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Delay-Independent Synchronization and Network Topology of Systems with Transmission Delay Couplings

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Abstract : We investigate the relationship between graph topology and delay-independent synchronization occuring in networks of the identical nonlinear systems with transmission delays. In this paper, we show that if networks contain a cycle subgraph of an odd number of nodes, partial synchronization corresponding to the equitable graph partition with the fewest cells occurs for a sufficiently large coupling strength regardless of the length of time-delay. The validity of the obtained results are examined through numerical simulations of Hindmarsh-Rose neuron systems networks.

Key Words : synchronization, time-delay, chaotic systems, graph theory.

1. Introduction

Synchronous behaviors appear in wide range of physical systems interconnected by exchanging their information or signals [1]. Synchronization in Huygens' pendulum clocks and flashing fireflies are typical examples. These are examples of synchronization in networks of periodic or almost periodic systems, but even if each system is chaotic, synchronization may emerge between the coupled systems. Therefore, synchronization in networks of chaotic systems is a recent hot topic in the interdisciplinary fields of applied physics, applied mathematics, mathematical biology and control science.

Synchronization conditions for networks of nonlinear systems including chaotic systems have been investigated by many researchers. The recent focus is shifting to the synchronization problem for large-scale networks with delayed couplings. In practical situations, the existence of data processing time and the limited transfer speed of signals cause time-delays in the interconnection between systems. Therefore, the effect of time-delay in synchronization is an important problem to be investigated in detail. Since, in general, the existence of time-delay tends to destabilize the system, the synchronization problem for delay-coupled systems has been investigated from a viewpoint of the delay-dependent stability of delay differential equations related to the synchronization error dynamics (cf. [2]–[5]). In networks of systems with transmission delays, however, synchronization may occur regardless of the length of time-delays. Wang and Slotine [6] show that the coupled integrator systems always achieve agreement for arbitrary length of time-delay. For networks of chaotic nonlinear systems, Yanagi and Oguchi [7] clarify network structures such that the synchronization error dynamics become delay-free differential equations leading to delay-independent synchronization. On the other hand, based on the delay-independent stability of delay systems, sufficient conditions for full synchronization and partial synchronization for general networks are derived in the form of linear matrix inequalities (LMIs) in [8] and [9]. However, the obtained LMI-based conditions do not explicitly indicate the relation between delay-independent synchronization and network topology.

In this paper, we attempt to characterize the structures of networks in which full or partial synchronization occurs independently of the length of time-delay. On the relationship between network topology and delay-independent synchronization, Yanagi and Oguchi [10] clarify the relationship between ring networks of systems with transmission delays and delay-independent synchronization by using the Lyapunov-Razumikhin approach. In this paper, we determine the synchronization patterns in more general networks via another approach based on the solvability of LMI condition derived in [9]. In particular, we show that if networks contain a cycle subgraph composed of an odd number of nodes, delay-independent partial synchronization occurs in the form of the equitable graph partition with the fewest cells. We also show that additionally if all the nodes have the same degree, full synchronization occurs in the network, no matter how long the transmission delay is, which is conjectured in [10]. The obtained results include the same consequences as those in [10] for ring networks as special cases.

This paper is organized as follows. In Section 2, we describe the coupled systems to be studied in this paper and briefly review the LMI condition for delay-independent synchronization in [9]. Based on the solvability of the LMI condition, Section 3 clarifies the relationship between delay-independent synchronization and network topology in bidirectional networks. The validity of the results is examined through numerical simulations in Section 4. Section 5 concludes this paper.

2. Preliminaries

2.1 Notation

The Euclidian norm of real vector x is written as $\|x\|$. The notation $\mathrm{col}(x_1, \ldots, x_n)$ denotes the column vector with scalar or vector elements $x_1, \ldots, x_n$. Let $\mathbf{1}_n$ stand for the n-dimensional column vector with all elements one. The n-dimensional identity matrix is denoted as I_n. Let $\mathbf{0}_{nm}$ denote the $n \times m$ matrix with all elements zero. The notation $\mathrm{diag}(A_1, \ldots, A_n)$ stands

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for the matrix with matrix elements $A_1, \ldots, A_n$ in the diagonal components. The notation $C^r(\mathcal{X}, \mathcal{Y})$ denotes the space of $r \geq 0$ times continuously differentiable functions mapping $\mathcal{X}$ to $\mathcal{Y}$. Let $(\nabla V(x))^\top = \partial V(x)/\partial x$ for a differentiable scalar function $V(x)$.

The communication topology to be considered in this paper is encoded in a weighted undirected graph $\mathcal{G} = (V, E)$ with a node set V and an edge set E. Here, a positive value w_{ij} is assigned to edge $(i, j) \in E$ as an edge weight if the edge exists. In a diagram of graph, each edge weight is written on the corresponding edge and we omit it if every edge has weight one. A path from node i_1 to node i_n is a sequence $\{i_1, i_2, \ldots, i_n\}$ such that $(i_l, i_{l+1}) \in E$ for $l = 1, 2, \ldots, n-1$. A cycle is defined as a path with $(i_n, i_1) \in E$. A graph is said to be connected if there exists a path from node i_1 to node i_n between any two nodes. A graph is said to be bipartite if nodes can be divided into two disjoint sets V_1 and V_2 such that every edge connects between a node in V_1 and one in V_2.

2.2 System Description

In this paper, we consider the following N identical nonlinear systems

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + Bu_i(t) \\ y_i(t) = Cx_i(t) \end{cases} \quad i = 1, 2, \ldots, N \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^m$ denote the state, input and output of system i, respectively. $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth vector field, and B and C are constant matrices with suitable dimensions. Here, we assume that CB is positive definite [11]. This assumption means that if the systems are single-input single-output systems, each system has the relative degree one. Under this assumption, the systems (1) can be transformed into

$$\begin{cases} \dot{z}_i(t) = q(z_i(t), y_i(t)) \\ \dot{y}_i(t) = a(y_i(t), z_i(t)) + CBu_i(t) \end{cases} \quad i = 1, 2, \ldots, N \tag{2}$$

where $z_i \in \mathbb{R}^{n-m}$, $q : \mathbb{R}^n \to \mathbb{R}^{n-m}$ and $a : \mathbb{R}^n \to \mathbb{R}^m$. The systems (2) are coupled via

$$u_i(t) = k \sum_{j=1,\, j\neq i}^{N} w_{ij}(y_j(t-\tau) - y_i(t)) \tag{3}$$

where $k > 0$ denotes the constant coupling strength, $\tau \geq 0$ is the time-delay, and w_{ij} is the weight of coupling between systems i and j, *i.e.* $w_{ij} = w_{ji} > 0$ if systems i and j exchange their delayed outputs with each other, and $w_{ij} = w_{ji} = 0$ otherwise. The network topology of the coupled systems (2), (3) is represented as an undirected graph $\mathcal{G} = (V, E)$ with the adjacency matrix $A = [w_{ij}]$ and the degree matrix $D = \mathrm{diag}(d_1, \ldots, d_N)$ where $w_{ii} = 0$ and $d_i = \sum_{j=1}^{N} w_{ij}$. The Laplacian matrix is defined by $L = D - A$. Throughout this paper, we assume that each system (2) satisfies the followings:

Assumption 1 Each system (2) is strictly C^1-semipassive with a radially unbounded positive definite storage function V, *i.e.* there exists a radially unbounded positive definite function $V \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ such that

$$\dot{V}(z_i, y_i) \leq y_i^\top u_i - H(z_i, y_i)$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is a scalar function which is positive outside some ball $\mathcal{B} = \{\mathrm{col}(z_i, y_i) \in \mathbb{R}^n \mid \|\mathrm{col}(z_i, y_i)\| < R\}$ with some $R > 0$.

Assumption 2 For dynamics of each z_i in (2), there exists a positive definite function $V_0 \in C^2(\mathbb{R}^{n-m}, \mathbb{R}_{\geq 0})$ such that

$$(\nabla V_0(z_i - z_j))^\top (q(z_i, y^*) - q(z_j, y^*)) \leq -\alpha \|z_i - z_j\|^2$$

for all $z_i, z_j \in \mathbb{R}^{n-m}$ and $y^* \in \mathbb{R}^m$ where $\alpha > 0$.

Assumption 1 guarantees that the coupled systems (2), (3) are ultimately bounded and all solutions converge to a ball with a certain radius for any values of k and τ in finite time [5]. Assumption 2 implies that $z_i = z_j$ is globally asymptotically stable fixing as $y_i(t) = y_j(t) = y^*(t)$. Then $z_i(t)$ and $z_j(t)$ converge to a unique solution determined by $y^*(t)$.

In what follows, we deal with the global asymptotic synchronization in the coupled systems (2), (3), *i.e.* systems i and j are globally asymptotically synchronized if $\|X_i(t) - X_j(t)\| \to 0$ as $t \to \infty$ for any initial conditions $X_i(\theta)$ and $X_j(\theta)$, $\theta \in [-\tau, 0]$ where $X_i(\theta) = \mathrm{col}(z_i(\theta), y_i(\theta))$.

2.3 Graph Partition and Synchronization Condition Based on LMI

To discuss synchronization patterns, we define an equitable partition of V. Here, a partition is a set π of non-empty and disjoint subsets $C_1, C_2, \ldots, C_W$ of V where $C_1 \cup C_2 \cup \cdots \cup C_W = V$, and the element $C_l \in \pi$ is called a cell. The following definition is a slight modification of the equitable partition for an unweighted graph introduced in [12].

Definition 1 A partition $\pi = \{C_1, C_2, \ldots, C_W\}$ of the node set V of a graph $\mathcal{G} = (V, E)$ is said to be equitable if the relation

$$\sum_{j\in C_k} w_{i_1 j} = \sum_{j\in C_k} w_{i_2 j}, \quad \forall i_1, i_2 \in C_l \tag{4}$$

holds for any $C_l, C_k \in \pi$.

The equitable partition means that every node in the same cell has the same total amount of weights of edges coming from neighbors in each cell. From the definition, nodes belonging to the same cells in an equitable partition have the same degree because the relation (4) necessary holds for any $l, k = 1, 2, \ldots, W$ including $l = k$. Additionally, there exists an equitable partition composed of just one cell including all the nodes if they have the same degree. As mentioned in [9], synchronization appears only in the patterns of equitable partitions. Therefore, we focus on equitable partitions in the rest of the paper.

Here, we briefly review a delay-independent synchronization condition given in the form of LMI in [9], which ensures that systems in the same cells of equitable partition synchronize regardless of the value of time-delay τ. Let $\mathcal{G} = (V, E)$ be a graph with an equitable partition $\pi = \{C_1, C_2, \ldots, C_W\}$ and let cell $C_l \in \pi$ have r_l nodes. For simplicity, we relabel cells and nodes as follows:

(I) Each cell in the first M cells $C_1, C_2, \ldots, C_M \in \pi$ contains more than one node and each cell in the other $W - M$ cells $C_{M+1}, C_{M+2}, \ldots, C_W \in \pi$ has only one node,

(II) Nodes are numbered from C_1 so that $1, 2, \ldots, r_1 \in C_1$, $r_1 + 1, r_1 + 2, \ldots, r_1 + r_2 \in C_2$, and so on.

Let s_l be the minimum index associated with a node in C_l. Define the matrices $H_0 = [h_{ij}^0] \in \mathbb{R}^{W\times N}$ and $H_1 \in \mathbb{R}^{(N-W)\times N}$ as

$$h_{lj}^0 = \begin{cases} 1 & \text{if } j = s_l \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

$$H_1 = \left(\begin{array}{ccc|c} h_1 & & \mathbf{0} & \\ & \ddots & & \mathbf{0} \\ \mathbf{0} & & h_M & \end{array}\right) \tag{6}$$

with block matrices $h_l = [\mathbf{1}_{r_l-1} \;\; -I_{r_l-1}]$ for $l = 1, 2, \ldots, M$. In addition, define the nonsingular matrix $H \in \mathbb{R}^{N\times N}$ as

$$H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}.$$

From the simple calculations, HDH^{-1} and HAH^{-1} are given in the following forms.

$$HDH^{-1} = \begin{pmatrix} \mathcal{D} & \mathbf{0} \\ \mathbf{0} & D_\pi \end{pmatrix},\; HAH^{-1} = \begin{pmatrix} \mathcal{A} & * \\ \mathbf{0} & A_\pi \end{pmatrix}$$

Here, $\mathcal{D} = \mathrm{diag}(d_{s_1}, d_{s_2}, \ldots, d_{s_W})$, $\mathcal{A}$ is a $W \times W$ matrix with $\sum_{j\in C_k} w_{s_l j}$ in the (l, k)-th entry, that is $\mathcal{D}$ and $\mathcal{A}$ are the degree matrix and the adjacency matrix of the graph representing the network of cells, respectively, $D_\pi = H_1 D H_1^+ = \mathrm{diag}(d_{s_1} I_{r_1-1}, d_{s_2} I_{r_2-1}, \ldots, d_{s_M} I_{r_M-1})$ and $A_\pi = H_1 A H_1^+$ where H_1^+ is a pseudo-inverse matrix of H_1 such that

$$H_1^+ = \left(\begin{array}{ccc} h_1^+ & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & h_M^+ \\ \hline & \mathbf{0} & \end{array}\right) \in \mathbb{R}^{N\times(N-W)}$$

with block matrices h_l^+ given by

$$h_l^+ = \begin{pmatrix} 0 & \cdots & 0 \\ -1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & -1 \end{pmatrix} \in \mathbb{R}^{r_l\times(r_l-1)}.$$

for $l = 1, 2, \ldots, M$. Here, h_l^+ is a pseudo-inverse matrix of h_l.

Theorem 1 Consider the coupled systems (2), (3) with a network represented by a weighted undirected graph $\mathcal{G} = (V, E)$. Suppose that Assumptions 1 and 2 hold, and let a partition π of V be equitable. If there exist positive definite matrices P and $S \in \mathbb{R}^{(N-W)\times(N-W)}$ satisfying

$$\begin{pmatrix} PD_\pi + D_\pi P - S & -PA_\pi \\ -A_\pi^\top P & S \end{pmatrix} > 0, \tag{7}$$

then there is a constant $\bar{k}_\pi > 0$ such that all systems corresponding to nodes in the same cells of π are globally asymptotically synchronized for $k > \bar{k}_\pi$ regardless of time-delay $\tau \geq 0$.

The proof of Theorem 1 is given in a similar way as the proof for $m = 1$ in [9]. Therefore the proof is omitted.

3. Patterns of Delay-Independent Synchronization and Network Topologies

This section clarifies the relationship between synchronization pattern and network topology by identifying equitable partitions π satisfying the LMI (7) with

$$P = (H_1 H_1^\top)^{-1} = \mathrm{diag}((h_1 h_1^\top)^{-1}, \ldots, (h_M h_M^\top)^{-1}), \tag{8}$$

$$S = (H_1 H_1^\top)^{-1} D_\pi = \mathrm{diag}(d_{s_1}(h_1 h_1^\top)^{-1}, \ldots, d_{s_M}(h_M h_M^\top)^{-1}). \tag{9}$$

Here, the matrices P and S in (8), (9) are positive definite as shown in the following. $h_l h_l^\top$ is written as

$$h_l h_l^\top = \begin{pmatrix} \mathbf{1}_{r_l-1} & -I_{r_l-1} \end{pmatrix}\begin{pmatrix} \mathbf{1}_{r_l-1} & -I_{r_l-1} \end{pmatrix}^\top = I_{r_l-1} + \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

and the eigenvalues of $h_l h_l^\top$ are r_l and 1 since the eigenvalues of the $(r_l - 1) \times (r_l - 1)$ matrix with all entries one are calculated as $r_l - 1$ and 0. Therefore, $h_l h_l^\top$ are positive definite, and so are P and S. Considering the pseudo-inverse matrix $H_1^\dagger$ of H_1 constructed by replacing h_l^+ in H_1^+ with

$$h_l^\dagger = \frac{1}{r_l}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1-r_l & 1 & \ddots & \vdots \\ 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & & 1 \\ 1 & \cdots & 1 & 1-r_l \end{pmatrix}$$

for $l = 1, 2, \ldots, M$, the following equations hold:

$$H_1 D H_1^+ = H_1 D H_1^\dagger,$$
$$H_1 A H_1^+ = H_1 A H_1^\dagger,$$
$$H_1^\dagger = H_1^\top (H_1 H_1^\top)^{-1}.$$

Substituting the equations (8) and (9) into LMI (7) and using the above equations, we obtain the matrix inequality

$$\begin{pmatrix} H_1^\dagger & \mathbf{0} \\ \mathbf{0} & H_1^\dagger \end{pmatrix}^\top \begin{pmatrix} D & -A \\ -A & D \end{pmatrix}\begin{pmatrix} H_1^\dagger & \mathbf{0} \\ \mathbf{0} & H_1^\dagger \end{pmatrix} > 0. \tag{10}$$

Here, the matrix

$$\bar{L} := \begin{pmatrix} D & -A \\ -A & D \end{pmatrix}$$

represents the Laplacian matrix of an undirected graph with $2N$ nodes obtained from the following procedures:

(i) Make node set $V' = \{1', 2', \ldots, N'\}$ with the same number of nodes as those of $V = \{1, 2, \ldots, N\}$ and construct $\bar{V} = V \cup V'$,

(ii) If there is an edge $(i, j) \in E$ with weight w_{ij}, add bidirectional edges (i, j') and (i', j) with weight w_{ij}.

Defining $\bar{E}$ as the set of edges added in (ii), the obtained undirected graph $\bar{\mathcal{G}} = (\bar{V}, \bar{E})$ has the Laplacian matrix $\bar{L}$. Then, $\bar{\mathcal{G}}$ is bipartite as there is no edge between any two nodes in V and also between any two nodes in V'. Figure 1 shows an example of $\bar{\mathcal{G}}$. In Fig. 1, $\bar{\mathcal{G}}$ is described by putting nodes of V and V', and nodes i and j' are connected if there exists an edge $(i, j) \in E$. The obtained $\bar{\mathcal{G}}$ is composed of two connected components $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$. In the following discussion, we find equitable partitions π that satisfy the matrix inequality (10) by analyzing $\bar{\mathcal{G}}$ and $\bar{L}$. Here, the graph $\bar{\mathcal{G}}$ has the following properties.

Lemma 1 For a connected weighted undirected graph $\mathcal{G} = (V, E)$, the graph $\bar{\mathcal{G}} = (\bar{V}, \bar{E})$ obtained by the procedures (i) and (ii) has the following properties.

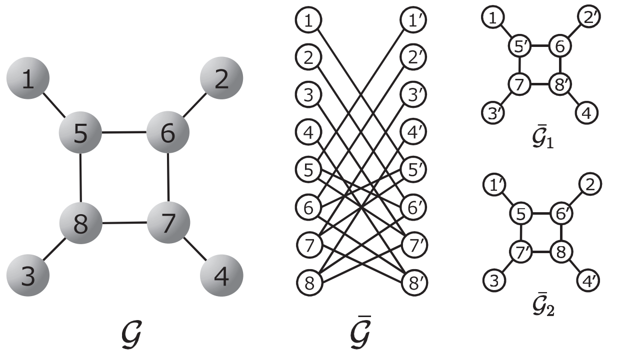

Fig. 1 Graph $\bar{\mathcal{G}}$ obtained by the procedures (i) and (ii) from the left-hand graph $\mathcal{G}$. The obtained $\bar{\mathcal{G}}$ consists of two components $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ shown in the right-hand graphs.

(L1) If $\mathcal{G}$ contains at least one cycle of an odd number of nodes, then $\bar{\mathcal{G}}$ is connected.

(L2) Otherwise, $\bar{\mathcal{G}}$ has exactly two connected components.

Proof If nodes i and j are adjacent in $\mathcal{G}$, there exist edges (i, j') and $(i', j) \in \bar{E}$. Hence, nodes i and j' are in the identical connected component and so are nodes i' and j. Since $\mathcal{G}$ is connected, every node in $\bar{\mathcal{G}}$ is in the component with either nodes i and j' or i' and j, and thus the number of connected components of $\bar{\mathcal{G}}$ is at most two.

Let $\mathcal{G}$ have a cycle of an odd number of nodes and have edges $(1, 2), (2, 3), \ldots, (N_o, 1) \in E$ with some odd number N_o. There are possibly two connected components in $\bar{\mathcal{G}}$ such that either one of the two contains node 1 and another contains node $1'$. From the procedure (ii), there exist edges $(1, 2'), (2', 3), (3, 4'), \ldots, (N_o, 1') \in \bar{E}$. Thus, $\bar{\mathcal{G}}$ is connected since nodes 1 and $1'$ belong to the identical component and we conclude (L1).

On the other hand, it is well known that a graph has no cycle of any odd number of nodes if and only if it is bipartite. Hence, if $\mathcal{G}$ has no cycle of any odd number of nodes, $\mathcal{G}$ has the partition $\tilde{\pi} = \{\tilde{C}_1, \tilde{C}_2\}$ of V such that any two adjacent nodes in $\mathcal{G}$ belong to the separate cells. Now, we define a new partition $\tilde{\pi}' = \{\tilde{C}'_1, \tilde{C}'_2, \tilde{C}'_3, \tilde{C}'_4\}$ of $\bar{V}$ such that if node $i \in V$ is in $\tilde{C}_1$ (or $\tilde{C}_2$), node $i \in \bar{V}$ belongs to $\tilde{C}'_1$ (or $\tilde{C}'_2$) and node $i' \in \bar{V}$ belongs to $\tilde{C}'_3$ (or $\tilde{C}'_4$). From the procedure (ii), $\bar{\mathcal{G}}$ has two connected components such that either one of the two is obtained by just replacing a node subset $\tilde{C}'_1$ in $\tilde{C}'_1 \cup \tilde{C}'_2$ with $\tilde{C}'_3$ and the other is obtained by replacing $\tilde{C}'_2$ with $\tilde{C}'_4$, *i.e.* $\tilde{C}'_2 \cup \tilde{C}'_3$ and $\tilde{C}'_1 \cup \tilde{C}'_4$. Therefore, we conclude (L2). □

The equitable partitions satisfying the matrix inequality (10) are chosen as follows.

Theorem 2 Consider a connected weighted undirected graph $\mathcal{G} = (V, E)$.

(A) If $\mathcal{G}$ contains at least one cycle of an odd number of nodes, the matrix inequality (10) always holds for any equitable partition π.

(B) Otherwise, the matrix inequality (10) always holds for an equitable partition π such that any cell in π is a subset of $\tilde{C}_1$ or $\tilde{C}_2$.

Proof First, we construct a partition of $\bar{V}$ corresponding to an equitable partition $\pi = \{C_1, C_2, \ldots, C_W\}$ for $\mathcal{G} = (V, E)$, and then we show that it is an equitable partition for $\bar{\mathcal{G}} = (\bar{V}, \bar{E})$. Now, we define sets of nodes in $\bar{V}$ such that if node $i \in V$ belongs to $C_l \in \pi$, node $i \in \bar{V}$ belongs to $\bar{C}_l$ and node $i' \in \bar{V}$ is in $\bar{C}_{W+l}$. Then the set $\bar{\pi} = \{\bar{C}_1, \ldots, \bar{C}_W, \bar{C}_{W+1}, \ldots, \bar{C}_{2W}\}$ is a partition of $\bar{V}$. The adjacency matrix of $\mathcal{G}$ can be written with blocks $A_{lk} \in \mathbb{R}^{r_l \times r_k}$ corresponding to cells in π as

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1W} \\ \vdots & \ddots & \vdots \\ A_{W1} & \cdots & A_{WW} \end{pmatrix}.$$

From the definition of the equitable partition, each block A_{lk} has a constant row sum. Here, the adjacency matrix of $\bar{\mathcal{G}}$ is

$$\bar{A} = \begin{pmatrix} \mathbf{0}_{NN} & A \\ A & \mathbf{0}_{NN} \end{pmatrix}.$$

Partitioning the matrix elements $\mathbf{0}_{NN}$ in $\bar{A}$ as

$$\mathbf{0}_{NN} = \begin{pmatrix} \mathbf{0}_{r_1 r_1} & \cdots & \mathbf{0}_{r_1 r_W} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{r_W r_1} & \cdots & \mathbf{0}_{r_W r_W} \end{pmatrix}$$

and substituting $\mathbf{0}_{NN}$ and A with blocks into $\bar{A}$, we see that each block in $\bar{A}$ corresponds to a cell in the partition $\bar{\pi}$ and has a constant row sum. Therefore, the partition $\bar{\pi}$ is equitable for $\bar{\mathcal{G}}$.

Next, paying attention to the relation between components constituting the enlarged graph $\bar{\mathcal{G}}$ and the equitable partition $\bar{\pi}$, we show that any cell in $\bar{\pi}$ consists of nodes in the identical connected component of $\bar{\mathcal{G}}$. As shown in Lemma 1, if $\mathcal{G}$ has a cycle of an odd number of nodes, $\bar{\mathcal{G}}$ should be connected, *i.e.* it cannot be decomposed into two or more components. Therefore, $\bar{\mathcal{G}}$ consists of only one component, and as a result, all the nodes must belong to the identical component. On the other hand, if $\mathcal{G}$ has no cycle of any odd number of nodes, $\bar{\mathcal{G}}$ has two components with node sets $\tilde{C}'_1 \cup \tilde{C}'_4$ and $\tilde{C}'_2 \cup \tilde{C}'_3$ where $\tilde{\pi}' = \{\tilde{C}'_1, \tilde{C}'_2, \tilde{C}'_3, \tilde{C}'_4\}$ is also a partition of $\bar{V}$ as mentioned in the proof of Lemma 1. From the construction of $\bar{\pi}$, if $C_l \subseteq \tilde{C}_1$ (or $\tilde{C}_2$), the inclusions $\bar{C}_l \subseteq \tilde{C}'_1$ (or $\tilde{C}'_2$) and $\bar{C}_{W+l} \subseteq \tilde{C}'_3$ (or $\tilde{C}'_4$) hold. As a result, each cell in $\bar{\pi}$ consists of nodes in either of the two connected components of $\bar{\mathcal{G}}$.

Finally, we show that the matrix inequality (10) holds for the equitable partition π in either case. Define $\bar{H}_0 = \mathrm{diag}(H_0, H_0)$ and $\bar{H}_1 = \mathrm{diag}(H_1, H_1)$ with the matrices $H_0 = [h^0_{ij}]$ defined in (5) and H_1 in (6) for π. In addition, define the matrix $\bar{H} \in \mathbb{R}^{2N \times 2N}$ as

$$\bar{H} = \begin{pmatrix} \bar{H}_0 \\ \bar{H}_1 \end{pmatrix}.$$

Since $\bar{\pi}$ is equitable for $\bar{\mathcal{G}}$, simple calculation gives the form

$$\begin{aligned} \bar{H}\bar{L}\bar{H}^{-1} &= \bar{H}\bar{D}\bar{H}^{-1} - \bar{H}\bar{A}\bar{H}^{-1} \\ &= \begin{pmatrix} \bar{\mathcal{D}} & \mathbf{0} \\ \mathbf{0} & \bar{D}_{\bar{\pi}} \end{pmatrix} - \begin{pmatrix} \bar{\mathcal{A}} & * \\ \mathbf{0} & \bar{A}_{\bar{\pi}} \end{pmatrix} =: \begin{pmatrix} \bar{\mathcal{L}} & * \\ \mathbf{0} & \bar{L}_{\bar{\pi}} \end{pmatrix} \end{aligned}$$

where $\bar{D} = \mathrm{diag}(D, D)$ is the degree matrix of $\bar{\mathcal{G}}$, $\bar{L}_{\bar{\pi}} = \bar{H}_1 \bar{L} \bar{H}_1^+$ with $\bar{H}_1^+ = \mathrm{diag}(H_1^+, H_1^+)$, and $\bar{\mathcal{L}} \in \mathbb{R}^{2W \times 2W}$ is the Laplacian matrix of the graph representing the interactions among cells of $\bar{\pi}$, *i.e.* $\bar{\mathcal{L}}_{ll} = d_{s_l} - \sum_{i \in C_l} w_{s_l i}$ and $\bar{\mathcal{L}}_{lk} = -\sum_{j \in C_k} w_{s_l j}$ for $l, k =$

$1, 2, \ldots, 2W$ and $l \neq k$. Since all cells in $\bar{\pi}$ consist of nodes contained in a single connected component of $\bar{\mathcal{G}}$, the number of components in the graph of cell-to-cell connections equals that of $\bar{\mathcal{G}}$. Hence, $\bar{\mathcal{L}}$ has the same number of zero eigenvalues as that of $\bar{L}$, which implies that the eigenvalues of $\bar{L}_{\bar{\pi}}$ are all positive.

Since $(H_1 H_1^\top)^{-1}$ is positive definite, there exists an orthonormal matrix P such that $P^\top (H_1 H_1^\top)^{-1} P = \Delta$ where Δ is the diagonal matrix whose diagonal entries are the eigenvalues of $(H_1 H_1^\top)^{-1}$. Taking $\bar{P} = \Delta^{\frac{1}{2}} P^\top$,

$$\begin{aligned}
&\mathrm{diag}(\bar{P}, \bar{P})\, \bar{L}_{\bar{\pi}}\, \mathrm{diag}(\bar{P}, \bar{P})^{-1} \\
&= \mathrm{diag}(\bar{P}H_1, \bar{P}H_1)\, \bar{L}\, \mathrm{diag}(H_1^{+}\bar{P}^{-1}, H_1^{+}\bar{P}^{-1}) \\
&= \mathrm{diag}(\bar{P}H_1, \bar{P}H_1)\, \bar{L}\, \mathrm{diag}(H_1^{\dagger}\bar{P}^{-1}, H_1^{\dagger}\bar{P}^{-1}) \\
&= \mathrm{diag}(\bar{P}H_1, \bar{P}H_1)\, \bar{L}\, \mathrm{diag}(\bar{P}H_1, \bar{P}H_1)^\top
\end{aligned} \tag{11}$$

holds. The symmetric matrix (11) is positive definite since all eigenvalues of $\bar{L}_{\bar{\pi}}$ are positive. Therefore, the matrix

$$\begin{aligned}
&\mathrm{diag}(\bar{P}, \bar{P})^\top \mathrm{diag}(\bar{P}H_1, \bar{P}H_1)\, \bar{L}\, \mathrm{diag}(\bar{P}H_1, \bar{P}H_1)^\top \mathrm{diag}(\bar{P}, \bar{P}) \\
&= \begin{pmatrix} H_1^\dagger & \mathbf{0} \\ \mathbf{0} & H_1^\dagger \end{pmatrix}^\top \begin{pmatrix} D & -A \\ -A & D \end{pmatrix} \begin{pmatrix} H_1^\dagger & \mathbf{0} \\ \mathbf{0} & H_1^\dagger \end{pmatrix}
\end{aligned}$$

is also positive definite, and the inequality (10) holds. □

This proof relies on the fact that $\bar{\mathcal{L}}$ has all zero eigenvalues of $\bar{L}$. Since $\bar{\mathcal{L}}$ represents a graph of cell-to-cell connection, if $\bar{\mathcal{G}}$ consists of two connected components, the graph of $\bar{\mathcal{L}}$ must equivalently have two connected components. As a result, each cell in the equitable partition $\bar{\pi}$ for $\bar{\mathcal{G}}$ must be composed of nodes in the identical component. The restriction on equitable partition in case (B) of Theorem 2 is due to this fact.

Yanagi and Oguchi [10] clarify that if the number of systems N in ring networks (Fig. 2) is odd, all systems are synchronized regardless of delay value and if N is even, all alternate systems are delay-independently partially synchronized, provided that the coupling strength k is larger than a threshold value. We see that Theorem 2 covers the cases of ring networks by choosing the equitable partition π with exactly one cell including all the nodes if N is odd and the equitable partition π with just two cells such that each cell has all alternate nodes if N is even.

If the graph $\mathcal{G}$ contains a cycle of an odd number of nodes, the equitable partition composed of the fewest cells in underlying all equitable partitions in $\mathcal{G}$ is taken as a candidate of (A) in Theorem 2. Therefore, synchronization emerges according to the pattern of the maximal equitable partition for sufficiently large coupling strength k. In addition, if all the nodes in $\mathcal{G}$ have the same degree, the maximal equitable partition has just one cell including all the nodes. Thus, all systems in the network are synchronized, in other words full synchronization is achieved, for sufficiently large k. This fact coincides with the conjecture stated in the last section of [10], and this paper gives the proof. We close this section with the following corollary.

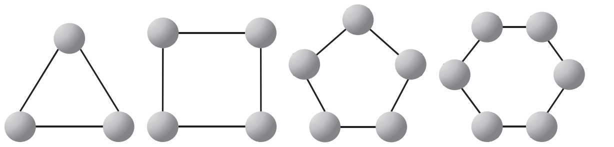

Fig. 2 Ring networks ($N = 3, 4, 5, 6$).

Corollary 1 Suppose that all the nodes in a connected weighted undirected graph $\mathcal{G}$ have the identical degree and that $\mathcal{G}$ has at least one cycle of an odd number of nodes. Then, the matrix inequality (10) always holds for the equitable partition π with only one cell including all the nodes in $\mathcal{G}$.

4. Numerical Simulations

To show the validity of Theorem 2 and Corollary 1, this section gives two numerical examples. Suppose that each node in networks is given by the following Hindmarsh-Rose neuron system [13]

$$\begin{cases} \dot{y}_i(t) = z_{i1}(t) - a y_i^3(t) + b y_i^2(t) - z_{i2}(t) + I + u_i(t) \\ \dot{z}_{i1}(t) = c - d y_i^2(t) - z_{i1}(t) \\ \dot{z}_{i2}(t) = r(s(y_i(t) - Q) - z_{i2}(t)) \end{cases}$$

where $a = 1$, $b = 3$, $I = 3.25$, $c = 1$, $d = 5$, $r = 0.005$, $s = 4$, $Q = -1.618$. Under these parameters and $u_i \equiv 0$, this system behaves chaotically. This system satisfies Assumptions 1 and 2 [5].

First, we consider the graph shown in Fig. 3. This graph has several cycle subgraphs of odd numbers of nodes, e.g. an induced subgraph composed of nodes $1, 2, 3, 4$ and 5. Additionally, all the nodes have degree 3. The simulation result is shown in Fig. 5. The gray regions mean sets of (k, τ) such that full synchronization is achieved. This result shows that full synchronization occurs for coupling strength k larger than around 0.9 and for sufficiently long time-delay, which supports the validity of Corollary 1.

Next, the bidirectional network given in Fig. 4 is considered. Since this graph has no odd cycle, there exists a partition $\tilde{\pi} = \{\tilde{C}_1, \tilde{C}_2\}$ consisting of $\tilde{C}_1 = \{1, 3, 6, 8\}$ and $\tilde{C}_2 = \{2, 4, 5, 7\}$ where any two adjacent nodes in the graph belong to the different cells. We find an equitable partition $\pi = \{\{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 8\}\}$ as a candidate of π in Theorem 2 (B). Figure 6 shows a simulation result for this network. The gray regions represent conditions inducing synchronization in the pattern of equitable partition π. For coupling strength over around 0.5, synchronization pattern of π occurs for sufficiently long time-delay. On the other hand, the black colored regions indicate sets of (k, τ) such that partial synchronization corresponding to another equitable partition $\hat{\pi} = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$ happens. The simulation result shows that this synchronization pattern collapses for long time-delay. The synchronization pattern of $\hat{\pi}$ may look different from π, but this pattern appears as a special case of π. In fact, each cell in π is a subset of a cell in $\hat{\pi}$. Since all systems in the same cells of π always synchronize for $k > 0.5$, this simulation result shows the validity of Theorem 2.

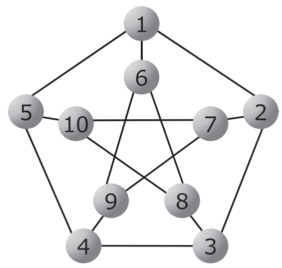

Fig. 3 Peterson graph.

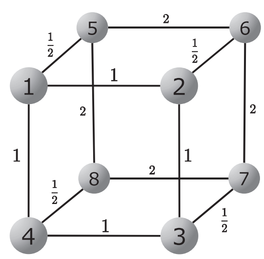

Fig. 4 Graph without odd cycle.

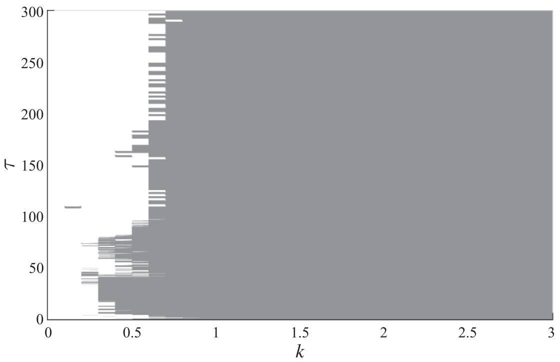

Fig. 5 Simulation result for the network in Fig. 3.

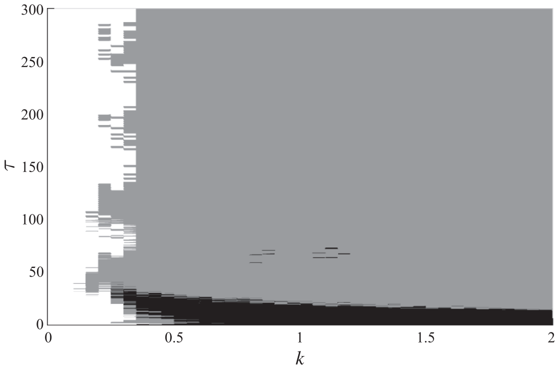

Fig. 6 Simulation result for the network in Fig. 4.

5. Conclusions

This paper discussed delay-independent synchronization in networks of identical nonlinear systems coupled with transmission delay couplings. Based on the solvability of LMI-based synchronization condition derived in [9], we clarified the relationship between delay-independent synchronization and network topology in networks of weighted undirected graphs. The patterns of synchronization are characterized depending on whether the network has a cycle subgraph of an odd number of nodes. If so, delay-independent synchronization emerges in systems included in the same cells of the equitable partition with the fewest cells for sufficiently large coupling strength. In addition, if all the nodes have the identical degree, full synchronization is achieved. The later statement was pointed out in [10], and this paper gives a proof. Moreover, if the network has no cycle subgraph of any odd number of nodes, delay-independent synchronization happens in systems included in the same cells of equitable partition with more subdivided cells.

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