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SOME ALGORITHMS FOR SOLVING EXTREME POINT MATHEMATICAL PROGRAMMING PROBLEMS*+

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SUMMARY

The paper presents some algorithms for solving the extreme point mathematical programming problem formulated in [5]. The algorithms discussed determine lower and upper bounds of the objective function and accomplish the search for the optimal solution. The direction of search depends upon the choice of the starting point. Algorithms making use of the two possible search directions are discussed in detail. In order to illustrate the operation of these algorithms, two numerical examples are solved. Some other algorithms, based on different approaches, are also briefly outlined.

1. INTRODUCTION

An extreme point mathematical programming problem, as formulated in [5], is a linear programming problem in which an optimal solution must be feasible for one set of constraints and be an extreme point of another set of constraints. It can be stated as follows:

PROBLEM I: maximize $\underline{c}\,\underline{x}$ (1)

subject to $A\,\underline{x} \le \underline{b}$ (2)

where $\underline{x}$ is an extreme point of

$D\,\underline{x} \le \underline{f}, \quad x \ge 0.$ (3)

Here $\underline{c}$ is a $1 \times n$, $\underline{x}$ is $n \times 1$, A is $m \times n$, $\underline{b}$ is $m \times 1$, D is $p \times n$, $\underline{f}$ is $p \times 1$ and $\underline{0}$ is $n \times 1$.

The main work in this area is due to Kirby et al. [4,5]. Gupta and Swarup [2] and Puri and Swarup [7] have extended their formulation to linear fractional functional programming problems. The authors of [4] have used their approach

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to solve some scheduling problems. They have also shown that zero-one programming problems are of this type. It has been found that the concepts of extreme point mathematical programming are of great help when solving critical path problem under assignment constraints. Such problems are of practical importance and some interesting results have been obtained [5].

In this paper, the extreme point mathematical programming problem is reconsidered and two algorithms to solve it are discussed. Some other formulations and suggestions are also discussed which might lead to efficient solution procedures in the future. The authors consider that this problem of mathematical programming is an important one and needs much more attention. This paper is an attempt in this direction. Some numerical illustrations are given in Section 5.

2. MATHEMATICAL DEVELOPMENT

PROBLEM II: maximize $\underline{c}\,\underline{x}$

subject to
$$A\,\underline{x} \leqslant \underline{b}$$
$$D\,\underline{x} \leqslant \underline{f}$$
$$\underline{x} \geqslant 0$$

PROBLEM III: maximize $\underline{c}\,\underline{x}$

subject to
$$A\,\underline{x} \leqslant \underline{b}$$
$$\underline{x} \geqslant 0$$

PROBLEM IV: maximize $\underline{c}\,\underline{x}$

subject to
$$D\,\underline{x} \leqslant \underline{f}$$
$$\underline{x} \geqslant 0$$

These problems can be arranged as shown in Tableau 1, where s_i $(i=1,\ldots m+1,\ldots,m+p)$ denote the slack variables. For the sake of simplicity it is assumed that both sets of constraints (2) and (3) are bounded. Thus the value z of the objective function (1) corresponding to any feasible solution to Problem I is finite. In other words, finite upper and lower bounds of this function can be established:

$$-\infty < z_{lb} < z < z_{ub} < +\infty \tag{4}$$

Tableau 1

Basic Variables	Constants	$x_1 \ldots x_n$	$s_1 \ldots s_m$	$s_{m+j} \ldots s_{m+p}$
s_1	b_1	$a_{11} \ldots a_{1n}$	$1 \ldots 0$	$0 \ldots 0$
$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
s_m	b_m	$a_{m1} \ldots a_{mn}$	$0 \ldots 1$	$0 \ldots 0$
s_{m+1}	f_1	$d_{11} \ldots d_{1n}$	$0 \ldots 0$	$1 \ldots 0$
$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
s_{m+p}	f_p	$d_{p1} \ldots d_{pn}$	$0 \ldots 0$	$0 \ldots 1$
$z_j - c_j$	0	$-c_1 \ldots -c_n$	$0 \ldots 0$	$0 \ldots 0$

It is clear that any of the values of the objective function (1) corresponding to the optimal solutions of Problems II, III and IV, if they exist, can be taken as the upper bound z_{ub}. In general, the lowest value of z_{ub} corresponds to the optimal solution of Problem II.

If a feasible solution to Problem I is given, the value of the objective function associated with this solution can be used as the lower bound. Such a solution can be obtained using the procedure outlined in Appendix. If neither the established lower bound nor the upper bound constitutes the optimal solution to Problem I, then the region defined by these bounds is to be searched for the optimal solution. There are two possible directions for this search. For the first one, we look for extreme points of the constraint set (3), i.e. feasible solutions to Problem IV, resulting in the least decrease of the value of the value of the objective function (1) when compared with the established z_{ub}. If the solution corresponding to such an extreme point is not feasible to Problem I, the value of the objective function (1) associated with it is assumed as a new upper bound.

The second approach consists in finding extreme points of (3) which result in the least increase of the value of the objective function (1) when compared with the established lower bound.

It is obvious that the higher the lower bound or the lower the upper bound, the computing effort needed to achieve the result is reduced.

It will be shown that using the so called restricted base entry procedure it is possible, in general, to increase the lower bound starting from the value corresponding to a feasible solution defined by Tableau 1. This follows from the results given in [5], where it was proved that an extreme point of (3) satisfying (2) is also an extreme point for Problem II. In other words, a feasible solution to Problem I is also a feasible solution to Problem II. Conversely, if a solution to Problem II is an extreme point of (3), then it is a feasible solution of Problem I. It should be noted that a solution to Problem II can have this property only if it has not more than p x-components different from zero.

Without loss of generality we may assume that the starting feasible solution to Problem II is given by $s_i = b_i$ $(i=1,\ldots,m)$, $s_{m+\ell} = f_\ell (\ell=1,\ldots,p)$; $x_1 = \cdots = x_n = 0$. In this case the lower bound for the objective function (1) is zero.

Denote by H_{II} the set of indices of the non-basic variables of Problem II with the property that $z_j - c_j < 0$ and compute the following quantities

$$\frac{b_r}{a_{rj}} = \min_i \left\{\frac{b_i}{a_{ij}}; a_{ij} > 0, j \varepsilon H_{II}\right\}; \quad \frac{f_t}{d_{tj}} = \min_\ell \left\{\frac{f_\ell}{d_{\ell j}}; d_{\ell j} > 0, j \varepsilon H_{II}\right\} \tag{5}$$

If there exists a $j \varepsilon H_{II}$, say s, such that

$$b_r/a_{rs} - f_t/d_{ts} \geqslant 0 \tag{6}$$

then performing the pivot operation on d_{ts} a new solution to Problem II is obtained, which is simultaneously a solution to Problem I. This property is due to the fact that when performing such a pivot operation, a feasible solution to Problem IV results, which determines a new extreme point of (3). If there is more than one j for which the inequality holds, any one can be used or typical rules of simplex method may be applied, i.e., that j for which $c_j - z_j$ is maximum is to be selected. The value of the objective function (1) corresponding to this solution is now taken as the new lower bound. This procedure is repeated until there is no $j \varepsilon H_{II}$ for which the inequality (6) is satisfied. If for a particular j there are several t for which the ratio f_t/d_{tj} is the same, then the simplex tie breaking rules are applied. However, it can happen that even for starting Tableau 1 of a given problem, there is no j satisfying this inequality. If this is the case, then an optimal solution to Problem IV is to be obtained.

As was mentioned earlier, the first of the presented approaches attempts to determine an extreme point of (3), which results in the least decrease in the value of the objective function (1) when compared with the established upper bound. Assume that the optimal solution to Problem IV is given. The value of the objective function (1) corresponding to this solution can be taken as the new upper bound. A solution to Problem IV resulting in the smallest deviation from the optimal value is called the second best solution. The method for finding such a solution is given by Hadley [3]. It consists of the following steps.

The quantities q_j are to be determined from

$$q_j = \frac{\bar{f}_t}{\bar{d}_{tj}} = \min_{\ell} \left\{ \frac{\bar{f}_\ell}{\bar{d}_{\ell j}};\ j \varepsilon H_{IV},\ d_{\ell j} > 0 \right\} \tag{7}$$

where $\bar{f}_t$, $\bar{d}_{tj}$ are elements of Tableau 1 corresponding to the optimal solution to Problem IV and H_{IV} is the set of indices j of the non-basic variables of Problem IV for which $z_j - c_j > 0$. If the optimal solution to Problem IV is not unique, i.e., $z_j - c_j = 0$ for some j, then these calculations should be performed for all tableaus giving optimal solutions. The pivot operation performed on an element $\bar{d}_{tj} > 0$, $j \in H_{IV}$ results in a decrease of the value of the objective function. The amount of this decrease is given by

$$\gamma_j = q_j(z_j - c_j), \quad j \in H_{IV} \tag{8}$$

It is clear that the second best solution to Problem IV can be obtained by performing the pivot operation on an element $\bar{d}_{ti}$, where i is determined by

$$\gamma_i = \min \gamma_j, \quad j \in H_{IV}$$

The quantity i need not be unique. If a resulting solution satisfies the constraint (2), then it is the optimal solution to Problem I. Otherwise the value of the objective function (1) corresponding to this solution can be assumed as a new upper bound. Denote this value by z^1_{ub}. It is shown in [5] that the third best solution to Problem IV, the next candidate to be the optimal solution to Problem I, can be determined similarly. As an illustration consider the following problem:

PROBLEM IV': maximize $\underline{c}\,\underline{x}$

subject to

$$D\,\underline{x} \leqslant \underline{f}$$
$$\underline{c}\,\underline{x} \geqslant z^1_{ub}$$
$$\underline{x} \geqslant 0$$

It is obvious that the optimal solution to Problem IV' is the second best solution to Problem IV. If the procedure discussed above is followed, it is possible to determine the second best solution to Problem IV'. However, this solution is the third best solution to Problem IV. Hence, the procedure discussed above allows any k-th best solution to Problem IV to be determined. In general the computational effort required to determine the k-th best solution /k=2,3.../ is much higher than that needed to obtain the best solution to the problem of similar dimension. This aspect needs further study based on computer calculations, because it can affect the effectiveness of algorithms using the suggested procedure. The authors intend to present a subsequent publication concerning the problem mentioned.

Similarly it is possible to find an extreme point of Problem IV resulting in the least increase of the value of the objective function (1) when compared with the established lower bound. In order to find such an extreme point, the following problem is considered

PROBLEM IV": maximize $\underline{c}\ \underline{x}$

subject to

$$D\ \underline{x} \leqslant \underline{f}$$

$$\underline{c}\ \underline{x} \geqslant z_{1b}$$

$$\underline{x} \geqslant 0$$

Problem IV" is obtained by adding the constraint $\underline{c}\ \underline{x} \geqslant z_{1b}$ to those of Problem IV. Hence, Problem IV" can have more extreme points than problem IV. The objective function assumes the value z_{1b} at these new extreme points. These points can be determined using the normal pivot operation, but two cases have to be considered. Let us assume that there exist nonbasic variables of Problem IV whose relative cost coefficients are equal to zero. When performing the pivot operation on an element of a column associated with such a variable, the value of the objective function is not changed. Hence, the tableau obtained determines the desired extreme point. However, it should be noted that the same result can be obtained by performing the pivot operation on an element of a column having a positive or negative value of the relative cost coefficient providing the value of the basic variable leaving the basis is equal to zero.

In order to simplify the notation, the following symbols are introduced

$$g_{ij}^{\alpha} = \begin{cases} \bar{d}_{ij}^{\alpha} & i=1,\dots,p \\ \bar{c}_{j}^{\alpha} & i=p+1 \end{cases} \qquad p_{i}^{\alpha} = \begin{cases} \bar{f}_{i}^{\alpha} & i=1,\dots,p \\ \bar{z}_{1b}^{\alpha} & i=p+1 \end{cases} \tag{9}$$

A bar is used to denote those elements of the tableau corr- esponding to feasible solutions to Problem IV", for which the objective function (1) assumes the value z_{1b}. The index α is used to enumerate these tableaux.

Using notation similar to that above,

$H_{IV}^{"\alpha}$ denotes the set of indices of the nonbasic variables of

Problem IV" such that

$$z_j^\alpha - c_j^\alpha < 0$$

$$q_j''^\alpha = \frac{p_r^\alpha}{g_{rj}^\alpha} = \min_{1\leq i\leq p+1} \left\{ \frac{p_i^\alpha}{g_{ij}^\alpha} ; g_{ij}^\alpha > 0, j \in H''^\alpha_{IV} \right\} \quad \text{and}$$

$$\gamma = \min_\alpha \left\{ \min_{j\in H''^\alpha_{IV}} q_j''^\alpha (z_j^\alpha - c_j^\alpha) \right\} \tag{10}$$

It is clear that the smallest deviation from the z_{lb} is obtained by performing the pivot operation on the element defined by γ.

3. THE SOLUTION PROCEDURE

The discussion presented leads to the following algorithms.

3.1 ALGORITHM 1

Step 1: Suppose a feasible solution to Problem I is given. Test this solution for optimality of Problem IV. If this is the case, then the obtained solution solves Problem I and the algorithm terminates. If not, use the value of the objective function corresponding to this solution as the lower bound and proceed to Step 2.

Step 2: Use the restricted base entry procedure to find a new extreme point of Problem IV, that is a feasible solution to Problem I. If this is not possible, proceed to Step 3. If a new extreme point is obtained, then return to Step 1.

Step 3: Use the tableau resulting from Step 2 to determine the optimal solution to Problem IV. Test this solution for feasibility with respect to (2). If this is the case, the solution obtained solves Problem I and the algorithm terminates. Otherwise use the value of the objective function corresponding to this solution as the upper bound z_{ub}^o and proceed to Step 4.

Step 4: Formulate Problem $(IV')^k$ k=0,1,... by adding to Problem IV the constraint $\underline{c}\,\underline{x} \geq z_{ub}^k$ [Problem $(IV')^{k=0}$ is equivalent to Problem IV] and proceed to Step 5.

Step 5: Determine the second best solution to Problem $(IV')^k$. Test it for feasibility with respect to (2). If this is the case, the solution obtained solves Problem I and the algorithm terminates. Otherwise, establish the value of the objective function corresponding to this solution as a new upper bound z_{ub}^{k+1}. Increase k by 1 and return to Step 4.

3.2 ALGORITHM 2

Steps 1 and 2 are the same as in Algorithm 1.

Step 3: Use the tableau obtained at Step 2 to determine the optimal solution to Problem II. Establish the value of the objective function corresponding to this solution as the new upper bound and proceed to Step 4.

Step 4: Formulate Problem $(IV'')^k$ k=0,1,... by adding to Problem IV the constraint $\underline{c}\,\underline{x} \geqslant v^k$ (v^o is the value of the lower bound obtained at Step 1) and proceed to Step 5.

Step 5: Determine an extreme point of problem $(IV'')^k$ that results in the least increase of the value of the objective function when compared with v^k. Denote the corresponding value of the objective function by v^{k+1}. Denote the corresponding value of the objective function by v^{k+1}. Test the obtained solution to Problem $(IV'')^k$ for feasibility with respect to (2). If this is the case, establish v^{k+1} as the new lower bound, increase k by one and return to Step 4. Otherwise, check whether v^{k+1} is greater or equal to the upper bound determined at Step 3. If this is the case, the solution obtained using the highest lower bound is optimal and the algorithm terminates. If not, increase k by one and return to Step 4.

It should be noted that there is always such an extreme point of Problem IV for which $v \geqslant z_{ub}$. This follows from the fact that the optimal value of the objective function corresponding to Problem II is less than or equal to that of Problem IV.

4. OTHER FORMULATIONS

It is well known [6] that if the polyhedron corresponding to the constraint set (3) is nonempty and bounded, then any element belonging to the polyhedron can be written in the form

$$\underline{x} = \sum_{i=1}^{r} \lambda_i \underline{x}^i, \quad \lambda_i \geq 0, \quad i=1,\ldots,r, \quad \sum_{i=1}^{r} \lambda_i = 1 \qquad (11)$$

where $\underline{x}^i$ are the extreme points of (3). By use of this property, Problem I can be written in the following form

$$\text{maximize} \quad \sum_{i=1}^{r} (\underline{c}\ \underline{x}^i)\lambda_i \qquad (12)$$

$$\text{subject to} \quad \sum_{i=1}^{r} (A\ \underline{x}^i)\lambda_i \leq \underline{b}$$

$$\sum_{i=1}^{r} \lambda_i = 1 \qquad (13)$$

$$\lambda_i \geq 0$$

$$\text{and} \quad \lambda_i = 0 \text{ or } 1 \text{ for all } i \qquad (14)$$

In order to simplify the notation, we let

$$\begin{cases} A\ \underline{x}^i = \underline{h}_i \\ \underline{c}\ \underline{x}^i = \underline{w}_i \end{cases} \qquad (15)$$

The problem then becomes

$$\text{maximize} \quad \sum_{i=1}^{r} \underline{w}_i\lambda_i \qquad (16)$$

$$\text{subject to} \quad \sum_{i=1}^{r} \underline{h}_i\lambda_i \leq \underline{b}$$

$$\sum_{i=1}^{r} \lambda_i = 1 \qquad (17)$$

$$\lambda_i \geq 0$$

$$\text{and } \lambda_i = 0 \text{ or } 1 \text{ for all } i \qquad (18)$$

Due to constraint (18), this problem is a typical 0,1 programming problem and it can be solved by any known integer programming methods [8]. However, ignoring (18), the resulting problem can be solved suing the decomposition principle [6] and the solution checked for feasibility with respect to (18). If it is satisfied, the algorithm terminates. Otherwise the second best solution to this problem must be found and the solution checked for feasibility. If a solution to the problem exists, then this algorithm should terminate in a finite number of steps. However, it should be mentioned that finding k-th best solution (k=2,3,...,r) to such problems is not easy to accomplish.

To find the second best solution using Hadley's procedure [3], the following expression should be evaluated:

$$\min_i \left[\min_l (\bar{b}_l/\bar{h}_{li})(z_i - w_i)\right] \quad (19)$$

However, it is difficult to determine $\min_l(\bar{b}_l/\bar{h}_{li})$ which requires generation of all the columns of the simplex tableau. This is not practical as it would do away with the advantages of the decomposition principle. Therefore, this problem needs further consideration.

5. NUMERICAL EXAMPLES

The algorithms 1 and 2 discussed in this paper are now illustrated with two numerical examples. The first one is taken from [5].

5.1 EXAMPLE 1:

maximize $x_1 + 20x_2$ (1')

subject to $x_1 + x_2 \leq 11$ (2')

$3x_1 + 5x_2 \leq 45$

where (x_1,x_2) is an extreme point of

$-5x_1 + x_2 \leq 1$ (3')

$2x_1 + x_2 \leq 22$

$x_1,x_2 \geq 0$

5.1.1 OPERATION OF ALGORITHM 1: For the example considered tableau 1 has the following form:

Tableau 1'

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3	s_4
s_1	11	1	1	1	0	0	0
s_2	45	3	5	0	1	0	0
s_3	1	-5	①	0	0	1	0
s_4	22	2	1	0	0	0	1
$z_j - c_j$		-1	-20	0	0	0	0

$z_{lb}=0$
$z_{ub}\leq\infty$

Following Steps 1 and 2 with the pivot operation performed on the encircled element of Tableau 1', Tableau 2' results.

Tableau 2'

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3	s_4
s_1	10	6	0	1	0	-1	0
s_2	40	28	0	0	1	-5	0
x_2	1	-5	1	0	0	1	0
s_4	21	⑦	0	0	0	-1	1
$z_j - c_j$	20	-101	0	0	0	20	0

$z_{lb}=20$
$z_{ub}\leq\infty$

It follows from Tableau 2' that there is no possibility of repeating Steps 1 and 2. Hence, Step 3 is to be performed. This results in Tableau 3'.

Tableau 3'

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3	s_4
s_1	-8	0	0	1	0	-1/7	-6/7
s_2	-44	0	0	0	1	-1	-4
x_2	16	0	1	0	0	2/7	5/7
x_1	3	1	0	0	0	-1/7	1/7
$z_j - c_j$	323	0	0	0	0	39/7	101/7

$z_{lb}=20$
$z_{ub}=323$

The solution obtained is not feasible for Problem I, so Steps 4 and 5 must be followed. Using Tableau 3' the quantities q_j are: $q_{s_3} = 56$ and $q_{s_4} = 21$. Hence $s_3 = 56(39/7) = 312$ and $s_4 = 21(101/7) = 303$. Therefore, in order to find the second best solution to Problem IV, the pivot operation is to be performed on the element 1/7 appearing in the column associated with the variable s_4. The result of this pivot operation is the same as shown in Tableau 2'. However, the solution associated with this tableau is feasible for Problem I. Hence it is the optimal solution to Problem I and the algorithm terminates.

5.1.2 OPERATION OF ALGORITHM 2: Steps 1 and 2 are the same for both algorithms, therefore Step 3 starts with Tableau 2'. The optimal solution to Problem II is given by Tableau 4'.

Tableau 4'

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3	s_4
s_1	10/7	0	0	1	-3/14	1/14	0
x_1	10/7	1	0	0	1/28	-5/28	0
x_2	57/7	0	1	0	5/28	3/28	0
s_4	11	0	0	0	-1/4	1/4	1
$z_j - c_j$	1150/7	0	0	0	101/28	55/28	0

$z_{1b} = 20$
$z_{ub} = 1150/7$

Following Step 4, Problem $(IV'')^o$ is then formulated as

PROBLEM $(IV'')^o$:

$$\max\ x_1 + 20x_2$$
$$\text{subject to } -5x_1 + x_2 \leq 1$$
$$2x_1 + x_2 \leq 22$$
$$x_1 + 20x_2 \geq 20$$

Performing the pivot operations in accordance with the rules described previously, it follows that the objective function of Problem $(IV'')^o$ has the value 20 at the following extreme points $x_1 = 420/38$, $x_2 = 18/39$ and $x_1 = 0$, $x_2 = 1$. However, it should be noted that the extreme point $x_1 = 0$, $x_2 = 1$ is associated with the following three feasible solutions to Problem $(IV'')^o$. Basic variables are underlined.

	x_1	x_2	s_3	s_4	s_5
1.	$\underline{0}$	$\underline{1}$	0	$\underline{21}$	0
2.	0	$\underline{1}$	$\underline{0}$	$\underline{21}$	0
3.	0	$\underline{1}$	0	$\underline{21}$	$\underline{0}$

It is easy to show that an extrema point of Problem IV, resulting in the least increase of the value of the objective function with respect to $v^o = 20$, is given by Tableau 5'.

Tableau 5'

Basic Variables	Constants	x_1	x_2	s_3	s_4	s_5	
s_5	303	0	0	39/7	101/7	1	
x_1	3	1	0	-1/7	1/7	0	
x_2	16	0	1	2/7	5/7	0	
$z_j - c_j$	323	0	0	39/7	101/7	0	$v^1 = 323$

However, the solution corresponding to Tableau 5' does not satisfy (2'). Moreover, the value of the objective function associated with this solution is greater than the upper bound determined at Step 3. Hence, the solution given by Tableau 2' is the optimal solution of this example and the algorithm terminates.

5.2 EXAMPLE 2:

$$\text{maximize} \quad 4x_1 + 7x_2 \qquad (1'')$$
$$\text{subject to} \quad 4x_1 + 7x_2 \leq 13 \quad (2'')$$
where (x_1, x_2) is an extreme point of
$$\begin{aligned} x_1 + x_2 &\leq 3 \\ x_2 &\leq 1 \qquad (3'') \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

5.2.1 OPERATION OF ALGORITHM 1: For this example, Tableau 1 has the following form:

Tableau 1"

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3	
s_1	13	4	7	1	0	0	
s_2	3	1	1	0	1	0	
s_3	1	0	(1)	0	0	1	
$z_j - c_j$	0	-4	-7	0	0	0	$z_{1b}=0$ $z_{ub}<\infty$

Following Steps 1 and 2, the pivot operation is performed on the encircled element of Tableau 1". As a result, Tableau 2" is obtained.

Tableau 2"

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3	
s_1	6	4	0	1	0	-7	
s_2	2	1	0	0	1	-1	
x_2	1	0	1	0	0	1	
$z_j - c_j$	7	-4	0	0	0	7	$z_{1b}=7$ $z_{ub}<\infty$

It follows from Tableau 2" that there is no possibility of repeating Steps 1 and 2. Hence, Step 3 follows. This results in Tableau 3".

Tableau 3"

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3	
s_1	-2	0	0	1	-4	-3	
x_1	2	1	0	0	1	-1	
x_2	1	0	1	0	0	1	
$z_j - c_j$	15	0	0	0	4	3	$z_{1b}=7$ $z^o_{ub}=15$

The solution given by Tableau 3" is not feasible with respect to (2"). Hence, the value of the objective function corresponding to this solution is established as the upper bound, $z^o_{ub} = 15$, and Step 4 and 5 follow. Using Tableau 3", the quantities q_i are $q_{s_2} = 2$ and $q_{s_3} = 1$. Hence $s_2 = 4 \cdot 2 = 8$ and $s_3 = 1 \cdot 3 = 3$. Therefore, in order to find the second best solution to Problem IV, the pivot operation is to be performed on the element 1 of the column associated with the variable s_3. The result of this operation is given by Tableau 4".

Tableau 4"

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3
s_1	1	0	3	1	-4	0
x_1	3	1	1	0	1	0
s_3	1	0	1	0	0	1
$z_j - c_j$	12	0	-3	0	4	0

The solution of Tableau 4" is feasible with respect to (2"); hence it is the optimal solution to this example and the algorithm terminates.

5.2.2 OPERATION OF ALGORITHM 2: Steps 1 and 2 are the same for both algorithms. Hence, the first tableau to be dealt with iw Tableau 2". The optimal solution to Problem II, required by Step 3, is given by Tableau 5".

Tableau 5"

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3	
x_1	6/4	1	0	1/4	0	-7/4	
s_2	1/2	0	0	-1/4	1	3/4	
x_2	1	0	1	0	0	1	
$z_j - c_j$	13	0	0	1	0	0	$v^o = z_{1b} = 7$ $z_{ub} = 13$

Following Step 4, we formulate

PROBLEM $(IV'')^o$: maximize $4x_1 + 7x_2$

$$\begin{aligned} \text{subject to}\quad & x_1 + x_2 \leq 3 \\ & x_2 \leq 1 \\ & 4x_1 + 7x_2 \geq 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Performing the pivot operations in accordance with the rules described previously, it follows that the objective function of Problem $(IV'')^o$ has the value 7 at the following extreme points $x_1 = 7/4$, $x_2 = 0$ and $x_1 = 0$, $x_2 = 1$. However it should be noted that the extreme point $x_1=0$, $x_2=1$ is associated with the following three feasible solutions to Problem $(IV'')^o$. Basic variables are underlined.

	x_1	x_2	s_2	s_3	s_4
1^o	$\underline{0}$	$\underline{1}$	$\underline{2}$	0	0
2^o	0	$\underline{1}$	$\underline{2}$	$\underline{0}$	0
3^o	0	$\underline{1}$	$\underline{2}$	0	$\underline{0}$

It is easy to show that an extreme point of Problem IV, resulting in the least increase in the value of the objective function with respect to $v^o = 7$, is given by Tableau 4". The obtained solution is feasible with respect to (2"). Therefore the value of the objective function $v^1 = 12$ is established as a new lower bound and we formulate

PROBLEM $(IV'')^1$ maximize $4x_1 + 7x_2$

$$\begin{aligned} \text{subject to}\quad & x_1 + x_2 \leq 3 \\ & x_2 \leq 1 \\ & 4x_1 + 7x_2 \geq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The objective function of Problem $(IV'')^1$ has the value 12 at the following extreme points $x_1 = 5/4$, $x_2 = 1$, and $x_1 = 3$, $x_2 = 0$. The extreme point $x_1 = 3$, $x_2 = 0$ is associated with the following three feasible solutions to Problem $(IV'')^1$. Basic variables are underlined.

	x_1	x_2	s_2	s_3	s_4
1^o	$\underline{3}$	$\underline{0}$	0	$\underline{1}$	0
2^o	$\underline{3}$	0	$\underline{0}$	$\underline{1}$	0
3^o	$\underline{3}$	0	0	$\underline{1}$	$\underline{0}$

An extreme point of Problem 1V which results in the least increase of the value of the objective function when compared with $v^1 = 12$ is given by Tableau 6".

Tableau 6"

Basic Variables	Constants	x_1	x_2	s_1	s_3	s_4	
s_4	3	0	0	4	3	1	
x_2	1	0	1	0	1	0	
x_1	2	1	0	1	1	0	
$z_j - c_j$	15	0	0	4	3	0	$v_2=15$

However the solution given by Tableau 6" is not feasible with respect to (2"). Moreover, the value of the objective function corresponding to this solution is equal to 15. Since this is greater than the upper bound $z_{ub} = 13$ established at Step 3 (Tableau 5"), the solution corresponding to the last value of the lower bound, given by Tableau 4", constitutes the optimal solution to Example 2 and the algorithm terminates.

6. CONCLUDING REMARKS

1. The algorithm presented in [5] requires the solution obtained to be tested for linear independence. The authors of [5] have recognized that this process is computationally lengthy and difficult. Using the algorithms discussed in this paper, such a problem does not arise.

2. Unfortunately, it is difficult to suggest which algorithm should be used for any given problem. However, if some information about the optimal solution exists, then we

can determine which direction of search, i.e. to start from lower or upper bound, is to be used. This choice determines the algorithm. However, if no information about the optimal solution is available, it is possible to use some combination of algorithms 1 and 2. This means the search for the optimum solution can be carried out in some sequence using the two directions.

3. It is difficult to estimate the computational requirement in each algorithm. However, the solution, if it exists, can be found in a finite number of iterations. In general, it is likely that Algorithm 2 will require more iterations than Algorithm 1, because the test for optimality can be lengthy.

4. The remaining formulation mentioned in Section 4 seems conceptually to be the most powerful. However, the authors are unable to give a detailed algorithm based on this formulation due to the difficulty of evaluating the expression (19). The authors consider that further work in this direction would be of interest. Such a formulation is of importance due to the fact that it can be easily generalized to the case where the solution of Problem 1 must be an extreme point of not only one but any of several given extreme point sets. It means that instead of $\underline{Dx} < \underline{f}$, we have $D_k\underline{x} < \underline{f}_k$, k=1,2,...,K, and x must be an extreme point of at least one such set.

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APPENDIX

If a "starting solution" to Problem II cannot be arranged in the form as given in Tableau 1, then it does not possess the property that a feasible solution to Problem II is also feasible for Problem I. In order to obtain a solution having this property or in other words to obtain a finite lower bound, artifical variables are used. They are introduced, if necessary, into the set of constraints (3) only and Problem II is solved with the Phase I objective function. The following notation is used:

(i) NB_{II} - the set of nonbasic variables of Problem II

(ii) H_{II} - the set of nonbasic variables of Problem II with the property $z_j - c_j < 0$

(iii) L - the set of those basic variables of Problem II that are negative. It should be noted that due to a procedure used the elements of this set are associated with the set of constraints (2) only.

(iv) R - the set of basic variables of Problem IV (including artificial ones)

(v) An extreme point of the set of constraints (3) is said to be flagged if it is infeasible with respect to the set of constraints (2).

(vi) F - the set of all flagged extreme points of the set of constraints (3).

(vii) x_{a_t} - the artificial variables

(viii) x_{ij} - the elements of tableau corresponding to the set of constraints (3)

(ix) $y_{\ell k}$ - the elements of tableau corresponding to the set of constraints (2)

A general procedure for obtaining a solution to Problem II having the discussed property is as follows.

Step 1: If there are $r \in R$, $s \in H_{II}$ such that the condition (6) holds and $y_{\ell s} < 0$, $\ell \in L$, then perform the pivot operation on the element x_{rs} and proceed to Step 2. If no such x_{rs} exist, then
(i) proceed to Step 3 when performing Phase I calculations
(ii) proceed to Step 6 when performing Phase II calculations and flag the extreme point of the set of constraints (3) corresponding to an obtained solution to Problem IV.

Step 2: If a solution obtained is feasible for Problem I, then the procedure proposed in the paper is to be followed; otherwise proceed to Step 1.

Step 3: Perform the pivot operation on the element x_{rs}, $r \in R$, $s \in H_{II}$ and proceed to Step 1.
If no such x_{rs} exist but a solution obtained is feasible for Problem IV, then proceed to Step 4; otherwise proceed to Step 5.

Step 4: Replace the Phase I objective function with that of Phase II, determine the sets NB_{II}, H_{II}, L, R and proceed to Step 1.

Step 5: Perform the pivot operation on the element x_{1s}, $s \in H_{II}$ where

$$\frac{x_{1o}}{x_{1s}} = \min_t \left\{ \frac{x_{to}}{x_{ts}} \; x_{ts} > 0 \right\}$$

If a solution obtained is feasible for Problem IV, then proceed to Step 4; otherwise proceed to Step 1.

Step 6: Determine a new basic feasible solution to Problem II that does not belong to the set F and proceed to Step 1. If there is no basic feasible solution to Problem IV other than those of the set F, then the algorithm terminates and Problem I has no feasible solution.

<u>EXAMPLE:</u> maximize $x_1 + 5x_2$ (A.1)

subject to $2x_1 + 4x_2 \leqslant 9$ (A.2)

$x_1 + 1/4\ x_2 \geqslant 1$

where (x_1,x_2) is an extreme point of

$x_1 + x_2 \leqslant 4$ (A.3)

$x_1 + x_2 \geqslant 2$

$x_1\ ,\ x_2 \geqslant 0$

The artificial variable x_a is introduced into the set of constraints (A.3) and Phase I calculations are to be performed. The initial tableau is

Tableau 1[a]

Basic Variables	Constants	x_1	x_2	s_1	s_2	s_3	s_4	x_a
s_1	9	2	4	1	0	0	0	0
s_2	-1	-1	-1/4	0	1	0	0	0
s_3	4	1	1	0	0	1	0	0
x_a	2	1	(1)	0	0	0	-1	1
$z_j - c_j$	-2	-1	-1	0	0	0	1	0

This solution is infeasible for Problem I, hence we find:
$NB_{II} = (x_1,x_2,s_4)$, $H_{II} = (x_1,x_2)$, $L = (s_2)$, $R = (s_3,x_a)$, $F = \phi$.
Following Step 1, the pivot operation is performed on the encircled element. This results in Tableau 2[a].

Tableau 2[a]

Basic variables	Constants	x_1	x_2	s_1	s_2	s_3	s_4	x_a	
s_1	1	-2	0	1	0	0	4	-4	
s_2	-1/2	-3/4	0	0	1	0	-1/4	1/4	
s_3	2	0	0	0	0	1	(1)	-1	
x_2	2	1	1	0	0	0	-1	1	
$z_j - c_j$	0	0	0	0	0	0	0	1	Phase I
$z_j - c_j$	10	4	0	0	0	0	-5	-	Phase II

The solution obtained is feasible for Problem IV. It follows from this tableau that for Phase II: $NB_{II} = (x_1,s_4)$, $H_{II} = (s_4)$, $L = (s_2)$ $R = (s_3,x_2)$, $F = \phi$. Following Step 1 and subsequently Step 6, the extreme point $(x_2 = 2,\ s_3 = 2)$ is flagged. The pivot operation is performed on the encircled element. As a result Tableau 3[a] is obtained.

Tableau 3[a]

Basic variables	Constants	x_1	x_2	s_1	s_2	s_3	s_4
s_1	-7	-2	0	1	0	-4	0
s_2	0	-3/4	0	0	1	1/4	0
s_4	2	0	0	0	0	1	1
x_2	4	(1)	1	0	0	1	0
$z_j - c_j$	20	4	0	0	0	5	0

The next pivot operation is performed on the encircled element of Tableau 3[a]. This results in Tableau 4[a].

Tableau 4[a]

Basic variables	Constants	x_1	x_2	s_1	s_2	s_3	s_4
s_1	1	0	2	1	0	-2	0
s_2	3	0	3/4	0	1	1	0
s_4	2	0	0	0	0	1	1
x_1	4	1	1	0	0	1	0
$z_j - c_j$	4	0	-4	0	0	1	0

The solution obtained has the desired property.