

Dynamic analysis of multi-link spatial flexible manipulator arms with dynamic stiffening effects

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Abstract The dynamics for multi-link spatial flexible manipulator arms is investigated. The system considered here is an N-flexible-link manipulator driven by N DC-motors through N revolute flexible-joints. The flexibility of each flexible joint is modeled as a linearly elastic torsional spring, and the mass of the joint is also considered. For the flexibility of the link, all of the stretching deformation, bending deformation and the torsional deformation are included. The complete governing equations of motion of the system are derived via the Lagrange equations. The nonlinear description of the deformation field of the flexible link is adopted in the dynamic modeling, and thus the dynamic stiffening effects are captured. Based on this model, a general-purpose software package for dynamic simulation of multi-link spatial flexible manipulator arms is developed. Several illustrative examples are given to validate the algorithm presented in this paper and to indicate that not only dynamic stiffening effects but also the flexibility of the structure has significant influence on the dynamic performance of the manipulator. © 2012 The Chinese Society of Theoretical and Applied Mechanics. [doi:10.1063/2.1206303]

Keywords flexible manipulator arm, multi-link, flexible joint, flexible link, dynamic stiffening

Modern robotics are now developing towards the high speed and precise operation, and the use of lightweight materials with distributed flexibility. The coupling problem between rigid-body motion of the structure and its flexibility brings a great challenge to the dynamic analysis and precise control of the robots, and in fact it has become a key problem in the robotics. Recently, the investigations on dynamics of a single-link manipulator arm have been paid attention to by many researchers.¹⁻⁸ Reference 1 investigated the coupling effect of a flexible link and a flexible joint. Solutions of the system dynamic equations showed that the frequencies and mode shapes of the system are parameterized in two ratios. One is the ratio of the moment of inertia of the link to that of the rotor of the actuator, and the other is the ratio of the bending stiffness of the link to the torsion stiffness of the joint. Reference 3 presented a dynamic model of a flexible-link and flexible-joint manipulator carrying a payload with rotary inertia. All of the dynamic coupling terms among the system reference rotational motion, joint torsional deformations and arm bending deformations were accounted for. References 5-7 presented the first-order approximation coupling model of a flexible beam, which was based on the theory of continuum medium mechanics and the theory of analytical dynamics. In addition, the second-order coupling terms of longitudinal displacement caused by transversal deformation were considered.

In addition to the dynamics of a single-link manipulator arm the dynamics of multi-link flexible manipulator arms has also been investigated by the

researchers.⁹⁻¹⁶ Book⁹ proposed a recursive Lagrange dynamic model for flexible manipulator arms, in which 4×4 homogenous transformation matrices were used to describe the kinematics of the system, and the approach of assumed modes was employed to describe the deformation of the flexible links. It was indicated that the resulting dynamic equations are efficient for computation. However, the dynamic model can not be accurately used for simulation considering the torsion of the link. Furthermore, Book did not present any simulation example in his paper. Reference 14 did a further work based on Ref. 9. Both of the bending and torsional flexibility of the link were taken into account. Dynamic simulation of a spatial flexible manipulator arm was carried out to validate the algorithm. However, the dynamic stiffening effect was not considered. Reference 15 studied the bending and torsional deformation of the flexible link, in which torsional effect and shear effect of the cross-section were considered, but the flexible effect and the mass of the joint were not taken into account. Although scholars have done a lot of works on the dynamic analysis of flexible robots, only few of the investigations considered both of the dynamic stiffening effects and the torsion of the link for flexible-link and flexible-joint robots.^{12,13}

In this paper, dynamic formulation for multi-link spatial flexible manipulator arms is proposed. The stretching deformation, bending deformation and the torsional deformation of the links are all considered. Furthermore, the flexibility and the mass of the joint are considered too. The dynamic stiffening effects are captured in the dynamic modeling. A general-purpose software package of multi-link spatial flexible manipulator arms is developed and several illustrative simulation examples are given to validate the dynamic model and

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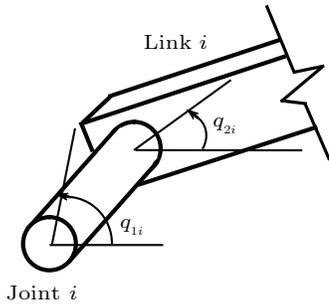


Fig. 1. The angular displacement of the flexible joint.

the corresponding software package.

In this paper, a spatial chain manipulator composed of N flexible link and N flexible joint is investigated. The flexible joint is simplified as a linear elastic torsion spring. As shown in Fig. 1, q_{1i} is the proximal angular displacement of the joint i , and q_{2i} is the distal angular displacement of the joint i . The flexible link is simplified as an Euler-Bernoulli beam, which ignores the shear deformation.

Assuming that the flexible link is straight before deformed, two coordinate systems are established each at the proximal and distal end of link i to express the transformation between different coordinate systems clearly.¹⁴ The definition of the coordinate systems $(X_b Y_b Z_b)_i$, $(X_d Y_d Z_d)_i$, $(H_x H_y H_z)'_i$ and $(H_x H_y H_z)_i$ can be seen in Ref. 14. When joint i is motionless, $(H_x H_y H_z)'_i$ is coincident with $(H_x H_y H_z)_{i-1}$, and matrix $\mathbf{H}\mathbf{H}_i^{i-1}$, the transformation matrix between them, is the function of q_{2i} . Matrix $\mathbf{d}\mathbf{H}_i$, the 4×4 homogeneous transformation matrix between frames $(X_d Y_d Z_d)_i$ and $(H_x H_y H_z)_i$, is a constant matrix. The transformation matrix $\mathbf{H}\mathbf{b}_i$ of $(H_x H_y H_z)'_i$ and $(X_b Y_b Z_b)_i$ is also a constant matrix.

Defining the joint-transformation matrix \mathbf{A}_i of joint i to be the transformation matrix from $(X_d Y_d Z_d)_{i-1}$ to $(X_b Y_b Z_b)_i$ one obtains that

$$\mathbf{A}_i = \mathbf{d}\mathbf{H}_{i-1} \mathbf{H}\mathbf{H}_i^{i-1} \mathbf{H}\mathbf{b}_i, \quad (1)$$

obviously, \mathbf{A}_i is a function of q_{2i} .

Defining \mathbf{E}_i to be the link-transformation matrix of link i from $(X_b Y_b Z_b)_i$ to $(X_d Y_d Z_d)_i$ and according to the assumption of small deformation of the links, \mathbf{E}_i can then be written as

$$\mathbf{E}_i = \mathbf{H}_i + \sum_{j=1}^{m_i} \delta_{ij} \mathbf{M}_{ij}, \quad (2)$$

in which

$$\mathbf{H}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{M}_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x_{ij} & 0 & -\theta_{zij} & \theta_{yij} \\ y_{ij} & \theta_{zij} & 0 & -\theta_{xij} \\ z_{ij} & -\theta_{yij} & \theta_{xij} & 0 \end{pmatrix}, \quad (3)$$

where x_{ij}, y_{ij}, z_{ij} are the x_{bi}, y_{bi}, z_{bi} components of the elastic linear displacement mode j of link i at the origin of the coordinate $(X_d Y_d Z_d)_i$, respectively. $\theta_{xij}, \theta_{yij}, \theta_{zij}$ are the x_{bi}, y_{bi}, z_{bi} rotation components of the elastic angular displacement mode j of link i at the origin of the coordinate $(X_d Y_d Z_d)_i$, respectively. δ_{ij} is the time-varying amplitude of mode j of link i , and m_i is the number of modes used to describe the deflection of link i .

Defining ${}^0\mathbf{W}_i$ or \mathbf{W}_i to be the 4×4 homogeneous transformation matrix from the base coordinate frame $(X_b Y_b Z_b)_0$ to $(X_b Y_b Z_b)_i$, one obtains that

$${}^0\mathbf{W}_i = \mathbf{W}_i = \mathbf{W}_{i-1} \mathbf{E}_{i-1} \mathbf{A}_i = \widehat{\mathbf{W}}_{i-1} \mathbf{A}_i, \quad (4)$$

where $\widehat{\mathbf{W}}_{i-1}$ is the 4×4 homogeneous transformation matrix from the base coordinate frame $(X_b Y_b Z_b)_0$ to the distal coordinate system $(X_d Y_d Z_d)_{i-1}$ of link $i-1$. $\widehat{\mathbf{W}}_i$ and \mathbf{W}_i can be derived by recursive formulation as Ref. 14.

To a generic point $P(\eta, 0, 0)$ on the link i , its homogeneous coordinates ${}^i\mathbf{h}_i(\eta)$ in the system $(X_b Y_b Z_b)_i$ in the deformed state can be approximated as

$${}^i\mathbf{h}_i(\eta) = [1 \quad \eta \quad 0 \quad 0]^T + \sum_{j=1}^{m_i} \delta_{ij} [0 \quad x_{ij}(\eta) \quad y_{ij}(\eta) \quad z_{ij}(\eta)]^T - \frac{1}{2} \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} \delta_{ij} \delta_{ik} [0 \quad \underline{x_{ijk}(\eta)} \quad 0 \quad 0]^T, \quad (5)$$

where the term with underline is the nonlinear strain coupling term, also known as the axial shortening due to the bending deformations of the link. When the flexible links are in motion at high speed, this term will bring the so called dynamic stiffening which will have a significant effect on the dynamic behavior of the flexible arms. To a single-link flexible manipulator arm, the rigid-flexible coupling dynamic model considering the dynamic stiffening effects has got perfect research. That also means the importance of dynamic stiffening effects in the dynamics of multi-link flexible manipulators. This underlined term can be detailed as

$$x_{ijk}(\eta) = \int_0^\eta \left(\frac{\partial y_{ij}(\xi)}{\partial \xi} \cdot \frac{\partial y_{ik}(\xi)}{\partial \xi} + \frac{\partial z_{ij}(\xi)}{\partial \xi} \cdot \frac{\partial z_{ik}(\xi)}{\partial \xi} \right) d\xi. \quad (6)$$

In terms of the fixed inertial coordinates of the base $(X_b Y_b Z_b)_0$, the position \mathbf{h}_i of the point is given as

$$\mathbf{h}_i = \mathbf{W}_i {}^i\mathbf{h}_i. \quad (7)$$

The velocity vector of the point in the inertial reference frame can be expressed as

$$\frac{d\mathbf{h}_i}{dt} = \dot{\mathbf{h}}_i = \dot{\mathbf{W}}_i {}^i\mathbf{h}_i + \mathbf{W}_i \dot{{}^i\mathbf{h}}_i. \quad (8)$$

Consequently, we can obtain the kinetic energy of link i in compact form as

$$K_i^b = K_{1i}^b + K_{2i}^b, \quad (9)$$

where K_{1i}^b is the kinetic energy of link i accounting for the rigid-body motion, the lateral and longitudinal deformation due to the flexibility, K_{2i}^b is the kinetic energy caused by torsional deformation motion which are defined as

$$K_{1i}^b = \frac{1}{2} \int_0^{L_i} \mu_i(\eta) \text{tr} \{ \dot{\mathbf{h}}_i \cdot \dot{\mathbf{h}}_i^T \} d\eta, \quad (10)$$

$$K_{2i}^b = \frac{1}{2} \int_0^{L_i} J_{xi}(\eta) \left(\frac{\partial \theta_{xi}(\eta, t)}{\partial t} \right)^2 d\eta, \quad (11)$$

where $\text{tr} \{ \cdot \}$ is the trace operator, μ_i is the mass per unit length, J_{xi} is the mass moment of inertia per unit length of the link about the neutral axis X , θ_{xi} is the torsional angle of link i , which can be expressed as

$$\theta_{xi} = \sum_{j=1}^{m_i} \delta_{ij} \theta_{xij}. \quad (12)$$

For the calculation of the kinetic energy of the joint i , we can lump its mass to link $i - 1$ in accord with the assumptions made in Ref. 17 for simplicity. Thus the kinetic energy need to be included only the part accounting for spinning kinetic energy of the rotor of joint i , namely

$$K_i^r = \frac{1}{2} J_{ri} \dot{\varphi}_i^2, \quad (13)$$

where J_{ri} is the moment of inertia of rotor i about its spinning axis, φ_i is the angular displacement of rotor i . The following relationship holds

$$\varphi_i = -n_i q_{1i}, \quad (14)$$

where n_i is the gear ratio of the joint.

Thus, the total kinetic energy of the system can be formed as

$$K = \sum_{i=1}^N (K_i^b + K_i^r). \quad (15)$$

The potential energy of the flexible manipulators mainly includes the elastic potential energy of the flexible links and flexible joints, and their gravitational potential energy. The elastic potential of the flexible link considered here has three parts: the axial deformation potential energy, the transversal deformation potential energy, and the torsional deformation potential energy.

$$V_i^b = \frac{1}{2} \int_0^{L_i} \left\{ E_i \left[A_{xi} \left(\frac{\partial x_i}{\partial \eta} \right)^2 + I_{yi} \left(\frac{\partial \theta_{yi}}{\partial \eta} \right)^2 + I_{zi} \left(\frac{\partial \theta_{zi}}{\partial \eta} \right)^2 \right] + G_i I_{xi} \left(\frac{\partial \theta_{xi}}{\partial \eta} \right)^2 \right\} d\eta, \quad (16)$$

where E_i is Young's modulus of the material, G_i is the shear modulus of the material; I_{xi} is the polar area moment of inertia of the link's cross section about the neutral axis, I_{yi} and I_{zi} are the area moments of inertia of the link's cross section about the y_i and z_i axes, respectively. It is worth mentioning that in Ref. 14 the axial deformation potential energy is ignored. Similar to the torsional angle θ_{xi} expressed in Eq. (12), θ_{yi} and θ_{zi} can also be expressed as

$$\theta_{yi} = \sum_{j=1}^{m_i} \delta_{ij} \theta_{yij}, \quad (17)$$

$$\theta_{zi} = \sum_{j=1}^{m_i} \delta_{ij} \theta_{zij}, \quad (18)$$

and x_i is the extension of the link, which can be expressed as

$$x_i = \sum_{j=1}^{m_i} \delta_{ij} x_{ij}. \quad (19)$$

With the assumption of linearly elastic torsional spring, we can get the potential energy of the joint as

$$V_i^r = \frac{1}{2} K t_i (q_{2i} - q_{1i})^2, \quad (20)$$

where $K t_i$ is stiffness coefficient of the torsional spring, the difference $q_{2i} - q_{1i}$ is the torsional angle of the spring.

Thus, the elastic potential energy of the system can be obtained as

$$V_e = \sum_{i=1}^N (V_i^b + V_i^r). \quad (21)$$

The gravitational energy of the system is

$$V_g = -\mathbf{g}^T \sum_{i=1}^N \mathbf{W}_i \mathbf{r}_i, \quad (22)$$

in which

$$\mathbf{g}^T = [0, g_x, g_y, g_z], \quad (23)$$

$$\mathbf{r}_i = M_i \mathbf{r}_{ri} + \sum_{k=1}^{m_i} \delta_{ik} \boldsymbol{\varepsilon}_{ik} - \frac{1}{2} \sum_{k=1}^{m_i} \sum_{l=1}^{m_i} \delta_{ik} \delta_{il} \boldsymbol{\kappa}_{ikl}, \quad (24)$$

here, M_i is the total mass of link i , $\mathbf{r}_{ri} = [1, r_{xi}, 0, 0]^T$ is the homogenous coordinates of the gravity center of link i (un-deformed) in the frame $(x_b y_b z_b)_i$.

$$\boldsymbol{\varepsilon}_{ik} = \int_0^{L_i} \mu_i(\eta) [0, x_{ik}, y_{ik}, z_{ik}]^T d\eta, \quad (25)$$

$$\boldsymbol{\kappa}_{ikl} = \int_0^{L_i} \mu_i(\eta) [0, \underline{x_{ikl}}, 0, 0]^T d\eta. \quad (26)$$

Then, we can get the total potential energy of the system as

$$V = V_e + V_g. \quad (27)$$

We use the Lagrange method to derive the dynamic equations of the system in accord with the methods of Refs. 9 and 14. By taking the generalized coordinates as $\mathbf{z} = [q_{11}, q_{21}, \delta_{11}, \delta_{12} \cdots \delta_{1m_1}, \cdots q_{1i}, q_{2i}, \delta_{i1}, \delta_{i2}, q_{1N}, q_{2N}, \delta_{N1}, \cdots \delta_{Nm_N}]^T$, and substituting the kinetic and potential energy expressions Eqs. (15) and (27) into the Lagrange equation of the second kind, and performing the required differentiation and algebraic manipulations, the dynamic equations can be written in compact form as

$$\mathbf{J}\ddot{\mathbf{z}} = \mathbf{R}, \tag{28}$$

where \mathbf{J} is the generalized mass matrix

$$\mathbf{J} = \begin{pmatrix} {}^1\mathbf{J}_1 & {}^2\mathbf{J}_1 & \cdots & {}^{N-1}\mathbf{J}_1 & {}^N\mathbf{J}_1 \\ {}^1\mathbf{J}_2 & {}^2\mathbf{J}_2 & \cdots & {}^{N-1}\mathbf{J}_2 & {}^N\mathbf{J}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {}^1\mathbf{J}_{N-1} & {}^2\mathbf{J}_{N-1} & \cdots & {}^{N-1}\mathbf{J}_{N-1} & {}^N\mathbf{J}_{N-1} \\ {}^1\mathbf{J}_N & {}^2\mathbf{J}_N & \cdots & {}^{N-1}\mathbf{J}_N & {}^N\mathbf{J}_N \end{pmatrix}, \tag{29}$$

and

$${}^h\mathbf{J}_j = \begin{pmatrix} J_r & 0 & 0 & \cdots & 0 \\ 0 & J_{jh} & J_{jh1} & \cdots & J_{jhm_h} \\ 0 & J_{j1h} & I_{j1h1} & \cdots & I_{j1hm_h} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & J_{jm_jh} & I_{jm_jh1} & \cdots & I_{jm_jhm_h} \end{pmatrix}. \tag{30}$$

The inertia coefficient J_r is expressed as follows: for $h = j$,

$$J_r = n_j^2 J_{rj}, \tag{31}$$

for $h \neq j$,

$$J_r = 0. \tag{32}$$

The inertia coefficient J_{jh} is

$$J_{jh} = 2\text{tr}\{\hat{\mathbf{W}}_{j-1} \mathbf{U}_j^j \tilde{\mathbf{F}}_h \mathbf{U}_h^T \hat{\mathbf{W}}_{h-1}^T\}. \tag{33}$$

The inertia coefficient J_{jkk} ($1 \leq k \leq m_h$) is expressed as follows: for $h = N; j = 1, 2, \cdots, N$,

$$J_{jkk} = 2\text{tr}\{\hat{\mathbf{W}}_{j-1} \mathbf{U}_j^j \tilde{\mathbf{W}}_h \mathbf{H}_{hk} \mathbf{W}_h^T\}; \tag{34}$$

for $h = j, j + 1, \cdots, N - 1; j = 1, 2, \cdots, N - 1$,

$$J_{jkk} = 2\text{tr}\{\hat{\mathbf{W}}_{j-1} \mathbf{U}_j^j \mathbf{F}_h \mathbf{M}_{hk}^T + {}^j\tilde{\mathbf{W}}_h \mathbf{H}_{hk}\} \mathbf{W}_h^T; \tag{35}$$

for $h = 1, 2, \cdots, j - 1; j = 2, 3, \cdots, N$,

$$J_{jkk} = 2\text{tr}\{\hat{\mathbf{W}}_{j-1} \mathbf{U}_j^j \mathbf{F}_h \mathbf{M}_{hk}^T \mathbf{W}_h^T\}. \tag{36}$$

Note that the inertia coefficient for the deflection variable δ_{hk} in the joint equation q_{2j} is the same as

that for the joint variable q_{2j} in the deflection equation δ_{hk} . So, there is

$$J_{jkk} = J_{jhhk}^T. \tag{37}$$

The inertia coefficient I_{jfhk} ($1 \leq f \leq m_j, 1 \leq k \leq m_h$) is expressed as follows: for $h = j = N$,

$$I_{jfhk} = 2\text{tr}\{\mathbf{N}_{jfk}\} + 2T_{jfk}; \tag{38}$$

for $h = j = 1, 2, \cdots, N - 1$,

$$I_{jfhk} = 2\text{tr}\{\mathbf{M}_{jf}^j \phi_j \mathbf{M}_{jk}^T + \mathbf{N}_{jfk}\} + 2T_{jfk}; \tag{39}$$

for $h = N; j = 1, 2, \cdots, N - 1$,

$$I_{jfhk} = 2\text{tr}\{\mathbf{W}_j \mathbf{M}_{jf}^j \mathbf{W}_h \mathbf{H}_{hk} \mathbf{W}_h^T\}; \tag{40}$$

for $h = j + 1, j + 2, \cdots, N - 1; j = 1, 2, \cdots, N - 1$,

$$I_{jfhk} = 2\text{tr}\{\mathbf{W}_j \mathbf{M}_{jf}^j ({}^j\phi_h \mathbf{M}_{hk}^T + {}^j\mathbf{W}_h \mathbf{H}_{hk}) \mathbf{W}_h^T\}. \tag{41}$$

\mathbf{R} is the generalized force matrix as

$$\mathbf{R} = [R_{r1}, R_{J1}, R_{111}, \cdots, R_{1m_1}, \cdots, R_{rj}, R_{Jj}, R_{j11}, \cdots, R_{jm_j}, \cdots, R_{rN}, R_{JN}, R_{N11}, \cdots, R_{Nm_N}]^T, \tag{42}$$

where,

$$R_{rj} = -q_{1j} K t_j + q_{2j} K t_j + F_j, \tag{43}$$

$$R_{Jj} = -2\text{tr}\{\hat{\mathbf{W}}_{j-1} \mathbf{U}_j \mathbf{Q}_j\} + \mathbf{g}^T \hat{\mathbf{W}}_{j-1} \mathbf{U}_j \mathbf{P}_j + q_{1j} K t_j - q_{2j} K t_j. \tag{44}$$

R_{jff} can be expressed as follows: for $j = 1, 2, \cdots, N - 1$,

$$R_{jff} = -2\text{tr}\left\{\mathbf{W}_j \mathbf{M}_{jf} \mathbf{A}_{j+1} \mathbf{Q}_{j+1} + \left[\ddot{\mathbf{W}}_{vj} \mathbf{H}_{jf} + 2\dot{\mathbf{W}}_j \left(\sum_{k=1}^{m_j} \mathbf{N}_{jfk} \dot{\delta}_{jk}\right) + \mathbf{W}_j \left(\sum_{k=1}^{m_j} \sum_{l=1}^{m_j} \mathbf{P}_{jfk l} \dot{\delta}_{jk} \dot{\delta}_{jl}\right)\right] \mathbf{W}_j^T\right\} - \sum_{k=1}^{m_j} \delta_{jk} K_{jfk} + \mathbf{g}^T \mathbf{W}_j \boldsymbol{\epsilon}_{jf} + \mathbf{g}^T \mathbf{W}_j \mathbf{M}_{jf} \mathbf{A}_{j+1} \mathbf{P}_{j+1} - \mathbf{g}^T \mathbf{W}_j \sum_{k=1}^{m_j} \delta_{jk} \boldsymbol{\kappa}_{jfk}; \tag{45}$$

for $j = N$,

$$R_{jff} = -2\text{tr}\left\{\ddot{\mathbf{W}}_{vj} \mathbf{H}_{jf} + 2\dot{\mathbf{W}}_j \left(\sum_{k=1}^{m_j} \mathbf{N}_{jfk} \dot{\delta}_{jk}\right) + \mathbf{W}_j \left(\sum_{k=1}^{m_j} \sum_{l=1}^{m_j} \mathbf{P}_{jfk l} \dot{\delta}_{jk} \dot{\delta}_{jl}\right) \mathbf{W}_j^T\right\} -$$

$$\sum_{k=1}^{m_j} \delta_{jk} K_{jfk} + \mathbf{g}^T \mathbf{W}_j \boldsymbol{\varepsilon}_{jf} - \mathbf{g}^T \mathbf{W}_j \sum_{k=1}^{m_j} \delta_{jk} \boldsymbol{\kappa}_{jfk}. \quad (46)$$

The concrete expressions of the terms in Eqs. (33)–(46), such as ${}^j \tilde{\mathbf{F}}_h, \mathbf{H}_{hk}, \mathbf{N}_{jfk} \dots$ are omitted here.

A general-purpose software package for the dynamic simulation of multi-link spatial flexible manipulator arms, which is written in C++, is developed here based on the proposed algorithm. In order to validate the algorithm and the package, and the effects of dynamic stiffening, flexibilities of the link and joint on the dynamics of the system, several dynamic simulations are given in the following section, where the specific parameters of the manipulator are referenced to the Canadarm2¹⁸ serving on the International Space Station.

In order to study the dynamic stiffening effect of the flexible manipulator with large overall motion, we will take a planar single-link flexible manipulator with uniform rotation as example. Herein, the length of the link is 7.11 m, the mass of the link is 314.88 kg, $EI = 3.8 \times 10^6 \text{ N} \cdot \text{m}^2$, $EA = 2.84 \times 10^7 \text{ N}$. In this example, the gravity and the flexible effect of the joint are ignored.

Figures 2(a) and 2(b) are the chordwise deformations of the manipulator considered and ignored dynamic stiffening effects when the large overall motion is at low speed (angular speed $\omega = 10 \text{ rad/s}$) and at high speed (angular speed $\omega = 80 \text{ rad/s}$), respectively. It is shown that when the large overall motion is at low speed, the chordwise deformations of the link tip are almost the same wherever the dynamic stiffening effects are considered or ignored. However, when the large overall motion is at high speed, the simulation result of the manipulator obtained without consideration of dynamic stiffening effects will be divergent while the simulation result of the manipulator obtained with the consideration of the dynamic stiffening effects is still convergent. That shows the significance of the dynamic stiffening effects on the dynamic performance of a flexible manipulator undergoing high speed large overall motion.

A two-link spatial flexible manipulator arm is shown in Fig. 3, the arm falls free by gravity from the horizontal position. The neutral axis of link 1 and link 2 are not in alignment, and there exists an offset of 0.475 m between them. So there will appear a torque to act on link 1 in the fall process, and consequently link 1 will perform torsion about its axis. In this case the manipulator will conduct a spatial motion. That is why the torsional deformation is necessary in the dynamic modeling. The moments of inertia of the joints are $n_1^2 J_{r1} = n_2^2 J_{r2} = 5 \text{ kg} \cdot \text{m}^2$, and the stiffness coefficients are $Kt_1 = Kt_2 = 1.33 \times 10^6 \text{ N/m}$.

The dimensions and material properties of the spatial flexible arms are given in Table 1. Compare the motion process of a flexible joint flexible link manipulator and a rigid joint flexible link manipulator, where

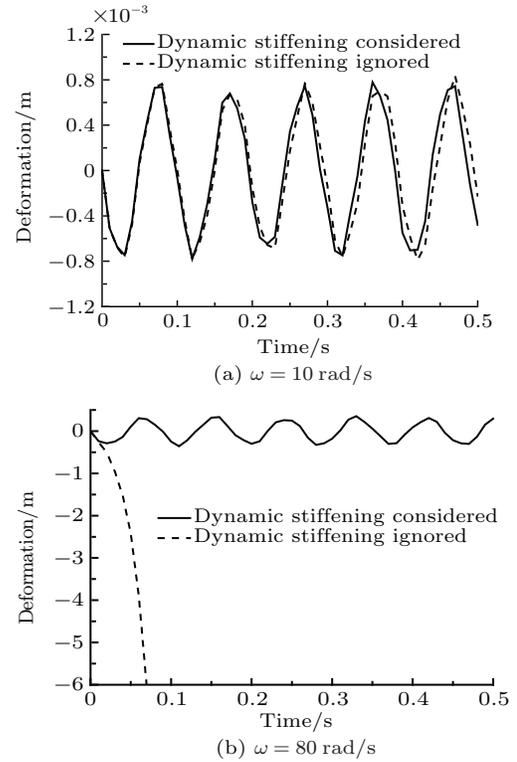


Fig. 2. The chordwise deformation of link tip.

Table 1. Data of the spatial flexible arm.

	Link 1	Link 2
Length/m	7.11	7.11
Mass/kg	314.88	314.88
The gravity center/m	(3.555,0,0)	(3.555,0,0)
$J_{xi}/(\text{kg} \cdot \text{m}^2)$	15.41	15.41
$E_i I_{yi} = E_i I_{zi}/(\text{N} \cdot \text{m}^2)$	3.8×10^6	3.8×10^6
$E_i A_{xi}/\text{N}$	2.84×10^7	2.84×10^7
$G_i I_{xi}/(\text{N} \cdot \text{m}^2)$	1.93×10^6	1.93×10^6
Modes	10	10

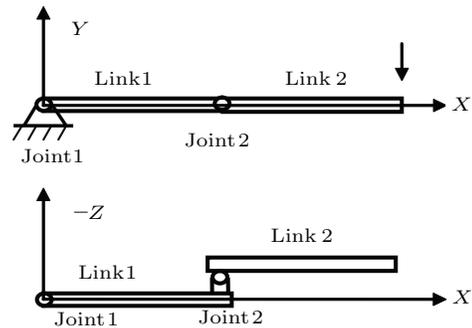


Fig. 3. Two-link spatial flexible manipulator arms.

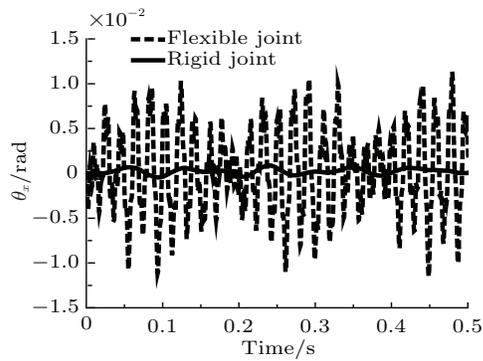


Fig. 4. The torsional angle of link 1.

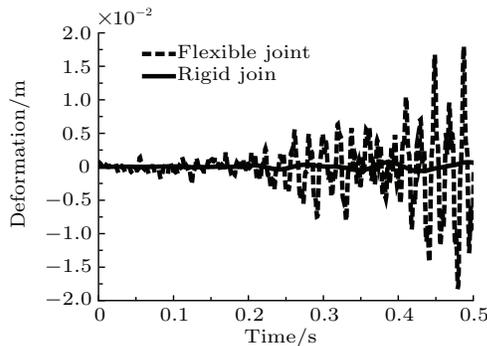


Fig. 5. The flapwise deformation of link tip.

the specific parameters of the two manipulators are the same.

Figure 4 shows the time history of the torsional angle of link 1 about its axis. As the arm falls, the torsional angle is lingering at the equilibrium position due to the flexible effect. When the joints are rigid, the torsional angle is changing in a small range. But when the joints are flexible, the changing range of the torsional angle increases obviously, which shows the effect of nonlinearity and instability. This means there exists the dynamic coupling between the torsional flexibility of the link and the flexibility of the joint.

Figure 5 is the flapwise deformation (direction Z) of link 2 tip. We can see that the tip deformation of the flexible link increases as the time changes, and presents significant nonlinear vibration. Also, the phenomenon is more pronounced when the joints are flexible. Certainly, we can get the same phenomenon on the longitudinal (direction X) and chordwise (direction Y) deformations.

Obviously, there is a significant difference between the dynamic performance of the flexible joint and flexible link manipulator and the rigid joint and flexible link manipulator. When the joints are flexible, the system will appear significant nonlinear. That is why we need to consider the flexibility of the joints in the precise dynamic modeling. In addition, the flapwise deformation of the flexible links, caused by the torsional effect of

the flexible joint, will have a significant influence on the system control.

In this paper, a further investigation into the dynamic modeling and simulation of the multi-link spatial flexible manipulator arms are presented based on Ref. 14. The stretching deformation, bending deformation and the torsional deformation of the link are all considered. Furthermore, the flexibility and the mass of the joint are considered, too. The complete governing equations of motion of the system are derived via the Lagrange equations. In the modeling the so called dynamic stiffening effects are included via the adoption of nonlinear deformation description. Based on this model, a general-purpose software package of multi-link manipulator arms is developed. Then examples are simulated using this software. The results show that the dynamic stiffening effects and flexible joints will all have a great influence on the dynamic performance of the manipulators. The dynamic model and the software package presented in this paper can be used in a variety of dynamic analysis for complicated mechanical system.

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