Abstract—This paper studies the kinematics and statics of under-constrained cable-driven parallel robots with three cables. A major challenge in the study of these robots is the intrinsic coupling between kinematics and statics, which must be dealt with simultaneously. This paper provides a general procedure that solves, in analytical form, the direct geometrico-static problem, which consists in determining the platform posture and the cable tensions once the cable lengths are assigned. The problem is proven to have up to 156 complex solutions.

I. INTRODUCTION

Cable-driven parallel robots (CDPRs) employ cables in place of rigid-body extensible legs in order to control the end-effector posture. A CDPR is referred to as fully-constrained if the posture of the end-effector is completely determined when actuators are locked and, thus, all cable lengths are assigned [1]. The minimum number of cables that are necessary to fully control the output motion is equal to the number $f$ of degrees of freedom (dofs) that the end-effector possesses with respect to the base. However, since cables may only exert tensile axial forces, a redundancy of control actions is usually necessary in order to guarantee that no cable becomes slack and, thus, full control is preserved for a generic loading condition [1], [2]. A CDPR is defined, instead, as under-constrained if the end-effector preserves some freedoms once actuators are locked and cable lengths are fixed [1]. Typically, this occurs when the end-effector is controlled by a number of cables smaller than $f$. The employ of CDPRs with a limited number of cables is justified in a number of applications (such as, for instance, rescue, service or rehabilitation operations), in which a limitation of dexterity is acceptable in order to decrease complexity, cost, set-up time, likelihood of cable interference, etc. It must also be observed that a theoretically fully-constrained CDPR operates, in considerable parts of its geometric workspace, as an under-constrained robot, namely when a full restraint may not be achieved because it would require a negative tension in one or more cables.

The above considerations motivate a careful study of under-constrained CDPRs. However, while fully-constrained robots have been extensively investigated [1]–[15], little attention has been dedicated to under-constrained ones [16]–[23]. When a fully-constrained CDPR operates in the portion of its workspace in which the required set of output wrenches is guaranteed to be applicable with purely tensile cable forces [10]–[13], [15], the posture of the end-effector is determined, in a purely geometrical way, by assigning cable lengths. Conversely, for an under-constrained CDPR, when the actuators are locked and the cable lengths are assigned, the end-effector is still movable, so that the actual configuration is determined by the applied forces. As a consequence, the end-effector posture depends on both cable lengths and equilibrium equations, and kinematics and statics (or dynamics) must be dealt with simultaneously. Moreover, as the pose depends on the applied load, it may change due to external disturbances, so that it is important to investigate equilibrium stability.

In [22], [23], a general methodology was proposed for the kinematic, static and stability analysis of general under-constrained $nn$-CDPRs, namely parallel robots in which a fixed base and a mobile platform are connected to each other by $n$ cables, with $n \leq 5$ and the anchor points on the base and the platform being distinct. In particular, a procedure was provided aimed at effectively solving, in analytical form, the inverse and direct position problems, namely, at finding the overall robot configuration and cable tensions when, respectively, either $n$ platform posture coordinates or the $n$ cable lengths are given, under the assumptions that a constant force is applied on the platform, cables are inextensible and massless, and interference problems are disregarded.

In a companion paper [24], the aforementioned methodology is applied to the inverse geometrico-static problem (IGP) of the general 33-CDPR. In this paper, the direct geometrico-static problem (DGP) of the 33-CDPR is, instead, tackled. The challenge consists in determining the platform posture and the cable tensions once the cable lengths are assigned. Section II presents the robot model. Sections III and IV formulate the geometrical and statical equations that govern the problem, whereas Section V provides the detailed procedures that solve it in analytical form. In Section VII, the main achievements of the paper are discussed.

II. GEOMETRICO-STATIC MODEL

A mobile platform is connected to a fixed base by three cables and is acted upon by a constant force $QL_e$ applied on a point $G$, e.g. the platform weight acting through its center of mass (Fig. 1). $Q$ is the magnitude of the force, whereas $L_e$ is the six-dimensional vector grouping the normalized Plücker coordinates of its line of action. $Oxyz$ is a Cartesian
coordinate frame fixed to the base, with i, j and k being unit vectors along the coordinate axes, whereas $Gx'y'z'$ is a Cartesian frame attached to the end-effector. The platform posture is described by $X = (x; \Phi)$, where $x = G - O$ and $\Phi$ groups the variables parameterizing the platform orientation with respect to $Oxyz$. If Rodrigues parameters are adopted, i.e. $\Phi = [e_1; e_2; e_3]$, the rotation matrix $R(\Phi)$ between the mobile and the fixed frame is given by

$$R = I_3 + 2\frac{\Phi \times \Phi}{1 + e_1^2 + e_2^2 + e_3^2},$$

where $\Phi$ denotes the skew-symmetric matrix expressing the operator $\Phi \times$. For the generic $i$th cable, $A_i$ and $B_i$ are, respectively, the anchor points on the base and the platform, $\rho_i$ is the cable length, $a_i = A_i - O$, $r_i = B_i - G$, $s_i = B_i - A_i$ and $u_i = (A_i - B_i)/\rho_i$. For apparent reasons, $\rho_i$ is assumed to be strictly positive, so that $s_i \neq 0$. If $b_i$ is the projection of $B_i - G$ on $Gx'y'z'$, then $r_i = R(\Phi)b_i$. $(\tau_i/\rho_i)L_i$ is the force exerted by the $i$th cable on the platform, with $\tau_i$ being the cable tension and $L_i/\rho_i$ the normalized Plücker vector of the cable line.

III. GEOMETRICAL CONSTRAINTS

When cable lengths are assigned, the set $C$ of the theoretical restraints imposed by the cables on the platform comprises 3 relations in $X$, i.e.

$$|s_i| = \sqrt{s_i \cdot s_i} = \rho_i, \quad i = 1 \ldots 3,$$

where

$$s_i = x + Rb_i - a_i.$$  

By subtracting the first one from the second and the third one, and by clearing the denominator $1 + e_1^2 + e_2^2 + e_3^2$, the following relations are obtained:

$$q_1 := H_{200}x^2 + H_{020}y^2 + H_{002}z^2 + H_{100}x + H_{010}y + H_{001}z + H_{000} = 0,$$

$$q_2 := I_{100}x + I_{010}y + I_{001}z + I_{000} = 0,$$

$$q_3 := K_{100}x + K_{010}y + K_{001}z + K_{000} = 0,$$

where all coefficients $H_{knn}$, $I_{knn}$ and $K_{knn}$ are quadratic functions of $e_1$, $e_2$ and $e_3$.

IV. STATICAL CONSTRAINTS

The platform equilibrium may be written as

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_e \end{bmatrix} \begin{bmatrix} (\tau_1/\rho_1) \\ (\tau_2/\rho_2) \\ (\tau_3/\rho_3) \\ Q \end{bmatrix} = 0,$$

with

$$\tau_i \geq 0, \quad i = 1 \ldots 3.$$  

Equations (5) amount to 6 scalar relations involving 9 variables, namely $x$, $\Phi$ and $\tau_i$, $i = 1 \ldots 3$. Following [22], cable tensions may be eliminated by observing that Eq. (5) holds only if

$$\text{rank}(M) \leq 3,$$

namely if $L_1$, $L_2$, $L_3$ and $L_e$ are linearly dependent. This is a purely geometrical condition, since $M$ is a $6 \times 4$ matrix only depending on the platform posture. By setting all $4 \times 4$ minors of $M$ equal to zero, a set of 15 scalar relations that do not contain cable tensions may be obtained$^1$.

If $O$ is chosen as the reduction pole of moments, $L_1$ and $L_e$ may be respectively expressed, in axis coordinates, as $-s_1; a_i \times s_i$ and $[e; x \times e]$, so that $M$ becomes

$$M(O) = \begin{bmatrix} -s_1 & -s_2 & -s_3 & e \\ 0 & -a_2 \times s_2 & -a_3 \times s_3 & x \times e \end{bmatrix}.$$  

The equations$^2$

$$p_1 := \text{det} M_{1234}(O) = 0,$$

$$p_2 := \text{det} M_{1235}(O) = 0,$$

$$p_3 := \text{det} M_{1245}(O) = 0,$$

$^1$In very special conditions, Eq. (7) is fulfilled because $L_1$, $L_2$ and $L_3$ become linearly dependent. In this case, equilibrium is possible only if $\text{rank}(M) \leq 2$, for, in any case, the external wrench must belong to the subspace generated by cable lines. Cases like the ones mentioned here, however, are sufficiently unlikely to occur not to be, in practice, of particular concern.

$^2$The notation $M_{kij,lkm}$ denotes the block matrix obtained from rows $h$, $i$ and $j$, and columns $k$, $l$ and $m$. When all columns of $M$ are used, the corresponding subscripts are omitted.
comprise the lowest-degree polynomials in $X$ among all minors of $M(\Phi)$. They are of degree 4 in the Rodrigues parameters and degree 2 in the components of $\mathbf{x}$, thus being of degree 6 in $M$. All other minors have degree ranging from 7 to 9 in $X$.

An additional sextic relation in $X$ emerges by setting $\det M_{j456}(O) = 0$ for $j = 1 \ldots 3$, so that

$$s_1 \cdot \det M_{456,234}(O) = 0, \quad (10)$$

and thus, since $s_1 \neq 0$,

$$p_4 := \det M_{456,234}(O) = 0. \quad (11)$$

Equation (11) is, indeed, of degree 4 in $\Phi$, degree 2 in $x$ and degree 6 in $X$.

For the purpose of this paper, it is worth deriving as many independent lowest-degree equations in $X$ as possible. Further sextics may be obtained as follows. Let $M$ be written by choosing a generic $P$ as the reduction pole of moments, namely as

$$M(P) = \begin{bmatrix} \cdots & s_i & \cdots & e \\ \cdots & (B_i - P) \times s_i & \cdots & (G - P) \times e \end{bmatrix}. \quad (12)$$

When $P = B_i$ or $P = A_i$, $i = 1 \ldots 3$, the moment vector in the $i$th column vanishes, so that setting $\det M_{j456}(B_i) = 0$ or $\det M_{j456}(A_i) = 0$ for $j = 1 \ldots 3$ yields, respectively,

$$s_i \cdot \det M_{456,km4}(B_i) = 0, \quad (13)$$

or

$$s_i \cdot \det M_{456,km4}(A_i) = 0, \quad (14)$$

with $k, m \in \{1, 2, 3\} - \{i\}$. This way, the following equations may be obtained:

$$p_5 := \det M_{456,234}(B_1) = 0, \quad (15a)$$

$$p_6 := \det M_{456,134}(B_2) = 0, \quad (15b)$$

$$p_7 := \det M_{456,134}(A_2) = 0, \quad (15c)$$

$$p_8 := \det M_{456,124}(B_3) = 0, \quad (15d)$$

$$p_9 := \det M_{456,124}(A_3) = 0. \quad (15e)$$

Analogously, by setting $P = G$, one obtains

$$p_{10} := \det M_{456,123}(G) = 0, \quad (16)$$

All polynomials $p_j$, with $j = 5 \ldots 10$, have degree 4 in the Rodrigues parameters and degree 2 in the components of $\mathbf{x}$. These are the only linearly independent sextics in $X$ that may be derived from the minors of $M$ by varying the moment pole.

V. DGP

Solving the DGP of the 33-CDPR requires solving, simultaneously, both the equations emerging from the geometrical constraints and those inferred from static equilibrium.

The 3 point-to-point distance relations in Eq. (4) represent the typical constraints governing the forward kinematics of parallel manipulators equipped with telescoping legs connected to the base and the platform by ball-and-socket joints. In particular, the DGP of the general Gough-Stewart manipulator depends on six equations of this sort, one of which is equivalent to Eq. (4a) and five more to Eqs. (4b)- (4c). This problem is known to be very difficult and it has attracted the interest of researchers for several years [25], [26]. The DGP of the 33-CDPR appears to be a even more complex task, since, in this case, three equations analogous to Eqs. (4b)-(4c), namely of degree 3 in $X$, are replaced by relationships that are, at least, of degree 6 in $X$. If the platform posture is described by Study homogeneous coordinates $X_s := (q_0, q_1, q_2, q_3, e_0, e_1, e_2, e_3)$, with

$$q_0 := e_0 g_0 + e_1 g_1 + e_2 g_2 + e_3 g_3 = 0, \quad (17)$$

the relations in Eqs. (4) become quadratic in $X_s$ [26], but the polynomials in Eqs. (9), (11), (15) and (16) remain of degree 6. The task does not appear to be significantly simplified. In the following, the number of complex solutions that the problem admits is determined by a hybrid approach based on Groebner bases and Sylvester’s dialytic method. Results are confirmed by homotopy continuation.

Let $J$ be the ideal generated by the polynomial set $J = \{q_1, q_2, q_3, p_1, \ldots, p_{10}\}$. $q_1$, $q_2$ and $q_3$ have, respectively, degree 4, 3 and 3 in the elements of $X$, whereas all other generators of $J$ have degree 6 in the same variables. In order to ease numeric computation via a computer algebra system, namely the Groebner Package provided within the mathematical software Maple$^{TM}$, all geometric parameters of the 33-CDPR are assumed to be rational. Accordingly, $\langle J \rangle \subset \mathbb{Q}[X]$, where $\mathbb{Q}[X]$ is the set of all polynomials in $X$ with coefficients in $\mathbb{Q}$. All Groebner bases are computed with respect to graded reverse lexicographic monomial orders (grolex, in brief)$^3$.

In general, a Groebner basis $G[J]$ of $\langle J \rangle$ with respect to grolex($z, y, x, e_1, e_2, e_3$) may be computed in a fairly expedited way. A key factor for the efficiency of such a computation is the abundance of generators available in $\langle J \rangle$, which significantly simplifies and speeds up calculation.

$G[J]$ comprises 137 polynomials, namely 2 of degree 3 in $X$, 41 of degree 4 in $X$ and 94 of degree 5 in $X$.

Once $G[J]$ is known, the number of complex roots in the variety $V(J)$ of $\langle J \rangle$ may be evaluated by the command $\text{PolynomialIdeals}[\text{NumberOfSolutions}]$ [27], [28]. In this case, the returned number is 156. In order to actually solve $J$, and thus eliminate unknowns, Groebner bases with respect to some elimination monomial orders are, however, needed. If $X_l$ is a list of $l$ variables in $X$ and $X_lX_l$ is the (ordered) relative complement of $X_l$ in $X$, a monomial order $>_l$ on $\mathbb{Q}[X]$ is of $l$-elimination type provided that any monomial involving a variable in $X_l$ is greater than any

$^3$Study coordinates are advantageously employed to solve the DGP of the Gough-Stewart manipulator.

$^4$The lexicographic monomial order is particularly suitable to solve systems of polynomial equations, for it provides polynomial sets whose variables may be eliminated successively. However, the Groebner bases that it provides tend to be very large and thus, even for problems of moderate complexity, they have little chance to be actually computable. Conversely, the graded reverse lexicographic order produces bases that are endowed with no particular structure suitable for elimination purposes, but it ordinarily provides for more efficient calculations.
monomial in $\mathbb{Q}[X \setminus X_I]$. If $G_{>l}[J]$ is a Groebner basis of $\langle J \rangle$ with respect to $>l$, then $G_{\geq l}[J \cap \mathbb{Q}[X \setminus X_I]]$ is a basis of the $l$th elimination ideal $\langle J_l \rangle := (J \cap \mathbb{Q}[X \setminus X_I])^{[29]}$. The $l$-elimination monomial order implemented in Maple is a product order that induces grevlex orders on both $\mathbb{Q}[X_I]$ and $\mathbb{Q}[X \setminus X_I]$. In this perspective, the FGLM algorithm [30], which converts a Groebner basis from one monomial order to another, may be called upon to compute elimination ideals of type $\langle J \rangle \cap \mathbb{Q}[X \setminus X_I]$, starting from $G[J]$. By this approach, a least-degree univariate polynomial in one of the original variables may be (theoretically) obtained.

Another method to compute a least-degree univariate polynomial of $\langle J \rangle$ emerges from the following observation. Let $N_{G[J]}$ be the number of generators in $G[J]$, with $G[J]$ being the Groebner basis of $\langle J \rangle$ with respect to grevlex $(X \setminus X_I)$. Furthermore, let $w$ be the last variable in $X \setminus X_I$. It is not difficult to verify that $G[J]$ comprises a number of monomials in $X \setminus X_I \setminus \{w\}$ which is exactly equal to $N_{G[J]}$. For example, the Groebner basis $G[J]$ of $\langle J \rangle \cap \mathbb{Q}[e_1, e_2, e_3]$ with respect to grevlex$(e_1, e_2, e_3)$ comprises 45 polynomials (9 of degree 8 in $\Phi$ and 30 of degree 9 in $\Phi$), including 45 monomials in $e_1$ and $e_2$ (of degree ranging from 0 to 8), whereas the Groebner basis $G[J]$ of $\langle J \rangle \cap \mathbb{Q}[e_2, e_3]$ with respect to grevlex$(e_2, e_3)$ contains 18 polynomials (15 of degree 17 in $\{e_2, e_3\}$ and 3 of degree 18 in $\{e_2, e_3\}$), including 18 monomials in $e_2$ (of degree ranging from 0 to 17). It follows that, if $w$ is assigned the role of ‘hidden’ variable, the resultant in $w$ of $J$ may be obtained from $G[J]$ by Sylvester’s dialectic method. Indeed, by writing the generators of $G[J]$ in the form
\begin{equation}
T(w)E_w = 0,
\end{equation}
where $T(w)$ is a $N_{G[J]} \times N_{G[J]}$ matrix that only depends on $w$ and $E_w$ is a $N_{G[J]}$ column vector comprising all monomials in $G[J]$ with variables in $X \setminus X_I \setminus \{w\}$, the sought-for resultant is
\begin{equation}
\det T(w) = \sum_{h=0}^{156} L_hw^h = 0,
\end{equation}
with the coefficients $L_h$ only depending on the input data, namely the robot geometry and the cable lengths. The degree of $\det T(w)$ is confirmed to be 156.

Table I reports, for the exemplifying 33-CDPR whose dimensions are given in Table II, the CPU time required to compute grevlex Groebner bases for the elimination ideals of $\langle J \rangle$, with $l = 0 \ldots 5$, on a PC with a 2.67GHz Xeon processor and 4GB of RAM. In particular, the third column reports the CPU time $T_{G[J]}$ required to get $G[J]$ both by computing $\langle J \rangle \cap \mathbb{Q}[X \setminus X_I]$ and, in parentheses, by computing $\langle J_{l-1} \rangle \cap \mathbb{Q}[X \setminus X_I]$. The elimination task proves to be, in general, computationally very expensive and time consuming. In particular, the ‘deeper’ the elimination process goes (i.e. the smaller the number of variables in $X \setminus X_I$), the longer is the time necessary to perform the computation and, above all, the larger is the amount of memory that is required. The latter issue is particularly critical. Indeed, for the example at hand, the last elimination ideal cannot be computed on the given PC, due to excessive memory usage. The fourth column reports the CPU time $T_{\langle J \rangle \cap \mathbb{Q}[e_3]}$ required to calculate $\langle J \rangle \cap \mathbb{Q}[e_3]$ by applying Sylvester’s dialectic method on $G[J]$, for $l = 0 \ldots 4$. In this case, computation time depends on the dimension of $T(w)$ and, thus, it normally decreases with the number of variables in $X \setminus X_I$. Memory requirements are modest and the order is ordinarily successful. It emerges from the above consideration that a hybrid approach, which eliminates a subset of variables by the FGLM algorithm and further applies Sylvester’s method on the Groebner basis of the corresponding elimination ideal, provides an effective strategy to compute a least-degree univariate polynomial in $\langle J \rangle$.

For the numeric solutions of the problem to be actually calculated, however, working with polynomials of degree as high as 156 is impractical and it poses substantial numerical problems. In this perspective, homotopy continuation offers a robust alternative [26]. If no information is a priori known about the roots in $\mathbb{V}$, the DGP of the 33-CDPR may be cast, on the basis of the degree of the polynomials contained in $J$, into the larger family of all polynomial systems made up out of $\Phi$ and the results must be successively sifted in order to retain only those that actually lie in the variety of $\mathbb{V}$. As expected, 156 solutions are finally obtained. If the roots in

### Table I

**Computation time to obtain Groebner bases of the elimination ideals of $\langle J \rangle$ for the example reported in Table II**

<table>
<thead>
<tr>
<th>$l$</th>
<th>$J_l$</th>
<th>$T_{G[J]}$ [min]</th>
<th>$T_{\langle J \rangle \cap \mathbb{Q}[e_3]}$ [min]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\langle J \rangle$</td>
<td>1.3</td>
<td>1919</td>
</tr>
<tr>
<td>1</td>
<td>$\langle J \rangle \cap \mathbb{Q}[y, x, e_1, e_2, e_3]$</td>
<td>19</td>
<td>2159</td>
</tr>
<tr>
<td>2</td>
<td>$\langle J \rangle \cap \mathbb{Q}[x, e_1, e_2, e_3]$</td>
<td>42 (27)</td>
<td>579</td>
</tr>
<tr>
<td>3</td>
<td>$\langle J \rangle \cap \mathbb{Q}[e_1, e_2, e_3]$</td>
<td>49 (24)</td>
<td>33</td>
</tr>
<tr>
<td>4</td>
<td>$\langle J \rangle \cap \mathbb{Q}[e_2, e_3]$</td>
<td>160 (80)</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>$\langle J \rangle \cap \mathbb{Q}[e_3]$</td>
<td>...</td>
<td>–</td>
</tr>
</tbody>
</table>

---

5. Computation time may significantly increase depending on the complexity of the coefficients of the polynomials in $J$.

6. In a computation performed on a more powerful workstation, Maple estimated a required memory usage of about 12GB, in order to derive $\langle J_5 \rangle$ from $\langle J_4 \rangle$. 

---

Table II

**Table II**

<table>
<thead>
<tr>
<th>$l$</th>
<th>$J_l$</th>
<th>$T_{G[J]}$ [min]</th>
<th>$T_{\langle J \rangle \cap \mathbb{Q}[e_3]}$ [min]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\langle J \rangle$</td>
<td>1.3</td>
<td>1919</td>
</tr>
<tr>
<td>1</td>
<td>$\langle J \rangle \cap \mathbb{Q}[y, x, e_1, e_2, e_3]$</td>
<td>19</td>
<td>2159</td>
</tr>
<tr>
<td>2</td>
<td>$\langle J \rangle \cap \mathbb{Q}[x, e_1, e_2, e_3]$</td>
<td>42 (27)</td>
<td>579</td>
</tr>
<tr>
<td>3</td>
<td>$\langle J \rangle \cap \mathbb{Q}[e_1, e_2, e_3]$</td>
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<td>4</td>
<td>$\langle J \rangle \cap \mathbb{Q}[e_2, e_3]$</td>
<td>160 (80)</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>$\langle J \rangle \cap \mathbb{Q}[e_3]$</td>
<td>...</td>
<td>–</td>
</tr>
</tbody>
</table>
Φ are computed via $G[J_3]$, the problem solutions may be completed by calculating the corresponding roots in $x$ as follows. $x$ and $y$ may be linearly eliminated from Eqs. (4b) and (4c), so that algebraic functions $x = x(z, Φ)$ and $y = y(z, Φ)$ may be derived. By way of them, $q_1$ and $p_1$ may be written as quadratic expressions in $z$, thus allowing $z$ and $z^2$ to be linearly computed. Back-substitution of $z$ in $x = x(z, Φ)$ and $y = y(z, Φ)$ completes the solution. Due to space limitations, only the real solutions of the example reported in Table II are listed in the table.

After an equilibrium configuration is found, it proves feasible only if it is stable and therein cable tensions are positive. Cable tensions may be computed by a suitable set of linear independent relations chosen within Eq. (5), whereas stability may be assessed by determining the definiteness of the reduced Hessian matrix $H$, as defined in [22]. In Table II, the symbols $>$, $≥$, $<$, $≤$ and $<>$ denote, respectively, a positive-definite, a positive-semidefinite, a negative-definite, a negative-semidefinite and an indefinite matrix.

It is worth observing that the procedures described so far are aimed to find an analytic solution to the problem and to ascertain its number of complex roots. Once the latter information is known, more efficient computational techniques may be used to numerically solve practical cases. For example, the complete family of 33-CDPR DGPs lies in a 21-dimensional parameter space, parametrized by the geometric quantities $a_k$, $b_k$ and $ρ_k$, $i = 1 \ldots 3$. Accordingly, when the 156 isolated roots of the DGP of a generic 33-CDPR are known, ‘parameter’ homotopy continuation may be applied to find the solutions for any other member of the family. In this case, only 156 paths need to be tracked and the algorithm may be quite fast [26]. Another possibly very efficient approach to solve the problem relies on techniques based on interval analysis. This method brings about the significant advantage of easily incorporating in the calculation the constraints (6), as well as uncertainties in the parameter values, physical bounds on variables ranges, etc. [32]. Such an approach will be subject of future research.

VI. EQUILIBRIUM CONFIGURATIONS WITH UNLOADED CABLES

Equation (2) represent a set of theoretical constraints. Indeed, since the actual constraints imposed by the cables are that

$$|s_i| ≤ ρ_i, \quad i = 1 \ldots 3,$$

(20)

the number of tensioned cables for which equality relations such as those in Eq. (2) hold is a priori unknown. Accordingly, the overall solution set must be obtained by solving the DGP for all possible constraint sets $\{s_j = ρ_j, j \in W\}$, with $W \subseteq \{1, 2, 3\}$ and $card(W) ≤ 3$, and by retaining, for each corresponding solution set, the solutions for which $|s_k| < ρ_k$, $k \not∈ W$ [22], [23]. In general, this amounts to solving 7 DGPs, namely, 1 DGP with 3 cables in tension, 3 DGPs with 2 cables in tension and 3 DGPs with 1 cable in tension. The solution of the problem with a single active cable is trivial, whereas the DGP of a CDPR suspended by 2 cables is solved in [23].

VII. CONCLUSIONS

This paper studied the kinematics and statics of under-constrained cable-driven parallel robots with three cables, in crane configuration. For such robots, kinematics and statics are intrinsically coupled and they must be dealt with simultaneously. This poses major challenges, since position problems gain remarkable complexity with respect to those of analogous rigid-link robots, such as the Gough-Stewart manipulator. This paper presented an original geometrico-static model that allowed the direct position analysis to be effectively performed in analytical form. The task consists in determining the platform posture and the cable tensions once the cable lengths are assigned. By a hybrid procedure relying on Groebner bases and Sylvester’s dialytic method, it was shown that the problem admits, in general, 156 complex solutions, with results being confirmed by homotopy continuation.

The mentioned hybrid procedure appears to be innovative, in order to obtain a least-degree univariate polynomial from a given polynomial ideal. Indeed, finding a Groebner basis suitable for elimination purposes may be a highly demanding task. Even by using computationally efficient monomial orders (such as grevlex) for initial computations and suitable algorithms (such as the FGLM one) to convert bases from the initial orders to the desired ones, memory usage and calculation times may be so large that performing a full elimination may easily prove unfeasible, even for problems of moderate complexity. The technique presented in this paper, encompassing three steps, considerably reduced computation requirements, in terms of both memory and time. First, a Groebner basis $G$ was calculated with respect to an efficient monomial order (such as grevlex). Then, a subset of the original unknowns was eliminated by computing, by way of the FGLM algorithm, a Groebner basis $G_1$ of a suitable elimination ideal. Finally, a least-degree univariate polynomial in one of the remaining unknowns was computed by applying Sylvester’s dialytic method to the polynomials of $G_1$. The method is tailored to the particular structure of the ideal emerging from the DGP of the 33-CDPR, but there are chances to generalize it to fit more general cases.

It must be observed that the reported number of solutions does not take into account constraints imposed by the stability of equilibrium and the sign of cable tensions. Once such constraints are imposed and the solutions are sifted, the number of feasible configurations drastically reduces.

ACKNOWLEDGMENT

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REFERENCES


TABLE II
REAL SOLUTIONS OF THE DGP OF A 3-CDPR

| Geometric dimensions and load: $a_2 = [10; 0; 0]$, $a_3 = [0; 12; 0]$, $b_2 = [1; 0; 0]$, $b_3 = [0; 1; 0]$, $(\rho_1, \rho_2, \rho_3) = (7.5, 10, 9.5)$, $Q = 10$. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Conf. $(\varepsilon_1, x_2, 3; x_1, y, z)$ | $(\varepsilon_2, 3; x_1, y, z)$ | $H_1$ | $H_2$ | $H_3$ |
| 1 | $(-2.220216376218525374, -5.90163268591521036, -0.471924164260346102)$ | $(+6.84, +3.05, +6.14)$ | $(+5.56, +3.05, +6.14)$ | $(+6.84, +3.05, +6.14)$ |
| 2 | $(-3.35509163773204616, +0.52533916614715099, +1.711202766207574689)$ | $(+5.26, +5.11, +5.81)$ | $(+5.26, +5.11, +5.81)$ | $(+5.26, +5.11, +5.81)$ |
| 3 | $(-2.667180926909149771, +0.4160373487571904226, +0.96555348628886102991)$ | $(-5.71, -4.85, -5.59)$ | $(-5.71, -4.85, -5.59)$ | $(-5.71, -4.85, -5.59)$ |
| 4 | $(-2.3997524860361477511, +3.4587652128866540, -0.86114045755031135)$ | $(-1.40, -0.90, -9.83)$ | $(-1.40, -0.90, -9.83)$ | $(-1.40, -0.90, -9.83)$ |
| 5 | $(-1.158549984627496690, -1.27325010592227311, -1.066002756029469830)$ | $(+6.76, +2.51, +4.86)$ | $(+6.76, +2.51, +4.86)$ | $(+6.76, +2.51, +4.86)$ |
| 6 | $(-0.5843498966623853987, -0.4877188332794653588, -1.2150961072659885404)$ | $(+5.46, +3.25, +5.50)$ | $(+5.46, +3.25, +5.50)$ | $(+5.46, +3.25, +5.50)$ |
| 7 | $(-0.5044773581184974044, +2.590309174688033712, +1.247759925956409397)$ | $(-4.61, +7.87, +9.12)$ | $(-4.61, +7.87, +9.12)$ | $(-4.61, +7.87, +9.12)$ |
| 8 | $(-0.325555841337169141, -0.8891989660641670137, -1.6130502813856035959)$ | $(+5.4345207732369320, -0.145550674282349313, +0.096921199952991064)$ | $(+3.024016548208678702, +0.7309738515377370356, +3.301925163795036092)$ | $(+5.90, +7.83, +9.56)$ |
| 9 | $(-0.6841447572486557310, -0.0996112288836193264, +0.526297692839509876)$ | $(+2.8410864910572365897, +4.813317787957522652, +4.453470255697577811)$ | $(+6.01, -7.61, -9.53)$ | $(+6.01, -7.61, -9.53)$ |


