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Generalization of Fourier-Laplace Transforms

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ABSTRACT

The Laplace transform is a widely used integral transform with many applications in physics and engineering. The Laplace transform is related to the Fourier transform, but whereas the Fourier transform expresses a function or signal as a series of modes of vibration, the Laplace transform resolves a function into its moments. Like the Fourier transform, The Laplace transform is used for solving differential and integral equations. This paper discusses an extension of Fourier –Laplace transform in the distributional generalized sense. The Twelve testing function space are defined by using Gelfand – Shilove technique. The paper describes the topological properties of S – type spaces for distributional Fourier – Laplace transform and also the results on strict inductive limit spaces.

Keywords: Fourier transform, Laplace transform, Fourier – Laplace transform Generalized function ,

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INTRODUCTION

Integral transformations have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics, and engineering science. Historically, the origin of the integral transforms including the Laplace and Fourier transforms can be traced back to celebrated work of Pieere – Simon – Laplace (1749 – 1827) on probability theory in the 1780s and to monumental treatise of Joseph Fourier (1768 – 1830)) on La Theorie Analytique de la Chaleur published in 1822 .

Fourier's treatise provided the modern mathematical theory of heat conduction, Fourier series , and Fourier integrals with many applications. Later on these transforms found wide applications in problems involving solution of linear differential equations and integral equations , linear boundary value , initial value problems and partial differential equations in the fields such as applied mathematics , mathematical physics , and engineering science. Laplace transform is also useful for evaluating certain definite integrals .[1]

Some of the recent and interesting applications are as follows which shows the versatilty of these transforms. J. Membrez et al [2] have used the Laplace transform to determine protein adsorption on porous beads. G. B. Davis. [3] used Laplace transform technique to find the analytical solution to single diffusion-convection equation over a finite domain. Li Ren et al. [4] applied it for solving convection dispersion equations. Fourier and Laplace transforms can be used in areas such as medical field for blood-velocity/time wave form over cardiac cycle from common femoral artery [5],in the analysis of functionally graded plates under thermo mechanical loading [6] and in probability theory for the integral expression for positive part moments (p>0) of random variables[7].

In the present paper, Fourier –Laplace transform is extended in the distributional sense.The plan of the paper is as follows. The definitions are given in section 2. In section 3, testing function space is defined by Gelfand – Shilov technique. Section 4 describes the Distributional Generalized Fourier Laplace transforms (FLT). In section 5 some results on countable union spaces are proved. The notations and terminologies are as per Zemannian [9].

2. Definitions

The Laplace transform with the parameter p of $f(x)$ denoted by $L[f(x)] = F(p)$ and is given by

$$L[f(x)] = \int_0^\infty e^{-px} f(x)dx = F(p) \tag{2.1}$$

The Fourier transform with parameter s of $f(t)$ denoted by $F[f(t)] = F(s)$ and is given by

$$F[f(t)] = F(s) = \int_{-\infty}^{\infty} e^{-ist} f(t)dt \tag{2.2}$$

The Conventional Fourier-Laplace transform is defined as

$$FL\{f(t,x)\} = F(s,p) = \int_{-\infty}^{\infty}\int_0^\infty f(t,x)e^{-ist} e^{-px} dtdx \tag{2.3}$$

3. Various Testing Function Spaces

3.1. The space $FL_{a,b,\alpha}$

It is given by

$$FL_{a,b,\alpha} = \left\{\phi : \phi \in E_+ / \gamma_{a,b,k,q,l}\, \phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k K_{a,b}(x) D_t^l D_x^q \phi(t,x)\right| C_{lq} A^k k^{k\alpha}\right\} \tag{3.1}$$

Where the constants A and C_{lq} depend on the testing function ϕ .

3.2. The space $FL_{a,b}^{\beta}$

This space is given by

$$FL_{a,b}^{\beta} = \left\{\phi : \phi \in E_+ / \sigma_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k K_{ab}(x) D_t^l D_x^q \phi(t,x)\right| C_{kq} B^l l^{l\beta}\right\} \tag{3.2}$$

The constants C_{kq} and B depend on ϕ .

3.3. The space $FL_{a,b,\alpha}^{\beta}$

This space is formed by combining the conditions (3.1) and (3.2)

$$FL_{a,b,\alpha}^{\beta}\left\{\phi : \phi\epsilon E_+ / \rho_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k K_{ab}(x) D_t^l D_x^q \phi(t,x)\right| \le C A^k k^{k\alpha} B^l l^{l\beta}\right\} \tag{3.3}$$

3.4. The space $FL_{a,b,\gamma}$

It is given by

$$FL_{a,b,\gamma} = \left\{\phi : \phi \in E_+ / \xi_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k K_{a,b}(x) D_t^l D_x^q \phi(t,x)\right| \le C_{lk} A^q q^{q\gamma}\right\} \tag{3.4}$$

3.5. The space $FL_{a,b,\alpha,m}$

It is defined as ,

$$FL_{a,b,\alpha,m} = \left\{\phi : \phi \in E_+ / \gamma_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k K_{a,b}(x) D_t^l D_x^q \phi(t,x)\right| \le C_{lq\delta}(m+\delta)^k k^{k\alpha}\right\} \tag{3.5}$$

For any$\delta > 0$, where m is the constant depending on the function ϕ .

3.6. The space $FL_{a,b}^{\beta,n}$

$$FL_{a,b}^{\beta,n} = \left\{\phi : \phi \in E_+ / \sigma_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k K_{a,b}(x) D_t^l D_x^q \phi(t,x)\right| \le C_{kq\epsilon}(n+\epsilon)^l l^{l\beta}\right\} \tag{3.6}$$

For any $\epsilon > 0$,where n is the constant depending on the function ϕ .

3.7. The space $FL_{a,b,\alpha,m}^{\beta,n}$

This space is defined by combining (3.5) and (3.6) as ,

$$FL_{a,b,\alpha,m}^{\beta,n} = \left\{\phi : \phi\epsilon E_+ / \rho_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k K_{a,b}(x) D_t^l D_x^q \phi(t,x)\right| \le C_{\delta\epsilon}(m+\delta)^k (n+\epsilon)^l k^{k\alpha} l^{l\beta}\right\} \tag{3.7}$$

for any $\delta > 0$, $\epsilon > 0$ and for given $m > o$ and $n > 0$.

3.8. The space $FL_{a,b,\gamma,p}$

This space is given by

$$FL_{a,b,\gamma,p} = \left\{\phi : \phi\epsilon E_+ / \xi_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k K_{a,b}(x) D_t^l D_x^q \phi(t,x)\right| \le C_{lkr}(p+r)^q q^{q\gamma}\right\} \tag{3.8}$$

For any $r > 0$,where p is the constant depending on the function ϕ .

3.9. The space $FL^{v}_{a,b,\alpha}$

It is given by

$$FL^{v}_{a,b,\alpha} = \left\{\phi : \phi \epsilon E_{-} / \gamma_{a,b,k,q,l}\phi(t,x) = \underset{\substack{-\infty<t<0 \\ 0<x<\infty}}{Sup}\left|(-t)^{k} K_{a,b}(x) D_{t}^{l} D_{x}^{q}\phi(t,x)\right| \leq C_{lq} A^{k} k^{k\alpha}\right\} \quad (3.9)$$

The smooth function $\phi(t,x)$ defined on I_2 is in $FL^{v}_{a,b,\alpha}$ if $\phi^{v}(t,x) = \phi(-t,x)$ is in $FL_{a,b,\alpha}$

3.10 The space $F^{v}L^{\beta}_{a,b}$

We define this space as ,

$$F^{v}L^{\beta}_{a,b} = \left\{\phi : \phi \epsilon E_{-} / \sigma_{a,b,k,q,l}\phi(t,x) = \underset{\substack{-\infty<t<0 \\ 0<x<\infty}}{Sup}\left|(-t)^{k} K_{a,b}(x) D_{t}^{l} D_{x}^{q}\phi(t,x)\right| \leq C_{kq} B^{l} l^{l\beta}\right\} \quad (3.10)$$

Here $\phi^{v}(t,x) = \phi(-t,x)$ is in $F^{v}L^{\beta}_{a,b}$.

3.11. The space $F^{v}L^{\beta}_{a,b,\alpha}$

Combining the conditions of (3.9) and (3.10) we get

$$F^{v}L^{\beta}_{a,b,\alpha} = \left\{\phi : \phi \epsilon E_{-} / \rho_{a,b,k,q,l}\phi(t,x) = \underset{\substack{-\infty<t<0 \\ 0<x<\infty}}{Sup}\left|(-t)^{k} K_{a,b}(x) D_{t}^{l} D_{x}^{q}\phi(t,x)\right| C A^{k} k^{k\alpha} B^{l} l^{l\beta}\right\} \quad (3.11)$$

Where the constants A, B, C depend on the testing function ϕ .

3.12. The space $F^{v}L_{a,b,\gamma}$

It is given by

$$F^{v}L_{a,b,\gamma} = \left\{\phi : \phi \epsilon E_{-} / \mu_{a,b,k,q,l}\phi(t,x) = \underset{\substack{0<t<\infty \\ -\infty<x<0}}{Sup}\left|t^{k} K_{a,b}(x) D_{t}^{l} D_{x}^{q}\phi(t,x)\right| \leq C_{lk} A^{q} q^{q\gamma}\right\} \quad (3.12)$$

4. Distributional Generalized Fourier –Laplace transforms (FLT)

For $f(t,x) \in FL^{*\beta}_{a,\alpha}$, where $FL^{*\beta}_{a,\alpha}$ is the dual space of $FL^{\beta}_{a,b}$ It contains all distributions of compact support. The distributional Fourier-Laplace transform is a function of $f(t,x)$ is defined as

$$FL\,\{f(t,x)\} = F(s,p) = \langle f(t,x),\ e^{-i(st-ipx)}\rangle \quad (4.1)$$

Where, for each fixed t $(0 < t < \infty)$, x $(0 < x < \infty)$, $s > 0$ and $p > 0$ the right hand side of (4.1) has a sense as an application of $f(t,x) \epsilon FL^{*\beta}_{a,\alpha}$ to $e^{-i(st-ipx)} \in FL^{\beta}_{a,\alpha}$.

RESULTS ON COUNTABLE UNION SPACE

5.1. Theorem: For a real numbers a_1, a_2, b_1 and b_2 such that $a_1 \leq a_2$ and $b_2 \leq b_1$ then $FL_{a_2,b_2,\alpha} \subset FL_{a_1,b_1,\alpha}$ and the induced topology on $FL_{a_2,b_2,\alpha}$ is weaker than the original topology that is $T_{a_1,b_1,\alpha} / FL_{a_2,b_2,\alpha} \subset T_{a_2,b_2,\alpha}$

Proof: Consider ,

$$\gamma_{a,b,k,q,l}\,\phi(t,x) = \underset{I}{Sup}\left|t^{k} K_{a_1,b_1}(x) D_{t}^{l} D_{x}^{q}\phi(t,x)\right|$$

$$\leq \underset{I}{Sup}\left|t^{k} K_{a_2,b_2}(x) D_{t}^{l} D_{x}^{q}\phi(t,x)\right| = \gamma_{a_2,b_2,k,q,l}\,\phi(t,x)$$

Hence $FL_{a_2,b_2,\alpha} \subset FL_{a_1,b_1,\alpha}$ if $a_1 \leq a_2$ and $b_2 \leq b_1$

Second part of the proof is simple and hence omitted. This completes the proof.

5.2 Theorem: If $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ $then$ $FL^{\beta_1}_{a,b,\alpha_1} \subset FL^{\beta_2}_{a,b,\alpha_2}$ and the topology of $FL^{\beta_1}_{a,b,\alpha_1}$ is equivalent to the topology induced on $FL^{\beta_1}_{a,b,\alpha_1}$ by $FL^{\beta_2}_{a,b,\alpha_2}$

Proof: Let $\phi \in FL^{\beta_1}_{a,b,\alpha_1}$ there fore

$\rho_{a,b,k,q,l}(\phi) = \underset{I}{Sup}\left|t^{k} K_{a,b}(x) D_{t}^{l} D_{x}^{q}\phi(t,x)\right| \leq C A^{k} k^{k\alpha_1} B^{l} l^{l\beta_1} \leq C A^{k} k^{k\alpha_2} B^{l} l^{l\beta_2}$,

Where $k, q, l = 0,1,2,3, \ldots \ldots \ldots$

Hence $\phi \in FL^{\beta_2}_{a,b,\alpha_2}$.

Consequently $FL^{\beta_1}_{a,b,\alpha_1} \subset FL^{\beta_2}_{a,b,\alpha_2}$ the topology of $FL^{\beta_1}_{a,b,\alpha_1}$ is equivalent to the topology $T^{\beta_2}_{a,b,\alpha_2} / FL^{\beta_2}_{a,b,\alpha_2}$ It is clear from the definition of the topologies of these spaces.

5.3 Theorem : $FL_{a,b} = \underset{\alpha_i,\beta_i=1}{\overset{\infty}{\cup}} FL_{a,b,\alpha_i}^{\beta_i}$ and if the space $FL_{a,b}$ is equipped with the strict $FL_{a,b}$ inductive limit topology defined by the injection maps from $FL_{a,b,\alpha_i}^{\beta_i}$ to $FL_{a,b}$ then the sequence $\{\phi_n\}$ in $FL_{a,b}$ converges to zero iff $\{\phi_n\}$ is contained in some $FL_{a,b,\alpha_m}^{\beta_m}$ and converges to zero .

Proof: Once we show that $FL_{a,b} = \underset{\alpha_i,\beta_i=1}{\overset{\infty}{\cup}} FL_{a,b,\alpha_i}^{\beta_i}$

clearly $\underset{\alpha_i,\beta_i=1}{\overset{\infty}{\cup}} FL_{a,b,\alpha_i,\beta_i} \subset FL_{a,b}$

For proving other inclusion, let $\phi \in FL_{a,b}$ then

$\rho_{a,b,k,q,l}(\phi) = \underset{I}{Sup} \left| t^k K_{a,b}(x) D_t^l D_x^q \phi(t,x) \right|$ is bounded by some number . We can choose the integers α_m and β_m such that $\rho_{a,b,k,q,l}(\phi) \le C A^k k^{k\alpha_m} B^l l^{l\beta_m}$

There fore $\in FL_{a,b,\alpha_m}^{\beta_m}$, for some integers α_m and β_m .

Hence $FL_{a,b} \subset \underset{\alpha_i,\beta_i=1}{\overset{\infty}{\cup}} FL_{a,b,\alpha_i}^{\beta_i}$ Thus $FL_{a,b} = \underset{\alpha_i,\beta_i=1}{\overset{\infty}{\cup}} FL_{a,b,\alpha_i}^{\beta_i}$.

5.4. Definition:

Let $\{a_n\}$ and $\{b_n\}$ be monotonic sequence, converging to $w +$ and Z $-$ respectively .

Now we define countable union space. $FL\,(w,z,\alpha) = \underset{n=1}{\overset{\infty}{\cup}} FL_{a_n,b_n,\alpha}$ (5.4.)

Concerning this space we prove next theorem.

5.5. Theorem: The space $FL(w,z,\alpha)$ is independent of the choice of the sequence $\{a_n\}$ and $\{b_n\}$. If $FL(w,z,\alpha)$ is equipped with the strict inductive limit topology defined by $FL_{a_n,b_n,\alpha}$ then the sequence $\{\phi_n\}$ in $FL(w,z,\alpha)$ converges to zero iff $\{\phi_n\}$ belongs to some $FL_{a_n,b_n,\alpha}$ and converges to zero in that space. Moreover $FL(w,z,\alpha)$ is complete.

Proof: Proof is easy and hence omitted.

CONCLUSION

In this paper Fourier-Laplace transform is generalized in the distributional sense . Twelve testing function spaces using Gelfand Shilov technique are developed. Topological properties are proved by using the testing function spaces.

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