta \| \psi \|_2 \right|^2 \\
&\leq -c_1 (z_2 - z_{ref}^2)^2 - c_3 (z_2 - z_{ref}^2)^2 - c_1 (z_2 - z_{ref}^2) \|\psi\|^2 + \frac{1}{2} \left| \Delta \| \psi \|_2 \right|^2.
\end{aligned}
\end{equation}

where $\Delta = \left[ k - \hat{k} \quad \eta - \hat{\eta} \quad f_d - \hat{f}_d \right]^T$ is the vector holding the discrepancy between actual system parameters and estimated parameters. Note that we have made use of the nonlinear damping term $-c_1 (z_2 - z_{ref}^2)^2 \|\psi\|^2$ to attain a negative quadratic term of $\psi$ and $\Delta$ (i.e., $-c_1 \left| z_2 - z_{ref}^2 \right| \|\psi\|^2_2 - \frac{1}{2} \left| \Delta \| \psi \|_2 \right|^2$)

and a positive term that is a function of $\Delta$ only ($\frac{1}{2} \left| \Delta \| \psi \|_2 \right|^2$), hence rendering $V_{sub}$ an iISS-Lyapunov function for the mechanical subsystem. Next, we define the Lyapunov function for the full system $V_{aug} = V_{sub} + \frac{(z_2^2 - \bar{u})^2}{2}$. Taking the derivative of $V_{aug}$ along the trajectories of the full system, leads to the following inequality:

\begin{equation}
\begin{aligned}
\dot{V}_{aug} \leq -c_1 (z_2 - z_{ref}^2)^2 - c_3 (z_2 - z_{ref}^2)^2 - c_1 (z_2 - z_{ref}^2) \|\psi\|^2 + \frac{1}{2} \left| \Delta \| \psi \|_2 \right|^2.
\end{aligned}
\end{equation}

where $\bar{u}$ writes as

\begin{equation}
\begin{aligned}
\bar{u} &= \frac{m(b+1)}{2} \left( \frac{k}{m} (x_0 - z_1) + \frac{\eta}{m} z_2 + \frac{f_d}{m} - \frac{a}{2m(b+1)} \tilde{z}_3^2 \tilde{u} \right) \left( \|\psi\|^2 - c_3 (z_2 - z_{ref}^2)^2 \right)
\end{aligned}
\end{equation}

By substituting the control input given in (4.18) into (4.23), we attain the following inequality:

\begin{equation}
\begin{aligned}
\dot{V}_{aug} \leq -c_1 (z_2 - z_{ref}^2)^2 - c_3 (z_2 - z_{ref}^2)^2 - c_1 (z_2 - z_{ref}^2) \|\psi\|^2 + \frac{1}{2} \left| \Delta \| \psi \|_2 \right|^2.
\end{aligned}
\end{equation}

Using the aforementioned definitions of the vectors $\psi$ and $\Delta$, and noting that $\Delta = \left[ \frac{k}{m} \quad \frac{\eta}{m} \quad \frac{f_d}{m} \right]^T$, we can further bound $\dot{V}_{aug}$ in the following way:

\begin{equation}
\begin{aligned}
\dot{V}_{aug} \leq -c_1 (z_2 - z_{ref}^2)^2 - c_3 (z_2 - z_{ref}^2)^2 - c_1 (z_2 - z_{ref}^2) \|\psi\|^2 + \frac{1}{2} \left| \Delta \| \psi \|_2 \right|^2.
\end{aligned}
\end{equation}

By making use of the nonlinear damping terms the same way as they have been utilized in deriving (4.22), we get

\begin{equation}
\begin{aligned}
\dot{V}_{aug} \leq -c_1 (z_2 - z_{ref}^2)^2 - c_3 (z_2 - z_{ref}^2)^2 - c_1 (z_2 - z_{ref}^2) \|\psi\|^2 + \frac{1}{2} \left| \Delta \| \psi \|_2 \right|^2.
\end{aligned}
\end{equation}
Finally, using the inequality (4.27), we have
\begin{equation}
\dot{V}_{awg} \leq c_1(z_2 - z_2^{ref})^2 + c_2(z_2 - \hat{u})^2 + \frac{1}{\tau_1^2} + \frac{1}{\tau_2^2} \| \Delta \|^2
\end{equation}
(4.28)

It is easy to see that the uncertain system can be expressed in the following nonlinear time-varying form:
\begin{equation}
\dot{e} = f(t, e, \dot{\Delta}),
\end{equation}
(4.29)

with
\[ e \in D_e, \Delta \in D_\Delta, \]
where
\[ e := [z_1 - z_1^{ref}, z_2 - z_2^{ref}, z_3 - \hat{u}]^T \]
and
\[ \Delta = [\Delta, \dot{\Delta}]^T. \]

Then, by considering the output map defined by
\[ h = [z_2 - z_2^{ref}, z_3 - \hat{u}]^T, \]
we can show that the system (4.29) with \( h \) is weakly zero-detectable (i.e. using an analysis of the zero-dynamics of (4.29) with \( h = \Delta \equiv 0 \)). Next, using the weakly-zero-detectability property together with inequality (4.28), we can conclude (via some additional steps, which are not included here due to space limitations, but will be included in a journal version of this work) that system (4.29) is LiISS with respect to the input \( \Delta \), implying that there exist functions \( \alpha \in \mathcal{K}, \beta \in \mathcal{KL} \) and \( \gamma \in \mathcal{K} \), such that, for all \( e(0) \in D_e \) and \( \Delta \in D_\Delta, e \) is defined and
\begin{equation}
\| e(t) \| \leq \beta(\| e(0) \|, t) + \alpha \left( \int_0^t \gamma(||\Delta||)ds \right)
\end{equation}
(4.30)
for all \( t \geq 0. \)

### 4.2.1 Estimation Module
The motivation behind proving that the system is LiISS with respect to \( (\Delta, \dot{\Delta}) \) is, if by an estimation method, the vectors \( \| \Delta_2 \| \) and \( \| \Delta_2 \| \) can be taken to 0, then we can claim via (4.30) that the system becomes stable. The advantage of using this method is that it provides modularity in the sense that the control law can be designed independently from the estimation law. In this way, it would be sufficient to design an estimation law that will take \( \| \Delta_2 \| \) and \( \| \Delta_2 \| \) to 0 sufficiently fast. To this purpose, we use a gradient descent-based filters [18]. We have three parameters that are varying over time \( k, \eta, f_d \). These parameters enter the dynamics through the following equation:
\begin{equation}
z_2 = f(z, u) + F(z, u)\dot{\theta}_e := -\frac{p_3^2}{m_1^2} \left[ \begin{array}{c} \frac{x_0 - x_1}{m} \\ \frac{x_1}{m} \end{array} \right]^T \begin{bmatrix} k & \eta \\ \eta & f_d \end{bmatrix},
\end{equation}
(4.31)

The main problem with estimation for the system at hand is there is only a single equation through which the uncertain parameters enter the dynamics (4.31). Hence, using the x-Swapping Scheme given in [18], we can only estimate one parameter at a time. To this purpose, we state the following assumption:

**Assumption 1.** The uncertain parameters \( k, \eta \) and \( f_d \) vary slowly, and over a given period of time, only a single parameter can change while the others stay constant.

We use the following equations for the estimation filters [18]:
\begin{equation}
\dot{\hat{\Delta}} = (A_0 - \lambda F(z, u)^T F(z, u) + F(z, u)^T \Delta_0) \Omega^T + F(z, u)^T
\end{equation}
(4.32)
\begin{equation}
\dot{\Delta}_0 = (A_0 - \lambda F(z, u)^T F(z, u) + F(z, u)^T) \Omega_0 - z + f(z, u)
\end{equation}
(4.33)
with the gradient law for updating estimated parameters:
\begin{equation}
\dot{\Omega} = \Gamma (\Omega - \Omega_0), \quad \dot{\Omega}_0 = \Gamma (\Omega_0 - \Omega_0^T)
\end{equation}
(4.34)

In the filter equations given in (4.32), \( A_0, P = P^T > 0 \) are constant, design matrices that satisfy the Lyapunov equation, \( PA_0 + A_0 P = -I \), and \( \Lambda \) is a design variable. Since we estimate only one parameter at a time, the equations become scalar for each parameter. The following equations are used for estimating \( k, \eta \) and \( f_d \) separately:
- For the parameter \( \hat{k} \):
\begin{equation}
\begin{aligned}
\dot{\hat{\theta}} &= (A_0 - \lambda \frac{x_0 - x_1}{m}) \Omega + \frac{x_1}{m} \\
\dot{\Omega} &= (A_0 - \lambda \frac{x_0 - x_1}{m}) \Omega + \frac{x_0 - x_1}{m} \\
\dot{\Omega}_0 &= (A_0 - \lambda \frac{x_0 - x_1}{m}) \Omega_0 - z + f(z, u)
\end{aligned}
\end{equation}
(4.35)

- For the parameter \( \hat{\eta} \):
\begin{equation}
\begin{aligned}
\dot{\hat{\eta}} &= (A_0 - \lambda \frac{x_0 - x_1}{m}) \Omega + \frac{x_0 - x_1}{m} \\
\dot{\Omega} &= (A_0 - \lambda \frac{x_0 - x_1}{m}) \Omega + \frac{x_0 - x_1}{m} \\
\dot{\Omega}_0 &= (A_0 - \lambda \frac{x_0 - x_1}{m}) \Omega_0 - z + f(z, u)
\end{aligned}
\end{equation}
(4.36)

**Lemma 4.3.** Consider the closed-loop dynamics given by (4.8), (4.18) and (4.19), with an unknown parameter \( k, \eta, \) or \( f_d \). Then, under Assumption 1, there exist positive gains \( c_1, c_2, c_3, k_1, k_2, k_3 \) such that the closed-loop dynamics given by (4.8), (4.18), (4.19) and the filters (4.34), (4.35), (4.36) are stable, and that the unknown parameter is asymptotically estimated.

**Proof.** The proof is straightforward from the result of Lemma 4.1, and the known convergence properties of the gradient descent-based filters [18].
5 Simulations

We show here the behavior of the proposed approach on the example of electromagnetic actuator presented in [11], where the model (3.6) is used with the numerical values of Table 1. The desired trajectory has been selected as the 5th order polynomial \(x_{ref}(t) = \sum_{i=0}^{5} a_i (t/t_f)^i\), where the \(a_i\)s have been computed to satisfy the boundary constraints \(x_{ref}(0) = 0.2, x_{ref}(t_f) = x_f, \dot{x}_{ref}(0) = \ddot{x}_{ref}(t_f) = 0, \dddot{x}_{ref}(0) = \dddot{x}_{ref}(t_f) = 0\), with \(t_f = 0.5 \text{ sec}, x_f = 0.85 \text{ mm}\). Due to space limitations, we only report hereafter the results of the ISS adaptive backstepping controller. However, we can underline here that the first adaptive controller leads to numerical results in concordance with the theoretical analysis, i.e., convergence of the armature velocity to the desired velocity, with bounded position tracking error and bounded uncertain parameters estimation errors. To test the ISS adaptive controller, we considered the following scenario: We considered uncertainties in the model appearing sequentially over time. First, at \(t = 0 \text{ sec}\), we considered that the parameter \(k\) has an error of 16%. Next, we consider that at \(t = 38 \text{ sec}\), the parameter \(\eta\) sustains an error of 50%, finally at \(t = 75 \text{ sec}\), we assume a disturbance force \(f_d\) of \(-50N\) (static friction force). We simulated the controller (4.18) and (4.19) with the gains \(c_1 = 100, c_2 = 100, c_3 = 50, \kappa_1 = \kappa_2 = \kappa_3 = 1\). For the filters (4.34), (4.35), (4.36), we used the gains \(A_0 = -0.5, P = 1, \lambda = 1, \Gamma = 100\). We underline here that, due to the structure of the model, we could estimate only one parameter at the time (see Section 4.2.1). We see clearly on Figures 1, 2, 3 that the numerical results are concordant with the theoretical analysis, since the estimated parameters converge all to their actual value. Furthermore, we see on Figures 4, 5 that we achieve very good tracking of both the desired position and the desired velocity trajectories.

Table 1: Numerical values of the mechanical parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m)</td>
<td>0.27 [kg]</td>
</tr>
<tr>
<td>(R)</td>
<td>6 [(\Omega)]</td>
</tr>
<tr>
<td>(\eta)</td>
<td>-0.25 [kg/(\text{sec})]</td>
</tr>
<tr>
<td>(x_0)</td>
<td>8 [mm]</td>
</tr>
<tr>
<td>(k)</td>
<td>75 [N/mm]</td>
</tr>
<tr>
<td>(a)</td>
<td>(14.96 \times 10^{-6} [N \text{m}^2/A^2])</td>
</tr>
<tr>
<td>(b)</td>
<td>(4 \times 10^{-5} [m])</td>
</tr>
</tbody>
</table>

Figure 1: Estimation of \(k\) over time

Figure 2: Estimation of \(\eta\) over time

Figure 3: Estimation of \(f_d\) over time

Figure 4: Moving element actual position vs. desired position
6 Conclusion

We have studied in this paper the problem of adaptive control for electromagnetic actuators. We have developed two trajectory tracking controller based on adaptive backstepping approaches. We have studied the stability properties of the proposed controller and shown the performance on a numerical example.

References


