Bounds for Ramsey numbers of complete graphs dropping an edge

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Abstract

Let $K_n - e$ be a graph obtained from a complete graph of order $n$ by dropping an edge, and let $G_p$ be a Paley graph of order $p$. It is shown that if $G_p$ contains no $K_n - e$, then $r(K_{n+1} - e) \geq 2p + 1$. For example, $G_{1493}$ contains no $K_{13} - e$, so $r(K_{14} - e) \geq 2987$, improving the old bound 2557. It is also shown that $r(K_{n+1} - e) \leq 4r(K_n - e) - 2$, implying that $r(K_{n-2}, K_n - e) \leq 4r(K_n - e) - 2$.

1. Introduction

Let $G$ and $H$ be graphs. The Ramsey number $r(G, H)$ is defined as the smallest positive integer $N$ such that for any graph $F$ of order $N$, either $F$ contains $G$ as a subgraph or $\overline{F}$ contains $H$ as a subgraph, where $\overline{F}$ is the complementary graph of $F$. We shall write $r(G, G)$ as $r(G)$.

Let $K_n - e$ be a graph obtained from $K_n$ by dropping an edge. Some exact values of $r(K_n - e, K_n - e)$ have been obtained such as $r(K_4 - e) = 10$ by Chvátal and Harary [1], $r(K_5 - e) = 22$ by Clapham et al. [2]. Also it is known that $45 \leq r(K_6 - e) \leq 70$, where the lower bound and upper bound are due to Exoo [3] and Huang and Zhang [5], respectively, and $r(K_7 - e) \leq 251$ was obtained by Shi and Zhang [10]. For $n \geq 8$, we do not have any lower bound of $r(K_n - e)$ better than that given by trivial relation

$$r(K_n - e) \geq r(K_{n-1}).$$

However, all known lower bounds for $r(K_n)$ with $n \geq 6$ are given by Paley graphs or their conjunctions in [9,7]. We shall use Paley graphs to give a lower bound for $r(K_n - e)$. We show...
that if $G_p$ contains no $K_t - e$, then $r(K_t - e) \geq 2p + 1$. For example, we already knew that $r(K_{14} - e) \geq r(K_{13}) \geq 2557$, and we shall improve it to 2987.

We also show an upper bound

$$r(K_2 + G) \leq 4r(K_2 + G) - 2.$$  

This gives $r(K_n - e) \leq 4r(K_{n-2}, K_n - e) - 2$.

2. Paley graphs

Let $q$ be a prime power and let $F(q)$ be a finite field of $q$ elements. An element $a \in F(q)$ is called a quadratic residue (mod $q$) if there exists $b \in F(q)$ such that $a = b^2 \pmod{q}$. We shall define a function on $F(q) \to F(q)$ as $\chi(x) = x^{(q-1)/2}$. Then from a well known theorem of Euler, for odd prime power $q$, the function $\chi(x)$ has the following properties:

$$\chi(x) = \begin{cases} 1 & x \neq 0 \text{ is a quadratic residue mod } q, \\ 0 & x = 0, \\ -1 & x \text{ is a quadratic non-residue mod } q. \end{cases}$$

The Paley graph $G_q$ is defined as follows. Let $q = 1 \pmod{4}$ be a prime. The vertex set of $G_q$ is $F(q)$, and its edge set is formed by connecting distinct vertices $x$ and $y$ with an edge if and only if $\chi(x - y) = 1$, namely, $x - y$ is a quadratic residue mod $q$. Note that $\chi(x - y) = 1$ if and only if $\chi(x - y) = 1 \pmod{4}$.

A graph $G$ of order $n$ is said to be a strongly regular graph with parameters $n, k, \lambda, \mu$, denoted by srg$(n, k, \lambda, \mu)$ for short, if it is $k$-regular, and any pair of vertices have $\lambda$ common neighbors if they are adjacent, or $\mu$ common neighbors if they are not adjacent. The following property of a Paley graph can be easily proved.

**Lemma 1.** Let $q = 4n + 1$ be a prime power. Then the Paley graph $G_q$ is a srg$(4n + 1, 2n, n - 1, n)$. Furthermore, it is self-complementary.

3. A lower bound for $r(K_n - e)$

If $G_q$ contains no $K_n - e$, then $r(K_n - e) \geq q + 1$ since $G_q$ is self-complementary. However, the value of $r(K_{p+1} - e)$ may be double that of $r(K_n - e)$. Generally, $G_p$ tends to give a smaller clique number than that of $G_q$. So we shall concentrate on Paley graphs of prime orders.

**Theorem 1.** Let $p = 1 \pmod{4}$ be a prime. If $G_p$ contains no $K_n - e$, then $r(K_{n+1} - e) \geq 2p + 1$.

Before giving the proof, we construct a graph $H_p$ from two copies of $G_p$. Similar constructions have been used by Shearer [9], Mathon [7], and McDiarmid and Steger [8].

Set the vertex sets of two copies of $G_p$ as

$$V = \{0, 1, \ldots, p - 1\} \quad \text{and} \quad V' = \{0', 1', \ldots, (p - 1)\},$$

respectively. Let $H_p$ be the graph on vertex set $V \cup V'$, which preserves the edges of both $G_p$. Then add the edges connecting any two vertices $x$ and $y'$, if and only if $x$ and $y$ are not adjacent in $G_p$. Writing this in detail, $H_p$ contains the edges

$$(x, x'), \quad x \in V$$

$$(x, y'), (x', y) \quad \text{if } (x, y) \notin E(G_p).$$
Note that all “crossing” edges of form \((x, x')\) are in the edge set of \(H_p\), which is illustrated in Fig. 1.

**Lemma 2.** The complement \(\overline{H_p}\) of the graph \(H_p\) is a subgraph of \(H_p\).

**Proof.** Let \(z \in F(p)\) with \(\chi(z) = -1\). Set a map \(\phi: V \cup V' \rightarrow V \cup V'\) as
\[
\phi(x) = zx \quad \text{for} \quad x \in V, \quad \text{and} \quad \phi(x') = (zx)' \quad \text{for} \quad x' \in V'.
\]

Clearly, \(\phi\) is a bijection. We shall show that if \((a, b)\) is not an edge of \(H_p\) with \(a \neq b\), then \((\phi(a), \phi(b))\) is an edge of \(H_p\).

**Case 1.** If \((x, y)\) is not an edge in \(V\), namely \((x', y')\) is not an edge in \(V'\) with \(x \neq y\), then \(\chi(x - y) = -1\), so
\[
\chi(\phi(x) - \phi(y)) = \chi(zx - zy) = \chi(z) \chi(x - y) = 1.
\]

Thus \((\phi(x), \phi(y))\) is an edge of \(H_p\).

**Case 2.** If \((x, y')\) is not an edge of \(H_p\) with \(x \in V\) and \(y' \in V'\), then \(x \neq y\) and \((x, y)\) is an edge of \(G_p\), hence \((zx, zy)\) is not an edge of \(G_p\), implying that \((zx, (zy)')\) is an edge of \(H_p\). Namely, \((\phi(x), \phi(y'))\) is an edge of \(H_p\). \(\square\)

The above proof in fact gives that the graph obtained from \(H_p\) by dropping crossing edges of form \((x, x')\) is isomorphic to \(\overline{H_p}\).

**Lemma 3.** If \(H_p\) contains \(K_s - e\), then \(G_p\) contains \(K_{s-1} - e\).

**Proof.** Let \(S\) be a vertex set with \(|S| = s\) such that \(K_s - e\) is a subgraph of \(H_p\) on \(S\) given as
\[
S = \{\bar{u}, \bar{v}, x_1, \ldots, x_m, y'_1, \ldots, y'_n\},
\]
where \(\bar{u}\) and \(\bar{v}\) are nonadjacent distinct vertices in \(K_s - e\), and \(x_i, y_j \in V\). To avoid the trivial case, we assume \(s \geq 4\). We shall write \(\bar{u}\) as \(u\) or \(u'\), and write \(\bar{v}\) as \(v\) or \(v'\) depending on whether \(\bar{u}\) and \(\bar{v}\) are in \(V\) or in \(V'\), respectively.

Note that \(u \neq v\). This is clear if both of \(\bar{u}\) and \(\bar{v}\) are in \(V\) or both in \(V'\). It is also clear if \(\bar{u} = u\) is in \(V\) and \(\bar{v} = v'\) is in \(V'\) as they do not have neighbors in common in \(H_p\). Since \(x_i\) and \(y'_j\) have neighbors in common, we known that \(x_i \neq y_j\) for any pair of \(i\) and \(j\). Furthermore, any pair of vertices in
\[
\{u, v, x_1, \ldots, x_m, y_1, \ldots, y_n\}
\]
are distinct. For any vertex \(a\), as \(\{\bar{u} - a, \bar{v} - a, x_1 - a, \ldots, x_m - a, (y_1 - a)'', \ldots, (y_n - a)''\}\) has adjacency the same as that of \(S\), so we may assume that \(x_1 = 0\) or \(y'_i = 0\), say \(x_1 = 0\), so
\[
S = \{\bar{u}, \bar{v}, 0, x_2, \ldots, x_m, y'_1, \ldots, y'_n\}.
\]
Fig. 2. The first process to get an induced $K_{s-1}-e$ in $G_p$.

Fig. 3. The second process for getting an induced $K_{s-1}-e$ in $G_p$.

Define $X = \{x_2, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$, and $X^- = \{x_2^-, \ldots, x_m^-\}$ and $Y^- = \{y_1^-, \ldots, y_n^-\}$.

Case 1. Both $\bar{u} = u$ and $\bar{v} = v$ are vertices of $V$. Then the subgraph induced by

$$\{u, v\} \cup X \cup Y = \{u, v, x_2, \ldots, x_m, y_1, \ldots, y_n\}$$

in $G_p$ is illustrated as follows in the left side, and the right side is that induced by $\{u^-, v^-\} \cup X^- \cup Y^-$. We shall give some explanation for the process illustrated in Fig. 2. The vertices $u$ and $v$ are represented by blacked dots to signify that they are in neighborhood $N(0)$, and so is the vertex set $X$ represented by a double cycles with the same center. A cycle representing the vertex set $Y$ is to signify that the set is out of the neighborhood $N(0)$.

Two parts connected by a real line are connected completely, and ones connected by a dotted line are not connected completely. For example, $u$ is adjacent to each vertex in $\{y_1', \ldots, y_n'\}$ in $H_p$, so it is non-adjacent to any vertex in $Y = \{y_1, \ldots, y_n\}$, and similarly $X$ and $Y$ are non-adjacent completely.

The connection of $u$ and $v$ is a dashed line, to signify that it is not under consideration.

For any non-zero elements $a$ and $b$ in $F(p)$ with $a \neq b$, it is easy to see that

$$\chi(a^{-1} - b^{-1}) = \chi(b - a)\chi(a^{-1})\chi(b^{-1}) = \chi(a - b)\chi(a)\chi(b).$$

Here three symbols $\chi(a - b)$, $\chi(a)$ and $\chi(b)$ determine the adjacency of $a^{-1}$ and $b^{-1}$. They are adjacent if and only if the number of negative symbols is even.

1. A blacked dot or a vertex in the double cycle means positive, and a hollow dot that will appear in Figs. 3 and 4 or a vertex in a cycle means negative.

2. A real line means positive and a dotted line means negative.

We list the negative signs of the involved symbols in Table 1 to show that $\{u^-, v^-\} \cup X^- \cup Y^-$ induces a subgraph of $K_{s-1}-e$ in $G_p$, where we call $\chi(a)$, $\chi(b)$, $\chi(a-b)$ involved symbols if the involved vertices are $a^{-1}$ and $b^{-1}$.

Case 2. $\bar{u} = u$ is a vertex of $V$ and $\bar{v} = v'$ is a vertex of $V'$, so

$$S = \{u, v', 0, x_2, \ldots, x_m, y_1', \ldots, y_n'\}.$$
Table 1
A subgraph of $K_{s-1} - e$ in $G_p$

<table>
<thead>
<tr>
<th>Involved vertices</th>
<th>Involved negative symbols</th>
<th>$\Pi$ (symbols)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i^{-1}$, $x_j^{-1}$</td>
<td>$\chi(y_i) \cdot \chi(y_j)$</td>
<td>$+$</td>
</tr>
<tr>
<td>$y_i^{-1}$, $y_j^{-1}$</td>
<td>$\chi(x_i - y_j)$</td>
<td>$+$</td>
</tr>
<tr>
<td>$u^{-1}$, $x_i^{-1}$</td>
<td>None</td>
<td>$+$</td>
</tr>
<tr>
<td>$u^{-1}$, $y_i^{-1}$</td>
<td>$\chi(u) \cdot (u - y_i)$</td>
<td>$+$</td>
</tr>
<tr>
<td>$v^{-1}$, $x_i^{-1}$</td>
<td>None</td>
<td>$+$</td>
</tr>
<tr>
<td>$v^{-1}$, $y_i^{-1}$</td>
<td>$\chi(v) \cdot (u - y_i)$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Fig. 4. The third process for getting an induced $K_{s-1} - e$ in $G_p$.

Note that $v$ is a vertex out of $N(0)$ as $v'$ is adjacent to the vertex 0, so we draw it as a hollow dot. The process for getting the subgraph on the right hand side is similar to that for Fig. 1; we thus omit the proof.

**Case 3.** Both $\bar{u} = u'$ and $\bar{v} = v'$ are vertices of $V'$, so

$$S = \{u', v', 0, x_2, \ldots, x_m, y'_1, \ldots, y'_n\}.$$ 

Then the subgraphs induced by $\{u, v\} \cup X \cup Y$ and by $\{u^{-1}, v^{-1}\} \cup X^- \cup Y^-$ in $G_p$ are illustrated in Fig. 4.

In conclusion, we have a subgraph $K_{s-1} - e$ in $G_p$, finishing the proof. $\square$

**Proof of Theorem 1.** The theorem follows from Lemmas 2 and 3 immediately. $\square$

**Lemma 4.** The Paley graph $G_p$ is edge transitive.

**Proof.** Let $(u, v)$ and $(x, y)$ be edges of $G_p$. It is easy to see that the map $\psi : V(G_p) \to V(G_p)$ with

$$\psi(z) = \frac{u - v}{x - y} (z - x) + u$$

is an automorphism of $G_p$ with $\psi(x) = u$ and $\psi(y) = v$. $\square$

Hence, if we can find

$$t = \max \{\omega(G_p), 2 + \text{ maximum order of clique in } N(u) \cap N(v)\},$$

where $u$ and $v$ are a pair of non-adjacent vertices of $G_p$ with $u \neq v$, then $G_p$ contains no $K_{t+1} - e$.

**Corollary 1.** If for some pair non-adjacent vertices $u$ and $v$ of $G_p$ with $u \neq v$,

$$\omega(G_p) = 2 + \text{ maximum order of clique in } N(u) \cap N(v),$$
then \( r(K_{r+2} - e) \geq 2p + 1, \) where \( t = \omega(G_p). \)

For \( p = 1493, \) Shearer \[9\] and Mathon \[7\] found that \( \omega(G_p) = 12. \) The vertices 0 and 2 are non-adjacent in \( G_p. \) We found that the maximum order of the clique in \( N(0) \cap N(2) \) is 10 and a maximum clique is as follows:

\[
\{282, 291, 297, 298, 313, 367, 479, 488, 757, 1074\}.
\]

The following result improves the old bound 2557 for \( r(K_{14} - e) \) to 2987.

**Corollary 2.** \( r(K_{14} - e) \geq 2987. \)

4. An upper bound for \( r(K_n - e) \)

Let \( G'' \) be a graph obtained from \( G \) by deleting two vertices of \( G. \) It is shown \[6\] that

\[ r(G) \leq 4 r(G, G'') + 2. \]

So we have \( r(K_n - e) \leq 4 r(K_{n-2} - e, K_n - e) + 2. \) However, if we delete two vertices with minimum degree, we shall have a smaller bound. The following is a generalization of a result of Exoo, Harborth, and Mengersen \[4\] who showed that \( r(K_{2,n}) \leq 4n - 2. \)

**Theorem 2.** Let \( G \) be a graph; then

\[ r(K_2 + G) \leq 4 r(G, K_2 + G) - 2. \]

In particular, \( r(K_n - e) \leq 4 r(K_{n-2}, K_n - e) - 2 \) for \( n \geq 2. \)

**Proof.** Let \( A = r(G, K_2 + G). \) Suppose, to the contrary, this is not valid. Then for \( N = 4A - 2, \) there is a red–blue edge coloring of \( K_N \) on vertex \( V \) that contains no monochromatic \( K_2 + G. \)

We now estimate the number of red edges between \( N_R(v) \) and \( N_B(v) \) for \( v \in V, \) where \( N_R(v) \) and \( N_B(v) \) are the red neighborhood and blue neighborhood of \( v, \) respectively. Set

\[ f(v) = \sum_{x \in N_R(v)} |N_R(x) \cap N_B(v)|. \]

Since for \( x \in N_R(v) \)

\[ d_B(v) = |N_R(x) \cap N_B(v)| + |N_B(x) \cap N_B(v)|, \]

and \( |N_B(x) \cap N_B(v)| \leq A - 1, \) so \( |N_R(x) \cap N_B(v)| \geq d_B(v) - A + 1. \) Hence

\[ f(v) \geq d_R(v)(d_B(v) - A + 1). \]

On the other hand,

\[ f(v) = \sum_{x \in N_B(v)} |N_R(x) \cap N_R(v)| \leq d_B(v)(A - 1). \]

Therefore, \( d_R(v)(d_B(v) - A + 1) \leq d_B(v)(A - 1) \) as \( d_R(v) + d_B(v) = 4A - 3, \) which yields

\[ d_R(v)d_B(v) \leq (A - 1)(4A - 3). \] (1)

For any vertex set \( U \subseteq V, \) consider the sum \( \sum_{x,y \in U, x \neq y} |N_B(x) \cap N_B(y)|, \) in which each vertex \( z \in V \) is counted \( \left( \frac{|N_B(z) \cap U|}{2} \right) \) times; thus we have

\[ \sum_{z \in V} \left( \frac{|N_B(z) \cap U|}{2} \right) \leq \left( \frac{|U|}{2} \right)(A - 1). \] (2)
Since either \( d_R(v) \geq 2A - 1 \) or \( d_B(v) \geq 2A - 1 \), we suppose that \( d_R(v) \geq 2A - 1 \) without loss of generality.

**Case 1.** \( d_R(v) = 2A - 1 \). Then \( d_B(v) = 2A - 2 \) so that (1) is violated.

**Case 2.** \( d_R(v) \geq 2A \). Take a subset \( U \subseteq N_R(v) \) with \( |U| = 2A \). Then for any \( z \in U, |N_B(z) \cap U| \geq A \), since otherwise \( |N_R(z) \cap U| \geq A \); hence \( z \) and \( v \), together with their common red neighbors, would form a red \( K_2 + G \). Similarly, for any \( z \in V \setminus U, |N_B(z) \cap U| \geq A + 1 \). So the left hand side of (2) is at least \( A(A - 1)(2A + 1) \). But the right hand side is just \( A(A - 1)(2A - 1) \), which is a contradiction, proving the upper bound. □

**Corollary 3.** For \( n \geq 2 \), \( r(K_{2,n}) \leq 4n - 2 \). If \( 4n - 3 \) is a prime power, then \( r(K_{2,n}) = 4n - 2 \).

**Proof.** Since \( r(K_{n}, K_{2}, n) = n \), the desired upper bound comes from the above theorem.

If \( q = 4n - 3 = 4(n - 1) + 1 \) is a prime power, then the Paley graph of order \( q \) is a strongly regular graph with parameters \( (4n - 3, 2n - 2, n - 2, n - 1) \). This graph and its complement do not contain \( K_{2,n} \) as it is self-complementary. Thus \( r(K_{2,n}) \geq 4n - 2 \), completing the proof. □

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**References**


