Anti-control of Hopf bifurcation in the new chaotic system with two stable node-foci

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ABSTRACT

In order to further understand a complex 3D dynamical system showing strange chaotic attractors with two stable node-foci near Hopf bifurcation point, we propose nonlinear control scheme to the system and the controlled system, depending on five parameters, can exhibit codimension one, two, and three Hopf bifurcations in a much larger parameter regain. The control strategy used keeps the equilibrium structure of the chaotic system and can be applied to degenerate Hopf bifurcation at the desired location with preferred stability.

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1. Introduction

For the investigation on generic 3D smooth quadratic autonomous systems, Sprott [1–3] found by exhaustive computer searching about 19 simple chaotic systems with no more than three equilibria. It is very important to note that some 3D autonomous chaotic systems have three particular fixed points: one saddle and two unstable saddle-foci (for example, Lorenz system [4], Chen system [5], Lü system [6], and the conjugate Lorenz-type system [7]). The other 3D chaotic systems, such as the original Rössler system [8], DLS [9] and Burke-Show system [10], have two unstable saddle-foci. Yang and Chen found another 3D chaotic system with three fixed points: one saddle and two stable fixed points [11].

Recently, Yang et al. [12] introduced and analyzed a new 3D chaotic system with six terms including only two quadratic terms in a form very similar to the Lorenz, Chen, Lü and Yang–Chen systems, but it has two very different fixed points: two stable node-foci. Therefore, it is very interesting to further find out the new dynamics of the system. This system has the form:

\[
\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= -cy - xz, \\
\dot{z} &= -b + xy,
\end{align*}
\]

where \(a\), \(b\) and \(c\) are real parameters.

The system possesses two equilibria \(E_1(\sqrt{b}, \sqrt{b}, -c)\) and \(E_2(-\sqrt{b}, -\sqrt{b}, c)\). It contains the chaotic system proposed by Munmuangsae and Srisuchinwong as \(c = 0\) [13], and the special system does not occur Hopf bifurcation. In particular, when \(c = 0\), we used the generalized Melnikov method, the parameter conditions could be obtained to guide system (1) with parameter perturbations to a low-periodic motion. Moreover, the existence conditions of periodic orbits and homoclinic orbits in the controlled system are given [14]. Actually, parameter \(c\) plays an important role in system (1) as described below.

We fix \(a = 10\) and \(b = 100\), and observe that system (1) has a chaotic attractor for parameter \(c = 11.2\) (see Fig. 1(a)), where the equilibria \(E_{1,2}\) both are two stable node-foci whose characteristic values are: \(\lambda_1 = -20.9778\), \(\lambda_{2,3} = -0.1111 \pm 9.7635i\). Moreover, for parameter \(c = 10\), system (1) also has a chaotic attractor (see Fig. 1(b)), where three characteristic values of the Jacobian of the linearized equation evaluated at the equilibria \(E_{1,2}\) are: \(\lambda_1 = -20\), \(\lambda_{2,3} = \pm 10i\). Therefore, system (1) has...
a chaotic attractor coexist with two non-hyperbolic equilibria. When $c = 11.2$ and 10, the different time series of state variable $z(t)$ are shown in Fig. 1(c) and (d), respectively. In addition, Fig. 2 shows the dependence of the Lyapunov exponents on the parameter $c$, where $a = 10$, $b = 100$ are fixed. In particular, for $c = 11.2$ and 10, the corresponding Lyapunov exponents and fractional dimensions of system (1) for initial value $(0.98, 0.82, 0.49)$ are given in Table 1.

The presence of bifurcation is of great importance in many physical, chemical, biological and geological nonlinear systems. For instance, the breakdown of the thermohaline ocean circulation in climate dynamics is caused by crossing Hopf bifurcations. In some practical circumstances, it may be advantageous to introduce certain bifurcations to the nominal branch of a system output. The new bifurcations may lead to new or more desirable operating conditions; they may also serve as warning signals for impending collapse or catastrophe. The task of creating a desired bifurcation at a desired location is referred to as the anti-control of bifurcation [15–17]. Aiming to contribute in the understanding of the dynamics of this new chaotic system, we present in this note an analytical bifurcation analysis of its controlled systems keeping original some properties.

The idea of this work is to design a control law such that our feedback nonlinear control system represents a perturbation of the chaotic system and such that our feedback system undergoes a controllable Hopf bifurcation. Very simple expressions for the so-called stability coefficients of the Hopf bifurcation are obtained through the design of simple control laws. To consider the problem of anti-control of Hopf bifurcation in the system (1), we introduce the feedback controllers

$$u = m(x - y) + n(x - y)^3,$$

so that the anti-controlled system takes the following form

![Fig. 1. Chaotic attractors of system (1) with initial values (0.98, -0.82, -0.49): (a) chaotic attractor when parameter values $(a, b, c) = (10, 100, 11.2)$; (b) chaotic attractor when parameter values $(a, b, c) = (10, 100, 10)$; (c) time series of state variable $z(t)$ of (a); (d) time series of state variable $z(t)$ of (b).](image)

![Fig. 2. (a) Lyapunov exponents of system (1): $a = 10$, $b = 100$ and $c \in [-10, 12]$; (b) Lyapunov exponents $\lambda(z_c)$ of system (4) with two stable node-foci, an enlargement of (a); $a = 10$, $b = 100$ and $c \in [10, 12]$.](image)
\[ \dot{x} = a(y - x), \quad \dot{y} = -cy - xz, \quad \dot{z} = -b + xy + u. \] (3)

Obviously the controller (2) keeps the equilibrium structure and does not change the divergence of the dynamic system (1). Due to the symmetry between \( E_1 \) and \( E_2 \), in what follows, we only consider the Hopf bifurcation at \( E_1 \) under the restriction \( b > 0 \).

It is organized as follows. In Section 2, we present the outline of the Hopf bifurcation methods about codimension one, two and three Hopf bifurcations, in particular, how to calculate the Lyapunov coefficients related to the stability of the equilibrium point. In Section 3, through a linear analysis of system (3), we obtain the Hopf conditions for two and three Hopf bifurcations, in particular, how to calculate the Lyapunov coefficients related to the stability of the equilibrium point. Finally, in Section 4, we make some concluding remarks.

### 2. Outline of the Hopf bifurcation methods

The beginning of this section is showing the projection method described in [18–20] for the calculation of the first, second and third Lyapunov coefficients associated to the Hopf bifurcations, denoted by \( l_1 \), \( l_2 \), and \( l_3 \), respectively. The method has been applied in some systems [21,22]. Consider the differential equation

\[ \dot{X} = f(X, \mu), \] (4)

where \( X \in \mathbb{R}^3 \) and \( \mu \in \mathbb{R}^6 \) are, respectively vectors representing phase variables and control parameters. Assume that \( f \) is a class of \( \mathcal{C}^\infty \) in \( \mathbb{R}^3 \times \mathbb{R}^6 \). Suppose that (4) has an equilibrium point \( X = X_0 \) at \( \mu = \mu_0 \), and denoting the variable \( X - X_0 \) also by \( X \), write

\[ F(X) = f(X, \mu_0), \] (5)

as

\[ F(X) = AX + \frac{1}{2} B(X, X) + \frac{1}{6} C(X, X, X) + \frac{1}{24} D(X, X, X, X) + O(||X||^8), \]

where \( A = f_s(0, \mu_0) \) and, for \( i = 1, 2, 3 \),

\[ B(X, Y) = \sum_{j=1}^{3} \frac{\partial^2 F_i(X, Y)}{\partial X_j \partial X_k} \bigg|_{X=0} X_j Y_k, \quad C(X, Y, Z) = \sum_{j=1}^{3} \frac{\partial^3 F_i(X, Y, Z)}{\partial X_j \partial X_k \partial Z_l} \bigg|_{X=0} X_j Y_k Z_l, \]

and so on for \( D_1 \) and \( D_2 \). Suppose that \( A \) has a pair of complex eigenvalues on the imaginary axis: \( \lambda_{2,3} = \pm i\omega_0(\omega_0 > 0) \), and these eigenvalues are the only eigenvalues with \( \text{Re} \lambda = 0 \). Let \( T^e \) be the generalized eigenspace of \( A \) corresponding to \( \lambda_{2,3} \). Let \( p, q \in \mathbb{C}^3 \) be vectors such that

\[ Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p, \quad \langle p, q \rangle = 1, \]

where \( A^T \) is the transposed of the matrix \( A \). Any vector \( y \in T^e \) can be represented as \( y = wq + \tilde{w} \bar{q} \), where \( w = (p, q) \in \mathbb{C} \). The two-dimensional center manifold associated to the eigenvalues \( \lambda_{2,3} \) can be parameterized by \( w \) and \( \tilde{w} \), by means of an immersion of the form \( X = H(w, \tilde{w}) \), where \( H: C^2 \to \mathbb{R}^3 \) has a Taylor expansion of the form

\[ H(w, \tilde{w}) = wq + \tilde{w} \bar{q} + \sum_{2 \leq j, k \leq 3} \frac{1}{j!} \sum_{2 \leq j, k \leq 3} h_{jk} \tilde{w}^j \tilde{w}^k + O(||w||^8), \]

with \( h_{jk} \in \mathbb{C}^3 \) and \( h_{jk} = \tilde{h}_{jk} \). Substituting this expression into (5) we obtain the following differential equation

\[ H_w w' + H_{w \tilde{w}} w = F(H(w, \tilde{w})), \]

where \( F \) is given by (5). The complex vectors \( h_{jk} \) are obtained solving the system of linear equations defined by the coefficients of (5), taking into account the coefficients of \( F \), so that system (5), on the chart \( w \) for a central manifold, writes as follows

\[ w = i\omega_0 w + \frac{1}{2} G_{21} \tilde{w} |w|^2 + \frac{1}{12} G_{22} \tilde{w} |w|^4 + \frac{1}{144} G_{44} \tilde{w} |w|^6 + O(|w|^8), \]
with \( G_{jk} \in C \). The first Lyapunov coefficient \( l_1 \) can be written as

\[
l_1 = \frac{1}{2} \text{Re}G_{21},
\]

where \( G_{21} = (q, C(q, q, q) + B(q, h_{20}) + 2B(q, h_{11})). \)

Defining \( \mathcal{H}_{32} \) as

\[
\mathcal{H}_{32} = 6B(h_{11}, h_{21}) + B(h_{20}, h_{30}) + 3B(h_{21}, h_{20}) + 3B(q, h_{22}) + 2B(q, h_{31}) + 6C(q, h_{11}, h_{11}) + 3C(q, h_{20}, h_{20}) + 3C(q, q, h_{21})
\]

\[
+ 6C(q, q, h_{21}) + 6C(q, h_{20}, h_{11}) + C(q, q, h_{30}) + D(q, q, q, h_{20}) + 6D(q, q, q, h_{20}) + 3D(q, q, q, h_{11}) + 3D(q, q, q, h_{20}) + E(q, q, q, q)
\]

\[
- 6G_{21}h_{21} - 3G_{21}h_{21},
\]

and \( G_{32} = (q, \mathcal{H}_{32}) \), the second Lyapunov coefficient \( l_2 \) is given by

\[
l_2 = \frac{1}{2} \text{Re}G_{32}.
\]

The third Lyapunov coefficient \( l_3 \) is defined by

\[
l_3 = \frac{1}{144} \text{Re}G_{43},
\]

where \( G_{43} = (q, \mathcal{H}_{43}) \). The expression for \( \mathcal{H}_{43} \) is too large to be put in print and can be found in [20].

A Hopf bifurcation point \((X_0, \mu_0)\) is an equilibrium point of (4) where the Jacobian matrix \( A \) only has a pair of purely imaginary eigenvalues \( \lambda_{2,3} = \pm iw_0(\mu_0 > 0) \), and the other eigenvalue with non-zero real part. At a Hopf point a two-dimensional center manifold is well defined, it is invariant under the flow generated by (4) and can be continued with arbitrary high class of differentiability to nearby parameter values.

A Hopf point is called transversal if the parameter-dependent complex eigenvalues cross the imaginary axis with non-zero derivative. In a neighborhood of a transversal Hopf point with \( l_1 \neq 0 \) the dynamic behavior of the system (4) reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to the following complex normal form

\[
w' = (\eta + iw)w + l_1w|w|^2,
\]

where \( w \in C, \eta, w \) and \( l_1 \) are real functions having derivatives of arbitrary higher order, which are continuations of \( 0, w_0 \) and the first Lyapunov coefficient at the Hopf point [18]. As \( l_1 < 0(l_1 > 0) \) one family of stable (unstable) periodic orbits can be found on this family of manifolds, shrinking to an equilibrium point at the Hopf point.

A Hopf point of codimension two is a Hopf point where \( l_1 \) vanishes. It is called transversal if \( \eta = 0 \) and \( l_1 = 0 \) have transversal intersections, where \( \eta = \eta(\mu) \) is the real part of the critical eigenvalues. In a neighborhood of a transversal Hopf point of codimension two with \( l_2 \neq 0 \) the dynamic behavior of the system (4) reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to

\[
w' = (\eta + iw)w + \tau w|w|^2 + l_2w|w|^4,
\]

where \( \eta \) and \( \tau \) are unfolding parameters.

A Hopf point of codimension three is a Hopf point where \( l_2 \) vanishes. It is called transversal if \( \eta = 0, l_1 = 0 \) and \( l_2 = 0 \) have transversal intersections, where \( \eta = \eta(\mu) \) is the real part of the critical eigenvalues. In a neighborhood of a transversal Hopf point of codimension three with \( l_3 \neq 0 \) the dynamic behavior of the system (4) reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to

\[
w' = (\eta + iw)w + \tau w|w|^2 + \alpha w|w|^4 + l_3w|w|^6,
\]

where \( \eta, \tau \) and \( \alpha \) are unfolding parameters. The bifurcation diagrams for \( l_3 \neq 0 \) can be found in [23].

3. Hopf bifurcations in the controlled system (3)

This subsection employs the 3D Hopf bifurcation theory and applies symbolic computations to perform the analysis of parametric variations with respect to dynamical bifurcations.

One first translate the equilibrium point \( E_1 \) to the origin \( O(0,0,0) \) under the following linear transformation

\[
\begin{align*}
x_1 &= x - x_0, \\
y_1 &= y - y_0, \\
z_1 &= z - z_0,
\end{align*}
\]

which transforms system (3) into the following form
\[
\begin{aligned}
\dot{x}_1 &= a(y_1 - x_1), \\
\dot{y}_1 &= -z_0 x_1 - c y_1 - x_0 z_1 - x_1 z_1, \\
\dot{z}_1 &= y_0 x_1 + x_0 y_1 + x_1 y_1 + m(x_1 - y_1) + n(x_1 - y_1)^3.
\end{aligned}
\]  

(6)

The Jacobian matrix \( A \) of system (6) at the equilibrium point \( O(0,0,0) \) is

\[
J(O) = \begin{pmatrix}
-a & a & 0 \\
-z_0 & -c & -x_0 \\
y_0 + m & x_0 - m & 0
\end{pmatrix},
\]

and its corresponding characteristic equation

\[
\lambda^3 + (a + c)\lambda^2 + \left(b - \sqrt{bm}\right)\lambda + 2ab = 0.
\]  

(7)

This indicates that the control gain \( n \) has no influence on the eigenvalues of matrix \( J(O) \). Taking \( m \) as the Hopf bifurcation parameter and suppose that Eq. (7) possesses one conjugate pair of pure imaginary eigenvalues \( \pm iw_0 \) \( \lambda_0 > 0 \), we can reach

\[
m = m_0 = \frac{\sqrt{b(c - a)}}{a + c}, \quad \lambda_0 = \sqrt{\frac{2ab}{a + c}}.
\]

Under this condition, the transversality condition

\[
\lambda'(m_0) = \frac{\sqrt{b(a + c)(a + b)}}{2[a^3 + b^2c + a^2(2b + c) + ab(2b + 2c)]} > 0,
\]

is also satisfied. The appearance of Hopf bifurcation also depends on the value of the first Lyapunov coefficient \( l_1 \),

\[
l_1 = \frac{(a + c)^3\{a^3 + 38a^3b + 2a^4c - 4a^2bc - 2ab^2c^2 - 2a^2c^3 - ac^4 + 12b^{3/2}\{a + c\}^3 + 8ab\}n}{8(a^3 + 2ab + 3a^2c + 3ac^2 + c^3)^2[(a + c)^4 + 8ab]}.
\]

Thus according to the critical conditions for Hopf bifurcation, we can obtain the following proposition. For convenience, we mark

\[
n_0 = \frac{-a^2 - 38a^2b - 2a^4c + 4a^2bc + 2ab^2c + 2a^2c^3 + ac^4}{12b^{3/2}[(a + c)^4 + 8ab]}.
\]

**Proposition 3.1.** If and only if \( (a, b, c, m, n) \in \{(a, b, c, m, n)\mid a > 0, b > 0, a + c > 0, m = m_0, n \neq n_0\} \), the controlled system (3) has a transversal Hopf point at \( E_t \). More specifically, if \( n < n_0 \) and \( m > m_0 \), but close to \( m_0 \), there exists a stable periodic orbit near the unstable equilibrium point \( E_t \); if \( n > n_0 \) and \( m < m_0 \), but close to \( m_0 \), there exists a unstable periodic orbit near the asymptotically stable equilibrium point \( E_t \).

**Remark 1.** In particular, if \( m = n = 0 \), the conditions for Hopf bifurcation is changed to

\[
\{(a, b, c)\mid c = a > 0, b > 0\}.
\]  

(8)

Then the following corresponding results are obtained:

\[
l_1 = \frac{bc^2}{(b + c^2)(b + 4c^2)^2} > 0, \quad \lambda'(c_0)|_{\lambda = 0} = -\frac{b}{2(4a^2 + b)} < 0.
\]

Therefore, the uncontrolled system (1) undergoes non-degenerate and subcritical Hopf bifurcation. If \( c > c_0 \), but close to \( c_0 \), there exists a unstable periodic orbit near the asymptotically stable equilibrium point \( E_t \). In particular, for \( \{(a, b, c)\mid a = 10, b = 100, 10 < c < 11.2\} \), we observe three attractors (each corresponds to different sets of initial condition), asymptotically stable equilibrium point, periodic orbits or chaotic attractors depending on the value of parameter \( c \) [12].

Compared with the uncontrolled system (1), the controlled system (3) can exhibit degenerate Hopf bifurcation in a much larger parameter region and make possible multiple periodic solutions bifurcate from the equilibrium point. In the following, we consider system (3) satisfying the special conditions (8) (i.e., Hopf bifurcation conditions of uncontrolled system (1)), and have the following conclusions.

**Proposition 3.2.** Define \( T = \{(a, b, c)\mid c = a, b > 0, m = 0\} \). If \( (a, b, c, m) \in T \), then Jacobian matrix of system (3) at \( E_t \) has one negative real eigenvalue \( \lambda_1 \) and a pair of purely imaginary eigenvalues \( \lambda_{2,3} \).
Theorem 3.1. Consider the system (3). The first Lyapunov coefficient at $E_1$ for parameter values in $T$ is given by

$$l_1 = \frac{bc(3b^{1/2}n + c^2(1 + 3\sqrt{bn}))}{(b + c^2)(b + 4c^2)^2}. \quad (9)$$

If $n \neq -\frac{c^2}{3\sqrt{bn}}$, system (3) has a transversal Hopf point at $E_1$. Moreover, if $n > -\frac{c^2}{3\sqrt{bn}}$, then $E_1$ is unstable (weak repelling focus) and for each $m < 0$, but close to 0, there exists a unstable limit cycle near the unstable equilibrium point $E_1$; if $n < -\frac{c^2}{3\sqrt{bn}}$, then the Hopf point at $E_1$ is asymptotically stable (weak attractor focus) and for each $m > 0$, but close to 0, there exists a stable limit cycle near the stable equilibrium point $E_1$.

The sign of the first Lyapunov coefficient is determined by the sign of the numerator of (9) since the denominator is positive. Observe that the first Lyapunov coefficient vanishes on the certain parameter values. In the following theorem we study the sign of the second Lyapunov coefficient where the first coefficient vanishes.

Theorem 3.2. Define $Q = \{(a, b, c, m, n) \mid a = c, b = 100, m = 0, n = -\frac{c^2}{30(100 + c^2)}, c \neq 5 \text{ and } c \neq 15\}$. If $(a, b, c, m, n) \in Q$, consider the system (3). The second Lyapunov coefficient at $E_1$ is given by

$$l_2|_{l_1=0} = \frac{g(c)}{8640000(25 + c^2)^3(100 + c^2)^2(75 - 20c + c^2)}, \quad (10)$$

where

$$g(c) = \frac{632812500000 + 774140625000c - 995625000000c^2 + 95578125000c^3 - 22495781250c^4 + 6416906250c^5}{1530040625c^6 + 182368750c^3 - 28934750c^2 - 448193750c^3 + 119892225c^{10} - 20994906c^{11}} + 4000000c^{12} - 260000c^{13} + 16000c^{14} - 800c^{15}.$$

If we set $h(c) = \frac{g(c)}{(25 + c^2)^3(100 + c^2)^2(75 - 20c + c^2)}$, whose graphics are shown in Fig. 3, $h(c)$ has the same sign with $l_2|_{l_1=0}$. When $h(c)$ is different from zero, the system (3) has a transversal Hopf point of codimension two at $E_1$. More specifically, if $c \in U$ then the system (3) has a transversal Hopf point at $E_1$ which is unstable since $l_2 > 0$; if $c \in S$ then the system (3) has a transversal Hopf point at $E_1$ which is asymptotically stable since $l_2 < 0$. Denoting

$$U = \{c \mid 0 < c < c_1 \text{ or } c_2 < c < 5 \text{ or } c_3 < c < 15\}, \quad S = \{c \mid c_1 < c < c_2 \text{ or } c_2 < c < c_3 \text{ or } 15 < c < +\infty\}.$$

Fig. 3. The segmented graphics of $h(c)$: (a) $c \in [0, 4.8]$; (b) $c \in [4.8, 5.2]$; (c) $c \in [5.2, 14.994]$; (d) $c \in (14.994, 15.004]$. 
Here, 
\[ c_1 = 2.973534232053151882810359 \ldots, \quad c_2 = 4.99201377061137248 \ldots, \]
and
\[ c_3 = 14.999857056504938495305127364 \ldots. \]
are the only three positive roots of \( h(c) = 0 \).

**Theorem 3.3.** For system (3) there are three parameter sets \( K_i = (a_i, b_0, c_i), i = 1, 2, 3, \) with
\[ a_1 = c_1 = 2.9735342432025 \ldots, \quad a_2 = c_2 = 4.9920137706 \ldots, \quad a_3 = c_3 = 14.999857056504 \ldots, \]
and \( b_0 = 100 \), where \( l_1 = 0 \) and \( l_2 = 0 \). For the sets of parameter values \( (a_i, b_0, c_i), i = 1, 2, \) system (3) has a transversal Hopf point of codimension three at \( E_1 \) which is unstable since \( l_1 < 0 \). Moreover, for the set of parameter values \( (a_3, b_0, c_3) \), system (3) has a transversal Hopf point of codimension three at \( E_1 \), which is unstable since \( l_2 > 0 \).

**Proof.** The algebraic expression for the third Lyapunov coefficient can be obtained in [18,19]. This is too long to be put in print. The sets \( k_i \) are the intersection of the surfaces \( l_1 = 0 \) and \( l_2 = 0 \). The existence and uniqueness of \( k_i \) with the above values has been established numerically. We only consider the parameters \( a_2 = c_2 = 4.9920137706 \) and \( b_0 = 100 \) for simplicity. For these values of the parameters one has
\[ p = (0.5i, 0.5i, 0.5), \quad q = (0.249999968 + 0.24960037i, -0.249999968 + 0.75039963i, 1), \]
\[ B(x, y) = (0, -x_1y_1 - x_1y_3 + x_2y_1 + x_1y_2), \quad C(x, y, z) = (0, 0, -0.03989779(x_1 - x_2)(y_1 - y_2)(z_1 - z_2)), \]
\[ h_{11} = (-0.01248, -0.01248, -0.0499999), \]
\[ h_{20} = (0.0334741 + 0.0141344i, -0.0532808 + 0.0275455i, 0.0333599 + 0.0499467i), \]
\[ G_{21} = -1.0407375808480790i, \]
\[ h_{21} = (-0.0000724813 + 0.0000934768i, 0.00040576 - 0.000718276i, 0.000624799 + 0.00033278i), \]
\[ h_{30} = (-0.000194235 - 0.000815189i, 0.000973039 + 0.00408377i, 0.00257021 - 0.000678581i), \]
\[ h_{31} = (-0.0147102 + 0.00348378i, -0.0271971 - 0.050833i, -0.0521208 + 0.0346218i), \]
\[ h_{40} = (5.57144 × 10^{-2} + 0.0000101954i, -0.0000811362 + 0.0000146596i, 0.0000208923 + 0.0000768410i^T), \]
\[ h_{22} = (0.0103969 - 0.0195264i, 0.0103969 - 0.0195264i, -0.0603007 + 0.0181556i), \]
\[ G_{32} = 0.0404805i, \]
\[ h_{32} = (-0.000771686 - 0.00188208i, 0.00103294 - 0.00135533i, 0.00323741 + 0.000261257i)^T, \]
\[ h_{41} = (0.000761692 - 0.000151785i, -0.00483548 + 0.00100335i, 0.000297999 + 0.00036599i)^T, \]
\[ h_{42} = (-0.0251182 + 0.00147962i, -0.022676 - 0.0617666i, -0.0622329 + 0.044876i)^T, \]
\[ h_{33} = (0.000578542 + 0.0000277593i, 0.000578542 + 0.0000277593i, -0.000465506 - 0.000713305i)^T, \]
\[ G_{43} = -0.20608 - 0.19453i. \]

By the above theorem and calculation, one has
\[ l_1(K_2) = \frac{1}{2} \text{Re}G_{21} = 0, \quad l_2(K_2) = \frac{1}{12} \text{Re}G_{32} = 0, \quad l_3(K_2) = \frac{1}{144}G_{43} = -0.00143111. \]

Similarly, we can get
\[ l_1(K_1) = \frac{1}{2} \text{Re}G_{21} = 0, \quad l_2(K_1) = \frac{1}{12} \text{Re}G_{32} = 0, \quad l_3(K_1) = \frac{1}{144}G_{43} = -4.8024 × 10^{-10}, \]
and
\[ l_1(K_3) = \frac{1}{2} \text{Re}G_{21} = 0, \quad l_2(K_3) = \frac{1}{12} \text{Re}G_{32} = 0, \quad l_3(K_3) = \frac{1}{144}G_{43} = 1953074.567065. \]

The other cases have the same type of proofs. Therefore, we can arrive at the above **Theorem 3.3.** □

4. Conclusion

In this paper, we have considered, via feedback control and symbolic computation, the problem of anti-control of Hopf bifurcation in the chaotic system coexist with two stable node-foci. By the numerical analysis, some global phenomena of the uncontrolled system suggested the existence of strange chaotic attractors with stable equilibria [12]. Although it is
out of the main purpose of this paper, we want to investigate it here because the existence of these attractors is in some sense related to the Hopf bifurcations which occur at the equilibria. An anti-controller \( u = m(x - y) + n(x - y)^3 \) is designed based on not changing divergence and the equilibrium of the original system and not raising the dimension of the anti-controlled system. The proposed anti-control scheme is effective and easy to manipulate with the aid of symbolic computation.

Through this analysis we obtain the parameter conditions for which the system presents Hopf bifurcations at these equilibria. Then we make an extension of the analysis to the more degenerate cases. The calculation of the first second and third Lyapunov coefficients, which makes possible the determination of the Lyapunov stability at the equilibria, can make the controlled system exhibit Hopf bifurcation in a much larger parameter region. Moreover, the anti-control gain \( n \) can show the criticality of Hopf bifurcation.

We believe that the unknown dynamical behaviors of the strange chaotic attractors with stable equilibria deserve further investigation and are very desirable for engineering applications such as secure communications in the near future.

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References