Lyapunov Based Reasoning Methods
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Abstract—Semiquantitative simulation is a novel approach for the analysis of uncertain dynamic systems that performs a comprehensive simulation study based on automated reasoning methods (for example with the SQSIM algorithm). Semiquantitative simulation of complex models is, however, hindered by the limited automated reasoning capabilities of the currently available semiquantitative simulation techniques. This paper describes the extension of semiquantitative simulation techniques on the basis of Lyapunov methods. This extension improves automated reasoning by utilizing generalized energy functions, called Lyapunov functions. Automated reasoning based on Lyapunov functions can be seen as a generalization of the energy considerations employed by engineers. It has the advantage that it can be used to analyze systems where it does not make sense to speak about energy in the physical sense. The difficult task of deducing a Lyapunov function for the semiquantitatively modeled dynamic system is solved by reformulating methods from nonlinear control theory. A procedure for an automatic deduction of a Lyapunov function and Lyapunov-based reasoning methods using this deduced Lyapunov function are given. The improved automated reasoning capabilities of our extended SQSIM simulation platform are demonstrated by example.

Index Terms—-Lyapunov analysis, qualitative simulation, semi-quantitative simulation, uncertain dynamic systems.

I. INTRODUCTION

A COMMON task in engineering is to design physical artifacts that exhibit certain dynamic behavior. For example, in control engineering, we design control devices so that the behavior of a controlled physical mechanism meets specified requirements. It is common to use simulation studies to verify that a design will produce an artifact with desired dynamic behavior. Difficulties arise from the fact that our knowledge about the world is often incomplete so that it is not possible to provide a precise and complete mathematical model of the dynamic system under investigation.

A novel approach for the task of analyzing ill defined dynamic systems is followed by research in qualitative reasoning (QR) [1], [2]. The goal of QR is to identify the structure of a dynamic system, i.e., to model it, and to predict and explain its possible behaviors. QR research is motivated by the observation that engineers often reason about the behavior of dynamic systems without knowing a precise and complete mathematical model. QR methods intend to replicate, in the computer, parts of human reasoning for the tasks of automated modeling [3], [4], intelligent numerical simulation [5], qualitative [6], [7] and semiquantitative simulation [8], [9], and causal explanation [10].

Our work deals with the QR subtask of semiquantitative simulation and extends the SQSIM framework [9], which is based on the QSIM algorithm of Kuipers [7]. Semiquantitative simulation provides a computer aided tool that allows the user to simulate ill-defined dynamic systems on a similar level of abstraction as engineers would reason about them. For example, consider the damped spring-mass system with constant external force $F$ shown in Fig. 1.

The schematic of the spring-mass system, together with basic physics knowledge, allows an engineer to model the system in qualitative terms. For example, no information is given concerning the stiffness of the spring. Nevertheless, it is possible to state qualitatively that the repelling force of the spring increases with the displacement. Qualitative relationships of this form provide the necessary level of detail that allows an engineer to deduce the possible behaviors of the device in qualitative terms. For example, consider an initially displaced mass with zero velocity. An engineer would use the qualitative model, which relates displacement, velocity, and acceleration, to deduce that the mass will move toward its position of rest. Depending on the damping, it can either move into its rest position without overshooting or oscillate.

A semiquantitative simulation deduces the possible set of behaviors as a sequence of qualitatively distinct states similar to the engineer’s prediction given above. However, it offers the capability to supplement the qualitative model with imprecise numerical information, such as numerical bounds for constants (e.g., for the external force $F$) or functional relationships (e.g., for the function relating deacceleration and displacement). The term semiquantitative model is used to indicate the combined qualitative and quantitative model of the system under investigation. A semiquantitative simulation deduces the set of possible behaviors for the model starting from an imprecise initial state. It provides a qualitative behavioral description that is supplemented with numerical interval information. A possible prediction of a behavior that starts at an initial displacement in the range $[1.0, 2.0]$ and tendency steady would be: a semiquantitative state with a displacement in the range $[0.0, 2.0]$ and tendency...
Semiquantitative simulation with the SQSIM algorithm is sound in the sense that all possible behaviors for the ill-defined device are predicted [11]. Furthermore, the automated reasoning-based simulation technique provides a justification for each predicted behavior, thus explains the prediction. These properties make semiquantitative simulation a suitable tool for verifying specific properties of ill-defined devices, for example, proving that a controlled tank system cannot overflow.

However, the inclusive property of deducing all possible behaviors does not come without a price. The predicted set of behaviors may contain genuinely impossible behaviors, which cannot be shown by real-world physical systems. For example, an increasing oscillatory behavior for the damped spring-mass system. These so-called spurious behaviors are a result of the limited automated reasoning capabilities. The prediction of spurious behaviors represents a severe limitation for the application of semiquantitative simulation and much research was devoted to developing advanced automated reasoning techniques that filter spurious behaviors out of the simulation result (e.g., [9], [12]–[15]). Nevertheless, many spurious behaviors remain undetected and spoil the simulation result.

The spurious behavior cited above, can be easily identified by an engineer by considering the dissipation of energy in the damped spring-mass system. In fact, the so-called kinetic energy filter for qualitative simulation [15] provides this energy-based automated reasoning functionality. However, the application of this powerful filter is limited to spring-mass systems and systems which share the same model structure so that a more flexible approach for energy-based reasoning is sought.

Lyapunov theory [16], [17] extends the energy concept to systems where it does not make sense to speak about energy in the physical sense. It allows to draw conclusions about possible behaviors of the system based on examining the time variation of a scalar, energy-like function, called Lyapunov function. The broad applicability of Lyapunov theory makes it a natural choice for extending the energy-based automated reasoning capability of semiquantitative simulation. However, applying Lyapunov theory requires experience with sophisticated techniques from control theory. In particular, finding a Lyapunov function for a system under investigation is a nontrivial task.

This paper describes a framework for performing an automated Lyapunov analysis and Lyapunov-based automated reasoning in the context of semiquantitative simulation. This functionality is developed in Section III after providing a comprehensive description of semiquantitative simulation in Section II. After reviewing Lyapunov theory in Section III-A and providing the necessary background from control theory in Section III-B, we proceed in showing the similarities of semiquantitative models and the standard nonlinear control loop in Section III-C. This similarity allows us to frame the task of finding a Lyapunov function for the semiquantitative model as control system’s analysis problem (Section III-D). Section III-E describes Lyapunov-based automated reasoning concepts that improve semiquantitative simulation by eliminating many spurious behaviors. The example Section IV illustrates our automated Lyapunov analysis and reasoning capability on the basis of the damped spring-mass system introduced above and a controlled tank mechanism.

II. SEMIQUANTITATIVE MODELING AND SIMULATION

This section describes the concepts and notions of semiquantitative modeling and simulation [9], which extend qualitative modeling and simulation [11] by incorporating imprecise numerical information. The spring-mass system (Fig. 1) that was informally introduced in Section I will be used as example throughout this section.

A. Semiquantitative Modeling

The rough schematic in Fig. 1, together with basic physics knowledge, allows us to provide a model for the spring-mass system in terms of the structural and qualitative relationships between the variables \( x(t), v(t) \) and \( a(t) \), which represent the physical entities displacement, velocity and acceleration, respectively

\[
\begin{align*}
\dot{x} & = v, \\
\dot{v} & = a, \\
a & = - f_1(x) - f_2(v) + f_3(F) \\
F & = \text{constant} \\
f_1 & \in M^+, f_2 \in M^+, f_3 \in M^+.
\end{align*}
\]

For example, (3) denotes the relationship between acceleration \( a \) and the variables displacement \( x \), velocity \( v \) and the constant force \( F \). The qualitative relationship \( f_1(x) \in M^+ \) specifies that the deacceleration due to spring tension increases with the displacement \( x \). \( M^+ \) denotes the set of nonlinear, time-invariant, continuously differentiable functions \( f(x) \) with the property \( df(x)/dx > 0 \).

The variables are characterized qualitatively in terms of their magnitude \((q_{mag})\) and direction of change \((q_{dir})\). This abstraction is based on the observation that the domain of a variable’s \( q_{mag} \) and \( q_{dir} \) can be partitioned into a finite number of qualitatively distinct regions. The simplest qualitative abstraction partitions a variable’s range into negative and positive values. However, it is often possible to identify additional system specific values, called landmarks, which represent qualitatively important points in the domain of a variable. For example, \( x_{-\infty} \) in Fig. 1 represents the positive mechanical limit for the displacement \( x \). Landmarks serve as boundaries for qualitative distinct regions, and we can think of them as symbolic names for particular real numbers whose numerical values are imprecisely known. The ordered set of landmarks specifies the quantity space of a variable. The quantity space of the displacement \( x \), for example, is defined by

\[x \ldots [\minf 0 x_{-\infty}]\]

with the default landmark 0, which separates the quantity space in positive and negative values, and \( \minf \), which stands for \(-\infty\). Variables that do not have system specific qualitatively important values (e.g., \( v \) and \( a \)) are specified using the default quantity...
space composed of the landmarks $\text{min} f$, 0, and $\text{inf}$, where $\text{inf}$ stands for $\infty$.

The qualitative value (qual) of a variable is described by the tuple $(\text{qval}, \text{qdir})$, where the direction of change (qdir) can be either inc for increasing, std for steady, or dec for decreasing. The magnitude (qval) of a variable is specified by landmarks or intervals formed by two adjacent landmarks of the associated quantity space. Examples for qualitative values are: $(0, \text{std})$ for $x$ at its rest position and $(0 \ x-\text{max}, \text{dec})$ for a positively displaced mass moving toward its rest position.1

The constraints (1)–(5) among the variables are only valid for displacements within the operating region $(\text{min} x, \text{max})$. We have to use another set of constraints whenever the mass hits the mechanical limit $x-\text{max}$. Region transition rules in the SQSIM framework take care of such situations and switch between sets of constraints and quantity spaces whenever variables cross the boundaries between operational regions.

Imprecise numerical information about the system under investigation can be incorporated to refine the coarse qualitative description introduced above. For our spring-mass system, for example, we know that the numerical value for the maximal positive displacement $x-\text{max}$ lies between 9.0 and 10.0. We incorporate this information by assigning a numerical interval to the landmark $x-\text{max} = [9.0 \ 10.0]$. The functional uncertainty of functions $f(x)$ of class $M^+$ is specified in more detail by static envelope functions $f^{-}(x)$, $f^{+}(x)$, and numerical bounds $s^{-}$, $s^{+}$ for the slope $df(x)/dx$ so that

$$f^{-}(x) \leq f(x) \leq f^{+}(x), \quad \frac{df(x)}{dx} \in [s^{-} \ s^{+}]. \quad (6)$$

For example, the function $f_2(x)$, which models the spring tension, is specified by the envelope functions $f_2^{-}(x) = 300x$, $f_2^{+}(x) = 320x$ for $x \geq 0$ and $f_2^{-}(x) = 320x$, $f_2^{+}(x) = 300x$ for $x < 0$, and the numerical bounds for the slope $s_2^{-} = 20.0$, $s_2^{+} = 40.0$.2 Whenever a function is known without uncertainty, we can express this fact by $f^{-} = f^{+}$. For example, the function $f_3(F)$ of class $M^+$ shall be defined by $f_3^{-}(F) = f_3^{+}(F) = 10000.0$.3

The set of variables, the quantity spaces, the constraints, the uncertain numerical information for the landmarks and the monotonic functions, and the region transition rules define the semiquantitative differential equation (SQDE) description of an uncertain dynamic system. The term semiquantitative differential equation can be justified by the fact that an SQDE describes a set of ordinary differential equations. We shall say that an SQDE abstracts an uncertain ordinary differential equation that describes a set of autonomous ordinary differential equations. We use the notation

$$\dot{x} = f(x), \quad f(x) \in F(x) \quad (7)$$

to specify the uncertain ordinary differential equation (ODE). The variable $x$ represents the state vector $x := [x_1, \ldots, x_n]^T$ with the state variables $x_j$, which are continuously differentiable functions of time.3 The function $f(x)$ denotes a nonlinear vector function that is an instance of the uncertain nonlinear vector function $F(x) := [f_1(x), \ldots, f_n(x)]^T$. This uncertain vector function is defined by the constraints and the uncertain numerical information of the SQDE. Its components $f_i(x)$ are composed using arithmetic operations ($+, -, \times, /$), uncertain constants $c_j$, and uncertain nonlinear functions of class $M^+$.4

We use the notation $F^j_i$ to represent the set of nonlinear functions of class $M^+$ that is specified by envelope functions and slope bounds $\{ f^{-}_j, f^{+}_j, s^{-}_j, s^{+}_j \}$. The set of vector functions with elements $F^j_i$ of class $M^+$ is denoted by $F^M := [f^M_1, \ldots, f^M_n]$. Constants $c_j \in c_j$ are defined by numerical intervals $c^{-}_j \leq c_j \leq c^{+}_j$. The uncertain function $F(x)$ is composed of continuously differentiable, functions of time, basic arithmetic operations, and functions of class $M^+$ that are continuously differentiable by definition. Therefore, we can conclude that the functions $f(x) \in F(x)$ are continuously differentiable in their domain $D_f \subseteq \mathbb{R}^n$ that can be drawn from the quantity spaces and region transition rules of the SQDE.

A similar concept for defining sets of ordinary differential equations is given by differential inclusions [18]. The set of uncertain ordinary differential equations that is defined by an SQDE, however, is more restrictive in the sense that the functions $f(x) \in F(x)$ are continuously differentiable and time-invariant, whereas the set-valued functions that define the right-hand side of a differential inclusion include discontinuous and time-variant functions.

B. Semiquantitative Simulation

A sequence of semiquantitative values represents a continuous behavior over time. The sequence describes alternately values at specific time points $t_j$ and behaviors within time intervals $(t_j, t_{j+1})$. Releasing the spring at its mechanical limit $x-\text{max}$, for example, will result in a continuous behavior of $x(t)$ that is abstracted by the following sequence of semiquantitative values

$$\{x-\text{max}, \text{std}, (0 \ x-\text{max}, \text{dec}), (0, \text{dec}), \ldots\}$.

SQSIM deduces the sequence of semiquantitative values for each variable of the SQDE. The simulation is based on constraint satisfaction [19] and interval arithmetic [20] and is performed in two steps.

1) Given an SQDE and partial information about the initial state, deduce all consistent initial semiquantitative states.

2) Deduce the possible sequences of semiquantitative states that start at the previously assigned initial states and that are consistent with the information given by the SQDE (constraints, quantity spaces, region transitions, and imprecise numerical information) and continuity.

1We shall use the variable name typeset in courier font to denote the qualitative value, e.g., $x$ stands for qual(x).

2We have to provide envelope functions for positive and negative velocities as the linear envelopes intersect each other at the origin $r = 0$. So-called sector conditions, which will be introduced in Section III, will provide a more compact definition for linear envelope functions.

3In fact, in order to allow the application of automated reasoning methods, we have to be more restrictive and avoid functions whose qualitative property, e.g., their direction of change, changes infinitely often within any finite interval of time. Functions that satisfy this stronger condition are called reasonable functions in QR literature [11].
The second simulation step is solved by repeatedly deducing the possible changes of the variables that will lead to a distinct semiquantitative state.

Let us demonstrate this inference process using our spring-mass example. We assume that the initial situation is described as an initial displacement $x$ in the range $[1.0, 2.0]$, zero initial velocity $v = 0$ and the external force $F = 0$. This specification defines the $\text{qmag}$ of the displacement $x = \langle 0, \text{max}, \text{?} \rangle$, velocity $v = \langle 0, \text{?} \rangle$ and the force $F = \langle 0, \text{?} \rangle$. The first simulation step determines the initial state as follows: SQSIM inserts a new landmark $x=0$ for the initial magnitude of $x$ and assigns the interval $[1.0, 2.0]$ to it. The magnitude of $v$ and (1) specify the $\text{qdir}(x) = \text{std}$. The right-hand side of (3) provides a negative magnitude, thus specifying $\text{qdir}(v) = \text{dec}$. Constraint (4) specifies $\text{qdir}(F) = \text{std}$ and completes the initial state

$$
\begin{align*}
  x &= \langle 0, \text{std} \rangle \\
  v &= \langle 0, \text{dec} \rangle \\
  F &= \langle 0, \text{std} \rangle.
\end{align*}
$$

(8)

SQSIM applies qualitative value transition rules to determine the possible state(s) of the system immediately after the initial time-point. These value transition rules impose continuity on the variable’s magnitude and direction of change [21]. For example, SQSIM applies the value transition $v = \langle 0, \text{dec} \rangle \rightarrow v = \langle \text{minf} 0 \rangle, \text{dec}$. All consistent value transitions lead to the following successor for the initial state (8)

$$
\begin{align*}
  x &= \langle 0, \text{dec} \rangle \\
  v &= \langle \text{minf} 0 \rangle, \text{dec} \\
  F &= \langle 0, \text{std} \rangle.
\end{align*}
$$

(9)

Repeating this inference process leads to a set of possible behaviors. Fig. 2 visualizes the simulation result as a branching tree. Possible behaviors are represented by paths from the root of the tree to the leaves (behavior numbers are given on the left). The semiquantitative states are indicated by nodes and alternate between time-point states (solid nodes) and states that describe the behavior within a time interval (the node number represents the state index). The behavior tree visualizes when qualitative discriminations occur and provides a rough classification based on the final state (e.g., a dotted circle indicates a quiescent state, a state where the $\text{qdir}$ of every variable is $\text{std}$).

Semiquantitative time plots provide the detailed information of a variable’s trajectory for a specific behavior. Fig. 3, for example, shows $x(t)$ for the behavior 4, a behavior that becomes overdamped after the first half-cycle of the oscillation where it enters a quiescent state at the time point referred to as $T_5$.

SQSIM inserts new landmarks whenever the $\text{qdir}$ of a variable becomes $\text{std}$ (e.g., $x=2$ in Fig. 3 represents the magnitude at the negative peak). This adds expressiveness as states can be compared to previous states within a behavior. However, it enables the possibility that the set of behaviors can grow infinitely. An alternative is an attainable envisionment simulation. This simulation predicts all possible states that can be reached from an initial state and links them into a directed graph. The graph is represented by a behavior tree, where the states at the leaves of the tree are either quiescent states or so-called cross-edge states that refer to other states within the tree. The envisionment simulation does not insert new landmarks so that it is guaranteed that the graph is finite, although it might be quite large and complex.

The SQSIM algorithm is sound in the sense that it deduces every possible behavior for an uncertain ODE [11]. It can be seen as a solver for an uncertain initial value problem

$$
\begin{align*}
  x &= f(x), x_0, f(x) \in F(x), x_0 \in \mathcal{D}_0
\end{align*}
$$

(10)

lies in the domain $\mathcal{D}_0$. This domain is defined by the numerical bounds $x_{0i}^- j$ and $x_{0i}^+ j$ for each state variable $x_j$ and defines a hyperrectangle or box

$$
\begin{align*}
  \mathcal{D}_0 := \{ x_0 | x_{0i}^- j, x_{0i}^+ j, j = 1, \ldots, n \}.
\end{align*}
$$

(12)

Besides performing dynamic simulation, it is possible to use semiquantitative inference to deduce the equilibrium points of an uncertain ODE. This can be done by performing a semiquantitative simulation with a partially defined initial state that specifies all state variables of the SQDE to be at rest (std). In our spring-mass example this would mean that we specify the variables $x$ and $v$ by $x = \langle ?, \text{std} \rangle, v = \langle ?, \text{std} \rangle$. Semiquantitative simulation completes this partial information and provides a set of quiescent states, which describes the possible equilibrium points of the model (the second simulation step terminates immediately as all initial states are quiescent).

The simulation for the spring-mass system with an external force $F = 0$ and the partially defined initial state described above deduces a unique quiescent state that describes an equilibrium point at the origin $x = 0, y = 0$. However, a simulation with $F \in [0.04, 0.06]$ deduces a quiescent state with a displacement $x \in [1.25, 2.6]$ and $v = 0$. This is due to the imprecision of $F$ and the $M^+$ function $f_1(x)$.
This example demonstrates that the equilibrium point of an uncertain ODE can only be determined as lying within a domain \( D_e \). Therefore, it is helpful to distinguish between two different types of equilibrium points of the uncertain ODE \( F(x) \) of the initial value problem (10), which is abstracted by an SQDE.

**Definition 1:** A point \( x_c \) in state space is called an **exact equilibrium point** of (13) if for all functions \( f(x) \in F(x) \) it is true that \( f(x_c) = 0 \).

**Definition 2:** A point \( x_c \) in state space is called an **uncertain equilibrium point** of (13) if for at least one function \( f(x) \in F(x) \) it is true that \( f(x_c) = 0 \).

We can specify equilibrium points by their **equilibrium domain** \( D_e \) that can be drawn from the deduced set of quiescent states. The domain is either defined by an exact equilibrium point, i.e., \( D_e := x_c \), or specified by a box in state space

\[
D_e := \{ x_c \mid x_{c,j} \in [x_{c,j}^-, x_{c,j}^+] ; \; j = 1, \ldots, n \}. 
\]

### C. Incompleteness

One of the major drawbacks of the SQSIM algorithm is the prediction of so-called **spurious behaviors**. These behaviors describe trajectories that are genuinely impossible for any system consistent with the SQDE model. The incompleteness is caused by limitations of the inference procedures. Fig. 4 shows a predicted spurious behavior for the spring-mass system. The trajectory describes an increasing oscillation, an impossible behavior for the damped spring-mass system.

Spurious predictions are not limited to behaviors that are qualitatively impossible (such as behavior 1). Weak numerical inference is also a source for impossible behaviors. Reconsider the behavior shown in Fig. 4. The semiquantitative behavior abstracts a set of trajectories \( x(t) \) that share the same qualitative behavior, i.e., they become overdamped after the first half-cycle. The simulation, however, does not provide a useful numerical range for the negative peak. The range for the landmark \( x=2 \) is only specified by the default negative range \((-\infty, 0)\), thus including spurious behaviors as the one shown in Fig. 5.

The spurious behaviors spoil the simulation result as we have to consider all predicted behaviors. The next section describes advanced inference methods based on Lyapunov theory that can reduce spurious behaviors significantly.

### III. LYAPUNOV’S DIRECT METHOD

Assumptions about the preservation or dissipation of energy are commonly used by engineers for the process of reasoning about the behaviors of physical mechanisms. Such an energy-based analysis is especially helpful for the analysis of systems that exhibit oscillatory behaviors. Damped and undamped spring-mass systems were thoroughly studied in QR research [3], [15], [22], [23], and it was shown that energy considerations are essential for a successful application of QR methods for the analysis of many systems [3]. Existing approaches for energy-based reasoning, however, are either limited to spring-mass systems and systems that share the same structure [15] or require the user to formulate the constraints that describe the energy assumptions. Our efforts are directed toward an automatic energy-based reasoning that is also applicable for mechanisms where it does not make sense to speak about energy in the physical sense. For this purpose we apply the generalized energy concept formulated by Lyapunov [16] that is based on examining the time variation of scalar, energy-like Lyapunov functions.

#### A. Lyapunov Functions

Let us first recall the concept of a Lyapunov function for an (exact) autonomous, nonlinear ODE

\[
\dot{z} = f(z) \tag{15}
\]

with an equilibrium point \( z_e \) at the origin of the state space, i.e., \( f(z_e) = 0 \).

A Lyapunov function for (15) is defined as follows:

**Definition 3:** The scalar, continuously differentiable function \( V(z) \) with the derivative

\[
\dot{V}(z) = \frac{\partial V}{\partial z} \dot{z} = \frac{\partial V}{\partial z} f(z) \tag{16}
\]

along the trajectories of (15) is called a **Lyapunov function** of (15) in the neighborhood \( D \) of the origin, if \( V(z) \) is positive definite in \( D \) and \( \dot{V}(z) \) is negative semidefinite in \( D \). Moreover, we shall call \( V(z) \) a **strong Lyapunov function** of (15) if \( \dot{V}(z) \) is negative definite in \( D \).

The following stability criteria for the equilibrium point of an autonomous nonlinear ODE are based on Lyapunov functions [17].
**Theorem 1:** The equilibrium point \( \mathbf{z} = \mathbf{0} \) of the system (15) is asymptotically stable, if there exists a strong Lyapunov function \( V(\mathbf{z}) \) for the system (15).

**Theorem 2:** The equilibrium point \( \mathbf{z} = \mathbf{0} \) of the system (15) is globally asymptotically stable, if there exists a radially unbounded strong Lyapunov function \( V(\mathbf{z}) \), i.e., \( V(\mathbf{z}) \to \infty \) as \( ||\mathbf{z}|| \to \infty \), for the system (15) in the entire state space \( \mathbb{R}^n \).

These theorems provide sufficient conditions for asymptotic stability but say nothing about how an appropriate Lyapunov function can be found for a particular system. In fact, finding a Lyapunov function is difficult and it is often the case that such a function can only be found using a method of systemized trial and error. Our intention of using Lyapunov functions for semi-quantitative simulation imposes an even higher burden as we have to find one Lyapunov function for all ODEs, which are abstracted by the SQDE. This task, which should be performed automatically by a modified SQSIM simulation system, seems to be hopeless. However, research in nonlinear control theory deals with a very similar problem. We show in the next subsections how theories of nonlinear control theory can be utilized for our task of finding a Lyapunov function for a class of uncertain ODE models, which are represented in the form of SQDEs.

### B. Lyapunov Analysis of Nonlinear Feedback Systems

Often, it is possible to represent physical systems in the form of the feedback loop given in Fig. 6. This representation separates the linear and nonlinear part of the model. The forward path specifies the linear part in the form of a linear time-invariant ODE, and the feedback path describes the static nonlinearities of the model.

The mathematical model of the feedback system is given by

\[
\begin{align*}
\dot{\mathbf{z}} &= \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u} \\
\mathbf{u} &= -f_z(\mathbf{y})
\end{align*}
\]  

(17)

with the state vector \( \mathbf{z} := [z_1, \ldots, z_n]^T \), the input vector \( \mathbf{u} := [u_1, \ldots, u_p]^T \), and the output vector \( \mathbf{y} := [y_1, \ldots, y_p]^T \). The forward path of the nonlinear feedback system is specified by the constant real matrices \( \mathbf{A} \), \( \mathbf{B} \), \( \mathbf{C} \), and \( \mathbf{D} \), and the nonlinear feedback connection is defined by the static nonlinear vector function

\[
f_z := [f_{z,1}(y_1), f_{z,2}(y_2), \ldots, f_{z,p}(y_p)]^T.
\]

(18)

The components \( f_{z,j} \) are so-called *sector-nonlinearities*, meaning that they satisfy the inequalities (sector-conditions)

\[
\alpha_j y_j^2 \leq f_{z,j}(y_j) \leq \beta_j y_j^2, \quad \forall j = 1, \ldots, p
\]

(19)

with constants \( \alpha_j \) and \( \beta_j \). The feedback system (17) has an equilibrium point at the origin \( \mathbf{z}_e = \mathbf{0} \) as sector-nonlinearities satisfy the condition \( f_{z,j}(\mathbf{0}) = 0 \). In our terminology we can say that the nonlinear feedback system (17) describes for given sector-conditions (19) an uncertain ODE with an equilibrium point at the origin.

Nonlinear control theory deals with the question whether the equilibrium point of (17) is asymptotically stable for the entire set of nonlinear vector functions specified by the sector-con-conditions (19). The Popov criterion and the Circle criterion (e.g., see [17]), provide sufficient conditions for asymptotic stability based on proving the existence of a Lyapunov function. The existence of a Lyapunov function is not enough for our purpose as we intend to use the Lyapunov function to improve semi-quantitative simulation. The simultaneous Lyapunov function method described in [24], however, meets our requirements. The method deduces one strong Lyapunov function for all ODE systems specified by (17)–(19). It solves this task by framing the problem as semidefinite programming. The next two subsections demonstrate the application of this method for SQDEs.

### C. Feedback Representation of SQDEs

The similarities of sector nonlinearities and \( M^+ \) functions allow us to use the comprehensive theory that was developed for nonlinear feedback systems of the form (17). For this purpose, we restrict the semi-quantitative analysis to uncertain ODEs that can be factorized in the form

\[
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v} \\
\mathbf{w} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{v} \\
\mathbf{v} = -f_x(\mathbf{w}), \quad f_x \in F^M_x,
\]

(20)

The main difference to the model (17) is that the components of the static nonlinear vector function

\[
f_x := [f_{x,1}(w_1), f_{x,2}(w_2), \ldots, f_{x,p}(w_p)]^T
\]

(21)

are of class \( M^+ \). The procedure to obtain the factorized form (20) from an SQDE specification

\[
\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f}(\mathbf{x}) \in \mathcal{F}(\mathbf{x})
\]

(22)

is as follows.\(^4\)

1) Identify all \( M^+ \) functions that are used to compose the right-hand side of (22) and arrange them in the form of the nonlinear vector function (21).

2) The output matrix \( \mathbf{C} \) and the direct transmission matrix \( \mathbf{D} \) are specified by the arguments of the \( M^+ \) functions.

3) The input matrix \( \mathbf{B} \) is specified by the arrangement of the \( M^+ \) functions in the right-hand side of (22).

\(^4\)It is clear that not every uncertain ODE can be transformed into the form given in (20). Uncertain ODEs with a right-hand side that contains a) products of state variables, e.g., \( x_i x_j \), b) products of state variables and \( M^+ \) functions, e.g., \( x_i f_j, f_j \in M^+ \), and c) products of \( M^+ \) functions, e.g., \( f_i f_j, f_i, f_j \in M^+ \) cannot be represented in the form (20). Furthermore, expressions using uncertain constants \( c_i \), e.g., additive \( c_i x_i \) and multiplicative \( c_i x_j \) expressions, must be reformulated using \( M^+ \) constraints as they represent the uncertainty of the model (20).
The dynamic matrix \( \mathbf{A} \) is specified by the right-hand side of (22) with all \( M^+ \) functions removed.

Let us demonstrate this transformation for the spring-mass system without external force

\[
\begin{align*}
\dot{\mathbf{x}} &= 0 \\
\dot{\mathbf{v}} &= -f_1(x) - f_2(v).
\end{align*}
\](23)

The state vector \( \mathbf{v} \) for the transformed system (20) shall be defined by \( \mathbf{v} := [x, v]^T \). The first step identifies the \( M^+ \) functions \( f_1(x) \) and \( f_2(v) \) and defines \( \mathbf{f}_x := [f_1, f_2]^T \). The arguments of \( \mathbf{f}_x \) specify the output vector \( \mathbf{w} = [x, v]^T \) so that \( \mathbf{C} = \mathbf{I} \) and \( \mathbf{D} = \mathbf{0} \) follow naturally. The following input matrix \( \mathbf{B} \) and the dynamic matrix \( \mathbf{A} \) reestablish the right-hand side of (23)

\[
\begin{align*}
-\mathbf{B}\mathbf{f}_x &= \begin{bmatrix} 0 & -f_2(v) \\ -f_1(x) & -f_2(v) \end{bmatrix} \\
\mathbf{A}\mathbf{x} &= \begin{bmatrix} x \vspace{1ex} v \\ 0 \vspace{1ex} 0 \end{bmatrix}.
\end{align*}
\]

The main consequence imposed by \( M^+ \) functions is that the uncertain ODE (20) does not necessarily have its equilibrium point at the origin. A commonly used technique in nonlinear system theory is to introduce a new state vector \( \mathbf{z} := \mathbf{x} - \mathbf{x}_e \) so that the equilibrium point is shifted into the origin. Let us demonstrate this state vector change for systems with an uncertain equilibrium point first. Consider a specific ODE system that is an instance of the uncertain ODE system (20) with the nonlinear vector function \( \mathbf{f}_x \) and the associated equilibrium point \( \mathbf{x}_e \). Let \( \mathbf{w}^u \) and \( \mathbf{v}^u \) denote values of the output vector \( \mathbf{w} \) and input vector \( \mathbf{v} \) at the equilibrium point \( \mathbf{x}_e \). By defining the new state vector \( \mathbf{z} := \mathbf{x} - \mathbf{x}_e \), output vector \( \mathbf{y} := \mathbf{w} - \mathbf{w}^u \), and input vector \( \mathbf{u} := \mathbf{v} - \mathbf{v}^u \) we shift the equilibrium point into the origin \( \mathbf{z}_e = \mathbf{0} \). The state vector change does not alter the matrices \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \) and \( \mathbf{D} \). However, the shifted nonlinear vector function \( \mathbf{f}_x^u \), which satisfies the condition \( \mathbf{f}_x^u(0) = \mathbf{0} \), must be specified by sectors for its components \( \mathbf{f}_{x_j}^u \) to fit into the feedback loop representation (17). Recall that the components \( \mathbf{f}_{x_j}^u \) of the vector function \( \mathbf{f}_x^u \) are specified by the envelope functions and the numerical bounds \( s^-_j \) and \( s^+_j \) for the slope. The slope bounds must hold for the shifted nonlinear vector function \( \mathbf{f}_x^u \) as well. Therefore, it is possible to specify the components of \( \mathbf{f}_x^u \) by linear static envelopes with sector-conditions (19) for \( \mathbf{f}_x \) that are specified by the slope bounds, i.e., \( \alpha_j = s^-_j \) and \( \beta_j = s^+_j \).

As the numerical bounds for the slope hold for every function \( \mathbf{f}_x \in \mathbb{F}^M \), we can conclude that we always end up defining a shifted nonlinear vector function \( \mathbf{f}_x^u \) with sector conditions defined by the numerical bounds for the slope.

The same procedure can be applied for an uncertain ODE with an exact equilibrium point. However, sectors for the components of the function \( \mathbf{f}_x \) that are based on the numerical bounds for the slope can be unnecessarily inclusive. The shifted envelope functions \( \mathbf{f}_{x_j}^\downarrow \) and \( \mathbf{f}_{x_j}^\uparrow \) often define tighter sectors \( [\alpha_j, \beta_j] \). The sectors can be deduced by evaluating the following two sector-inequalities for each \( j = 1, \ldots, p \)

\[
\begin{align*}
\alpha_j y_j^\downarrow \leq y_j f_{x_j}(y_j) &\leq \beta_j y_j^\downarrow, \\
\alpha_j y_j^\uparrow \leq y_j f_{x_j}(y_j) &\leq \beta_j y_j^\uparrow.
\end{align*}
\](24)

D. Simultaneous Lyapunov Functions

Having described a method for the representation of SQDEs in the form of the nonlinear feedback loop (17) we shall now provide a method for the deduction of a Lyapunov function for the nonlinear feedback loop. The method is based on [24] and calculates a strong quadratic Lyapunov function

\[
\dot{V}(\mathbf{z}) = -\mathbf{z}^T \mathbf{P} \mathbf{z}
\](25)

with a positive definite symmetric matrix \( \mathbf{P} \) for the nonlinear feedback loop representation of the SQDE, if one exists.\(^5\)

The key-idea is to transform the nonlinear ODE (17) into a linear time-varying ODE. For this purpose, we define the components of a time-varying gain \( \mathbf{k}(t) \) to specify the time-varying matrix \( \mathbf{K}(t) := \text{diag}(\mathbf{k}(t)) \) for a given trajectory \( \mathbf{z}(t) \) of the system by

\[
\mathbf{k}_j(t) := \begin{cases} f_{x_j}^u(y_j(t)), & y_j(t) \neq 0 \\
\alpha_j, & y_j(t) = 0. \end{cases}
\](26)

Furthermore, we use the time-varying gain \( \mathbf{k}(t) \) to specify the time-varying matrix \( \mathbf{K}(t) := \text{diag}(\mathbf{k}(t)) \). It is clear that the gain \( \mathbf{k}(t) \) is defined differently for each particular trajectory \( \mathbf{z}(t) \). Nevertheless, it is always true that the components \( \mathbf{k}_j(t) \) satisfy the conditions

\[
\alpha_j \leq \mathbf{k}_j(t) \leq \beta_j, \forall j = 1, \ldots, p.
\](27)

We shall say that the gain \( \mathbf{k}(t) \) can take on values in the box

\[
\mathcal{D}_k := \{ \mathbf{k} \mid k_j \in [\alpha_j, \beta_j], j = 1, \ldots, p \}.
\](28)

\( M^+ \) functions imply strictly positive bounds for the sector conditions, i.e., \( 0 < \alpha_j < \beta_j \). Therefore, we can always invert \( \mathbf{K}(t) \) and combine the output and feedback equation of (17) to

\[
\begin{align*}
[\mathbf{K}(t)]^{-1} &+ \mathbf{D} \mathbf{u} = -\mathbf{C} \mathbf{z} \\
\mathbf{u} &= -[\mathbf{K}(t)]^{-1} \mathbf{D}^{-1} \mathbf{C} \mathbf{z}
\end{align*}
\](29)

if

\[
\det [\mathbf{K}(t)]^{-1} + \mathbf{D} \neq 0, \forall \mathbf{z}(t) \in \mathcal{D}_k.
\](31)

We shall say that the system (17) is well posed if (31) is satisfied. If we define the \( 2^p \) vertex matrices \( \mathbf{K}^{(j)} \) by taking the gain \( \mathbf{k} \) at the vertices of \( \mathcal{D}_k \) it is possible to formulate the following theorem:

\textbf{Theorem 3 (Well Posed): A necessary and sufficient condition for (31) is that all} \( 2^p \) \textbf{numbers} \( \epsilon^{(j)} \) \textbf{are defined} \( \det [\mathbf{K}^{(j)}]^{-1} + \mathbf{D} \neq 0, j = 1, \ldots, 2^p \) \textbf{have the same sign and are not equal to zero.}

\textbf{Proof:} The proof is given in Appendix. \( \blacksquare \)

A well posed system allows us to reformulate the Lyapunov analysis of the nonlinear ODE (17) as the Lyapunov analysis of a linear time-variant ODE

\[
\dot{\mathbf{z}} = \mathbf{A} - \mathbf{B}[\mathbf{K}(t)]^{-1} \mathbf{D}^{-1} \mathbf{C} \mathbf{z} =: \mathbf{A}(t)\mathbf{z}.
\](32)\(^5\)

\( \text{We shall denote positive definiteness by } \mathbf{P} \succ 0. \)
The derivative of the quadratic Lyapunov function candidate $V(z) = z^T P z$ along the trajectories of (32) is given by

$$\dot{V}(z) = \dot{z}^T P \dot{z} + z^T (A(t))^T P \dot{A}(t) z.$$  

(33)

It is now our goal to find a positive definite matrix $P$ so that (33) is negative definite for all $K(t)$ in the box $\mathcal{D}_0$. For this purpose, we specify the vertex-matrices of $A(t)$ by

$$A^{(j)} := A - B \left[ (K^{(j)})^{-1} + D \right]^{-1} C, \quad j = 1, \ldots, 2^p$$

(34)

and formulate the following theorem.

**Theorem 4 (Simultaneous Lyapunov Functions):** Consider a well posed nonlinear feedback system (17) where $f(y)$ satisfies the sector condition (19). There exists a strong quadratic Lyapunov function $V(z) = z^T P z$ in $\mathbb{R}^n$, if and only if

$$(A^{(j)})^T P + PA^{(j)} - \varsigma P \preceq 0, \quad \forall j = 1, \ldots, 2^p.$$  

(35)

**Proof:** The proof is given in Appendix.

This theorem reformulates the problem of finding a quadratic Lyapunov function for the nonlinear feedback loop (17) into the following linear matrix inequality (LMI) problem. Given the $2^p$ matrices $A^{(j)}$, find a matrix $P > 0$ so that the $2^p$ linear matrix inequalities (35) hold. Powerful solvers for LMI problems, such as the MatLab LMI control toolbox [25] or SP [26] are based on Semidefinite programming [27], [28]. The following optimization problem (semidefinite program) solves the LMI problem stated above

$$\begin{align*}
\text{minimize} & \quad \varsigma \\
\text{subject to} & \quad P > 0 \\
& \quad (A^{(j)})^T P + PA^{(j)} - \varsigma P \preceq 0, \quad j = 1, \ldots, 2^p.
\end{align*}$$

The matrix $P$ defines a strong quadratic Lyapunov function for the system (17) in $\mathbb{R}^n$ if the semidefinite program is solved for $\varsigma < 0$. Otherwise ($\varsigma \geq 0$), there does not exist a single quadratic Lyapunov function for (17) as Theorem 4 provides a necessary and sufficient condition.

Our extended SQSIM simulation platform performs the described analysis automatically. Based on an SQDE model of the system under investigation, it checks whether the system can be represented in the form of (20) and performs the transformation whenever possible, it calculates the equilibrium point(s), performs the state variable change, and deduces a quadratic Lyapunov function for the system if one exists. The SQDE together with the Lyapunov function is then provided to the semiquantitative simulation engine.

**E. Reasoning Based on Quadratic Lyapunov Functions**

A quadratic Lyapunov function for the system under investigation allows to calculate bounds for the state variables. This fact can be used to formulate very efficient inference techniques. Behaviors predicted by semiquantitative simulation that reach the bounds (e.g., behaviors describing diverging trajectories) can be identified as spurious behaviors and filtered out of the simulation result. Let us outline this approach for an uncertain initial value problem (10) with a globally asymptotically stable exact equilibrium point $x_e$ first. The Lyapunov function

$$V(x) = (x - x_e)^T P (x - x_e)$$

(36)

describes for constant values $V(x) = c$ ellipsoids in state space whose centers lie at the equilibrium point $x_e$. Let $V_{\text{max}}$ denote the smallest ellipsoid so that $\mathcal{D}_0$ lies entirely inside the ellipsoid, i.e.,

$$V_{\text{max}} \geq (x_0 - x_e)^T P (x_0 - x_e), \quad \forall x_0 \in \mathcal{D}_0.$$  

(37)

Equation (36) defines a Lyapunov function for all ODE’s of the uncertain initial value problem (10). Therefore, it is possible to conclude that every trajectory of the uncertain initial value problem stays inside the ellipsoid $V(x) = V_{\text{max}}$. The value $V_{\text{max}}$ can be obtained by evaluating the Lyapunov function at the $2^n$ vertices $x_0^{(1)}, \ldots, x_0^{(2^n)}$ of $\mathcal{D}_0$ and taking the maximum $V_{\text{max}} = \max_{\ell=1,\ldots,2^n} V(x_0^{(\ell)})$ [29]. Every trajectory predicted by semiquantitative simulation that reaches this ellipsoid at $t > t_0$ is spurious. However, the representation of a region in the form of an ellipsoid is not helpful for our purpose as SQSIM uses boxes to represent numerical uncertainty. Therefore, we approximate the ellipsoid with a bounding-box $\mathcal{D}_b$, which contains the ellipsoid defined by $V_{\text{max}}$. Fig. 7 demonstrates the construction of a bounding-box for a second order system with an exact equilibrium point at the origin.

Systems with an uncertain equilibrium point require a slightly modified approach. We have to find the smallest ellipsoid that, centered at any point of $\mathcal{D}_e$, contains the box $\mathcal{D}_0$. It is possible to show that we can restrict our search to ellipsoids that are centered at the $2^n$ vertices $x_e^{(1)}, \ldots, x_e^{(2^n)}$ of $\mathcal{D}_e$, so that the value of $V_{\text{max}}$ can be evaluated by taking the maximum of the $2^n$ Lyapunov functions $V(x_0^{(\ell)}, x_e^{(\ell)})$ [29].

The bounds for the state variables $x_{b_{ij}}^-$ and $x_{b_{ij}}^+$ can be calculated via fundamental properties of ellipsoids as

$$x_{b_{ij}}^- = x_{e_{ij}}^* - \sqrt{V_{\text{max}} P_{i j}} / \det P$$

$$x_{b_{ij}}^+ = x_{e_{ij}}^* + \sqrt{V_{\text{max}} P_{i j}} / \det P$$

(38)
where $P_{ij}$ denotes the cofactor for the element $p_{jj}$ of the matrix $\mathbf{P}$. This leads to the bounding box

$$D_b := \left\{ \mathbf{x} \mid x_j \in [x_{b,j}^- , x_{b,j}^+] , \ j = 1, \ldots, n \right\}. \tag{39}$$

The bounds $x_{b,j}^-$ and $x_{b,j}^+$ specify experiment specific qualitatively important values. We represent the bounds by bounding landmarks in the quantity spaces of the state variables. Every predicted behavior that reaches a bounding landmark can be identified as spurious and filtered out of the simulation result. This concept is the basis for a simple and efficient filter for spurious behaviors that checks for states where the \textit{crucial} state is specified by a bounding landmark.

Bounding box filtering cannot identify spurious behaviors that stay within the box. However, certain behaviors of this type, such as the one shown in Fig. 8, can be identified using the following concept. Consider the two time-points $t_i, t_j$, where all state variables except $x_k$ are at their equilibrium value, i.e., $x_i(t_i) = x_j(t_j) = x_{e,k}$. The expression for the Lyapunov function at these time-points can be reduced to

$$V = (\mathbf{x} - \mathbf{x}_e)^T \mathbf{P} (\mathbf{x} - \mathbf{x}_e) = p_{kk} (x_k - x_{e,k})^2. \tag{40}$$

The values of the Lyapunov function at $t_i$ and $t_j$ satisfy the condition $V(t_j) < V(t_i)$. This condition can be written as

$$ (x_k(t_j) - x_{e,k})^2 < (x_k(t_i) - x_{e,k})^2 \tag{41}$$

because the coefficient $p_{kk}$ of the positive definite matrix $\mathbf{P}$ is always positive. We can identify some spurious behaviors that violate (41) by checking the ordering of the landmarks for $x_k(t_j)$ and $x_k(t_i)$ with respect to the landmark for $x_{e,k}$. Behaviors that satisfy one of the following two conditions

$$x_{e,k} < x_k(t_j) \leq x_k(t_i)$$

$$x_k(t_j) \leq x_k(t_i) < x_{e,k} \tag{42}$$

violate (41) and can be filtered out of the simulation result. We test these conditions symbolically and not numerically because landmarks with the symbolic ordering $\mathbf{lm}^{-1} < \mathbf{lm}^{-2}$ can have overlapping numerical ranges [e.g., see Fig. 9, where $x_{-2}$ and $x_{-5}$ with the symbolic ordering $X_{-5} < X_{-2}$ have the same numerical range $(-2.0, 0.0)$].

IV. EXAMPLES

A. Spring-Mass System Revisited

We want to demonstrate the Lyapunov reasoning methods with the damped spring-mass system first. This allows us to compare our inference methods to the kinetic-energy filter [15], which can handle second order systems of this specific type. The transformation of the SQDE (23) leads to the feedback system representation

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{z}$$

$$\mathbf{u} = - [ f_{M,1}^M(y_1), \ f_{M,2}^M(y_2) ]^T \tag{43}$$

with the state vector $\mathbf{z} := [x, v]^T$ and the sector $[30.0, 32.0]$ for $f_{M}^M$. The function $f_{M,2}^M$ shall be defined by the sector $[0.18, 0.24]$. Our extended SQSIM system calculates the strong quadratic Lyapunov function

$$V(\mathbf{z}) = \mathbf{z}^T \begin{bmatrix} 6.6421 \cdot 10^{-1} & 1.9284 \cdot 10^{-3} \\ 1.9284 \cdot 10^{-3} & 2.1426 \cdot 10^{-2} \end{bmatrix} \mathbf{z}. \tag{44}$$

This Lyapunov function, the initial state $x_0 \in [1.0, 2.0]$, $v_0 = 0$ and the equilibrium state at the origin implies the ranges for $x$ and $v$

$$x(t) \in (-2.0, 2.0) \quad v(t) \in (-11.1, 11.1).$$

A semiquantitative simulation with our extended SQSIM system deduces a set of behaviors that describes one oscillatory behavior with decreasing amplitude (see Fig. 9) and oscillatory behaviors with decreasing amplitude that become overdamped after a finite number of half-cycles. A semiquantitative simulation using the kinetic-energy filter, on the other hand, provides qualitatively the same set of possible behaviors but with less informative numerical ranges for the state variables

$$x(t) \in (-\infty, 2.0) \quad v(t) \in (-3560, 0).$$
B. Controlled Tank System

We shall now use the extended semiquantitative inference procedures to analyze a controlled fluid tank system (Fig. 10), where the inflow $z_i$ is provided by a fluid pump. We use a PI-controller to keep the fluid level at a specific set-point $x_s$. Although this system is rather simple from the control engineering point of view, it is important to note that current semiquantitative simulation techniques on the basis of the kinetic-energy filter can handle such a system only in revised and simplified form.

The uncertain mathematical model of the fluid tank with a PI-controller for a fixed set-point $x_s = 20.0$ is given by

$$\begin{align*}
\text{fluid-tank}: & \dot{x}_1 = -f_{z_1}(x_1) + f_{x_2}(u) \\
\text{PI-controller}: & e = x_s - x_1 \\
& \dot{x}_2 = e \\
& u = K \left[ e + \frac{1}{T_i}x_2 \right]
\end{align*}$$

where

- $x_1$ fluid level;
- $x_2$ integral part of the controller;
- $e$ control error;
- $u$ control signal.

The PI-controller is specified by the parameters $K = 0.14906$ and $T_i = 15.628$. The parameters $K$ and $T_i$ were determined using an eigenvalue placement procedure on the basis of a linear model derived from a nominal nonlinear mathematical model for the tank.

The uncertainty of the system should lie in the inexact knowledge of the maximal fluid level $x_{11} = \text{fluid-tank}$, the outflow characteristic of the tank $f_{z_1}(x_1) \in M^+$, and the relationship between the control signal $u$ and the inflow $z_i = f_{x_2}(u)$. The function $f_{x_2}(u)$ is of class $M^+$ for control signals $u$ within the range $0.8 \leq u \leq 5.0$. The fluid pump requires a certain level of activation to overcome static friction, thus imposing the lower limit $u = 0.8$. The upper limit is imposed by the limited operational range of the power amplifier that is used to drive the pump. The envelope functions and slope bounds for $f_{x_2}(u)$ within the operational range are given by

$$\begin{align*}
f_{x_2}^{-}(u) &= 2.0 \sqrt{u - 0.6} - \sqrt{0.2} \\
f_{x_2}^{+}(u) &= 2.0 \sqrt{u - 0.8} \\
s_i^- = 0.40, & s_i^+ = 10.0.
\end{align*}$$

The imprecise numerical information for the outflow characteristic $f_{z_1}(x_1)$ is specified by

$$\begin{align*}
f_{z_1}^{-}(x) &= 0.385 \sqrt{x + 0.25} - 0.5 \\
f_{z_1}^{+}(x) &= 0.385 \sqrt{x + 0.25} + 0.5 \\
s_i^- = 0.0285, & s_i^+ = 0.39.
\end{align*}$$

The main difficulty in abstracting the uncertain ODE (45) by an SQDE is due to the saturating relationship between the control signal and the relative inflow of the tank. This situation can be handled by SQSIMs ability to switch between SQDEs during simulation. In our case we can model the system by three SQDEs, one for each mode ($u \in [0.8, 5.0], \ u < 0.8, \ u > 5.0$). The three uncertain ODEs that abstract to the SQDEs $\text{PI-tank}$, $\text{PI-tank-u0}$, and $\text{PI-tank-u5}$ are

$$\begin{align*}
\text{PI-tank}: & \dot{x}_1 = -f_{z_1}(x_1) + f_{x_2}(u) \\
& u \in [0.8, 5.0], \ e = f_{x_2}(x_1) \\
& \dot{x}_2 = e \\
& u = f_{x_4}(e) + f_{x_5}(x_2) \quad (48) \\
\text{PI-tank-u0}: & \dot{x}_1 = -f_{z_1}(x_1) \\
& u < 0.8, \ e = f_{x_2}(x_1) \\
& \dot{x}_2 = e \\
& u = f_{x_4}(e) + f_{x_5}(x_2) \quad (49) \\
\text{PI-tank-u5}: & \dot{x}_1 = -f_{z_1}(x_1) + z_i^{**} \\
& 5.0 < u, \ e = f_{x_2}(x_1) \\
& \dot{x}_2 = e \\
& u = f_{x_4}(e) + f_{x_5}(x_2). \quad (50)
\end{align*}$$

The value for the inflow $z_i$ at the mode $u > 5.0$ is defined by $z_i^{**} \in [f_{z_2}(5.0), f_{x_2}^{+}(5.0)] = [3.3006, 4.0988]$. The exact relationships of the PI-controller are modeled as exact monotonic functions $f_{x_4}(x) \in M^-$, $f_{x_5}(x) \in M^+$ that are defined by

$$\begin{align*}
t_{x_4}^M &= \{f_{x_4} = f_{x_4}^{+} = 20.0 - x_1, s_i^0 = s_i^+ = -1.0\} \\
t_{x_5}^M &= \{f_{x_5} = f_{x_5}^{+} = 0.14906e, s_i^0 = s_i^+ = 0.14906\} \\
t_{x_5}^{M^+} &= \{f_{x_5} = f_{x_5}^{+} = 9.5376 \cdot 10^{-3} x_2 \\
& s_i^0 = s_i^+ = 9.5376 \cdot 10^{-3}\}.
\end{align*}$$

It is our intention to evaluate whether the system can overflow when filled from empty. For this purpose we perform a semiquantitative simulation with an initial state $x_0 = 0$. In the first step the simulation system deduces an uncertain equilibrium point

$$x_{e1} = 20.0, x_{e2} \in [146.04, 271.27].$$

The Lyapunov analysis uses the main SQDE ($\text{PI-tank}$), which can be represented as

$$\begin{align*}
\dot{z} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} u \\
\mathbf{u} &= - \begin{bmatrix} f_{z_1}, f_{z_2}, f_{z_3}, f_{z_4}, f_{z_5} \end{bmatrix}^T.
\end{align*}$$

$M^-$ denotes a negative $M^+$ function, i.e., $-f(\cdot) \in M^+ \iff f(\cdot) \in M^-$.
The direct transition matrix \( D \neq 0 \) is used to express the nested monotonic constraints. The functions \( f_{x,j} \) represent the shifted monotonic functions. These functions are defined by the sectors 

\[
\begin{align*}
\{0,0.285, 0.39\} & \text{ for } f_{x,1} , \\
\{0.40, 10.0\} & \text{ for } f_{x,2} , \\
\{-1.0, -1.0\} & \text{ for } f_{x,3} , \\
\{0.14906, 0.14906\} & \text{ for } f_{x,4} , \\
\{9.53676, 10^{-3}\} & \text{ for } f_{x,5}. 
\end{align*}
\]

On the basis of this representation it is possible to calculate a strong quadratic Lyapunov function

\[
V(z) = z^T \begin{bmatrix} 6.6573 \cdot 10^{-1} & -2.3000 \cdot 10^{-2} \\
-2.3000 \cdot 10^{-2} & 6.2718 \cdot 10^{-3} \end{bmatrix} z.
\]  

(54)

The Lyapunov function, the initial state, and the uncertain equilibrium state specifies a bounding box \( D_b \) that defines, together with the quantity spaces of the state variables, the following ranges for \( x_1 \) and \( x_2 \)

\[
x_1(t) \in [0.0, 48.673] \quad x_2(t) \in (-149.36, 506.68],
\]  

(55)

The upper bound for the fluid level (48.673) specifies a bounding landmark outside of the range of the tank (the maximal fluid level is specified by the numerical range \([43.0, 45.0]\)). Therefore, we cannot exclude the possibility of an overflow based on this information. However, an attainable envisionment simulation of the system does prove that the tank cannot overflow. The simulation only predicts behaviors with fluid levels within the range of the tank. We can validate this result by evaluating whether a semiquantitative state with a fluid level \( x_1 \in [43.0, 45.0] \) and an integral part within the valid range \( x_2 \in (-149.36, 506.68] \) is compatible with the information given by the SQDEs and continuity. Interval arithmetic allows us to determine the numerical range for the control signal as

\[
u = f_{xA}(f_{x}^{24}(x_1)) + f_{x}^{25}(x_2) = [-5.1511, 1.9764],
\]  

(56)

This range defines a relative inflow \( z_1 = f_{x}^{24}(u) = [0.0, 2.4092] \) that implies the range

\[
\dot{x}_1 = -f_{x}^{24}(x_1) + z_1 = [-2.7823, 0.17021]
\]  

(57)

for the derivative of the fluid level \( x_1 \). This range indicates that a consistent semiquantitative state with \( \text{mag}(x_1) = \text{xi-full} \) can only take on a decreasing rate of change, i.e., \( q\text{dir}(x_1) = \text{dec} \). However, reaching \( \text{xi-full} \) from a lower fluid level requires, by continuity of the derivative of the fluid level, a semiquantitative state with a direction of change \( q\text{dir}(x_1) = \text{inc} \) or \( q\text{dir}(x_1) = \text{std} \). Both situations do not agree with the numerical information given above. This proves the impossibility of an overflow of the controlled tank.

The set of predicted behaviors is visualized as attainable envisionment graph in Fig. 11. Important for our analysis is that a) the graph contains two quiescent states that describe the equilibrium point (52) and b) every branch that contains transition states (indicated by a circle with vertical bar) contains two transition states (one for the transition \( \text{PI-tank to PI-tank-u0} \) or \( \text{PI-tank-u5} \), respectively, and one for the reverse transition). This indicates that the system cannot get stuck in a state where the inflow \( z_1 \) is saturated. Fig. 12 visualizes the fluid level for the branches 1, 3, and 9 of the graph. These branches describe behaviors where the system cycles around the equilibrium point (branch 1) or reaches the equilibrium point without (branch 9) or with a single overshoot (branch 3). The behaviors do not cause a saturation of the inflow and are predicted using the main SQDE \( \text{PI-tank only}. \) Fig. 13 shows the time-plots for \( x_1 \), \( z_2 \), and \( u \) for the saturating behavior that is represented by branch 4 in the envisionment graph.

\[\text{Fig. 11. Attainable envisionment graph for the PI-controlled tank.}\]

The deduction of a Lyapunov function for a given system under investigation is generally difficult. However, the similarities of the system’s representation used in semiquantitative simulation and the representation used in nonlinear control theory allows us to reformulate the deduction problem so that theories from the latter can be applied. In this way it was possible to formulate a flexible method for the deduction of quadratic Lyapunov functions that can be applied automatically by our extended SQSIM simulation system. Lyapunov-based inference concepts for semiquantitative simulation, which are based on a quadratic Lyapunov function for the system under investigation, were formulated and demonstrated by example.

Our approach extends the applicability of energy-based inference to systems where it does not make sense to speak about energy in the physical sense. This broadens the applicability of semiquantitative simulation as we provide a powerful method for eliminating spurious behaviors for a large class of systems. The paper shows the basic Lyapunov-based reasoning concepts that account for the majority of improvements due to Lyapunov analysis in semiquantitative simulation. Additional Lyapunov-based filtering techniques that provide dynamic envelopes for the state variables, and a reliable stability test for quiescent states that is based on Lyapunov’s indirect method are given in [29].

The deduction of a Lyapunov function and the inference procedures derived from this additional information scale well to higher order systems. In fact, their application allowed the semiquantitative simulation of an oscillatory third order system for the first time [29]. The underlying SQSIM algorithm, however, does not share this property and runs into complexity problems when applied to higher order systems. Many behaviors predicted for higher order systems that agree with the Lyapunov information are artifacts of the simulation algorithm that are in-
Fig. 12. Semiquantitative time-plots for the fluid level \( r_1 \). (a) Behavior 1. (b) Behavior 3. (c) Behavior 9.

Fig. 13. Semiquantitative time-plots for behavior-branch 4.

relevant for the behavioral description. This suggests additional research dealing with the complexity and ambiguity problems in qualitative and semiquantitative simulation that extends the methods described in [30].

APPENDIX

PROOFS

Proof [Well Posed]: Let
\[
\xi(k_1, \ldots, k_p) := \det \left[ (K(t))^{-1} + D \right]
\]
denote the left-hand side of (31). Necessary should be clear, since the image of \( D_k \) under \( \xi \) is connected, and therefore every interval that contains numbers of different sign contains zero as well. To prove sufficiency we shall show that both the maximum and the minimum of \( \xi \) over \( D_k \) are achieved at vertices of \( D_k \). This can be done by exploiting the special structure of the matrix \( M := (K(t))^{-1} + D \). Each gain \( k_j(t) \) appears only once on the diagonal element of \( m_{jj} \). Let us assume that the maximum of \( \xi \) is achieved at \( k^* := [k_1^*, \ldots, k_p^*]^T \). Then it is possible to show that \( k_j^* \in \{ \alpha_j, \beta_j \} \) by the following argument: Assume that \( k_1 \) can be varied, whereas the others are fixed at the values \( k_2, \ldots, k_p \). The determinant of the matrix \( M \) can be calculated by
\[
\xi(k_1, k_2^*, \ldots, k_p^*) = \frac{1}{k_1^*} \xi_0 + \xi_1,
\]
where \( \xi_0 \) and \( \xi_1 \) are determined by the cofactors for the elements \( m_{ij}, j = 1, \ldots, n \). Because \( k_1 \neq 0 \) we can conclude that (58) can achieve its extremal values only at the endpoints of the interval \( \{ \alpha_1, \beta_1 \} \), i.e., \( \{ \alpha_j, \beta_j \} \), or if \( \xi_0 = 0 \) that indicates that (58) is independent of \( k_1 \), we can set \( k_1^* \) to \( \alpha_1 \). Repeating this argument for every \( k_j \) proves sufficiency of the theorem.

Proof [Simultaneous Lyapunov Functions]: Necessity of (35) follows from the fact that (33) must be negative definite for every \( K(t) \), including \( K(0) \). Sufficiency can be shown using an argument similar to the one used for proving well posedness. For any \( z \neq 0 \) we can write (33) in the form
\[
\psi(k_1, \ldots, k_p) := 2z^TP \left[ A - B (K(t))^{-1} + D \right]^{-1} C z
\]
where \( \psi \) is negative definite on \( D_k \). Let us assume that the maximum of \( \psi \) is achieved at \( k^* := [k_1^*, \ldots, k_p^*]^T \). We shall show that \( k_j^* \in \{ \alpha_j, \beta_j \} \) by the following argument: Assume that \( k_1 \) can be varied, whereas the others are fixed at the values \( k_2, \ldots, k_p \). By exploiting the special structure of the matrix \( M := (K(t))^{-1} + D \) we can write for (59) using (58)
\[
\psi(k_1, k_2^*, \ldots, k_p^*) = \frac{1}{k_1^*} \xi_0 + \psi_1
\]
where
\[
\psi_1 = \frac{1}{k_1^*} \xi_1.
\]
Since the denominator of (60) cannot vanish for \( k_1 \neq [\alpha_1, \beta_1] \) due to well posedness we can conclude that the maximum of \( \psi \) can only be achieved at an endpoint of the interval \( \{ \alpha_1, \beta_1 \} \), i.e., \( k_1^* \in \{ \alpha_2, \beta_2 \} \), or (60) is independent of \( k_1 \) so that we can set \( k_1^* \) to \( \alpha_1 \). We can show that the maximum of \( \psi \) and therefore of \( V \) is achieved at a vertex of \( D_k \) by repeating this argument for all other \( k_j \).

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