OBSERVER-BASED FAULT DETECTION AND ISOLATION FOR STRUCTURED SYSTEMS

C. Commault, J-M. Dion, O. Sename and R. Motyeian

Abstract

Fault Detection and Isolation (FDI) problems are here considered for linear systems with faults and disturbances. We design a set of observer-based residuals, in such a way that the transfer from the disturbances to the residuals is zero and the transfer from the faults to the residuals allows fault isolation. We are interested in obtaining a transfer function from faults to residuals with either a diagonal structure (i.e. a dedicated structured residuals set) or a triangular one. We deal with this problem when the system under consideration is structured, that is, the entries of the system matrices are either fixed zeros or free parameters. To a structured system one can associate in a natural way a directed graph. We can then provide necessary and sufficient conditions under which the FDI problems have a solution for almost any value of the free parameters. These conditions are simply expressed in terms of input-output paths in the associated graph.

I. Introduction

This paper is concerned with the Fault Detection and Isolation (FDI) problem for linear systems with faults and disturbances. This problem has received considerable attention in the past ten years [1], [2], [3], [4]. We consider a model-based approach for FDI where two steps are distinguishable: the generation of the residuals which are sensitive to the faults, and the isolation of the faults.

Many works have been done on observer-based approaches, using either robust design [1], [2], [3], [5] or a structural approach [5], [1], [6], [7].

In this paper we consider the observer based FDI problem using structured residual sets that allow fault isolation. We are interested in obtaining a transfer from faults to residuals with either a diagonal structure (i.e. a dedicated structured residual set) or a triangular one. We will give for these problems intrinsic solvability conditions depending on the internal structure of the system and not on the specific values of the parameters. We look for internal structures which are well suited for diagnosis. Our approach is mainly at an analysis level. If the solvability conditions are fulfilled, it is then necessary to use existing synthesis methods to reach a practical solution [1]. Notice that such methods will actually require the knowledge of the value of parameters. If the solvability conditions are not fulfilled, the graph analysis will give precious hints on how to modify the sensor localizations to reach these conditions.

An interesting tool for this purpose is the notion of structured system. The FDI problem is then studied here for linear structured systems which represent a large class of parameter dependent linear systems in state space form [8]. Such linear systems are represented by matrices whose entries are either fixed...
zeros or free parameters. For these systems, generic properties are studied, that is properties which are true for almost all values of the parameters. The analysis of such systems gives nice graph conditions to generically solve classical control problems [9], [10]. Moreover it allows to use some efficient algorithms to check the solvability of such problems [11]. Using a structural approach introduced by Lin [8], we will here derive some graph solvability conditions for the FDI problem. This work follows [12], [13], [14] where preliminary results were given. We give first the necessary and sufficient conditions to get a triangular structure, with internal stability. These conditions are given in terms of number of disjoint paths in the system associated matrix graph. Then the bank of observers based FDI problem with disturbances such that the transfer matrix between faults and residuals is required to be diagonal is considered. We prove that the necessary and sufficient condition for the diagonal case with a bank of observers is the same as the condition for the triangular one with a unique observer. We show on a significant example that the diagonal FDI problem can be solved using a bank of observers while no solution exists with a unique observer.

In a recent paper [7], Hastrudi Zad and Massoumnia studied generic conditions under which diagnosis is possible. Our paper generalizes this work in several senses. First we consider here that the system may be affected by disturbances. Much more important, we consider here the internal structure of the system while in [7] all the system parameters are free. Although we use different detection filters, our approach leads to the same FDI solvability conditions as in [7] when we assume no fixed zero in the state space representation and no disturbance.

The outline of this paper is as follows. The problem is formulated in section 2. The linear structured systems are presented in section 3. Solvability conditions for the unique observer-based diagonal and triangular FDI problems with disturbances are given in section 4. Solvability conditions for the bank of observer-based diagonal FDI problem with disturbances, are given in section 5. An illustrative example is presented in section 6. Some concluding remarks end the paper.

II. PROBLEM FORMULATION

Let us consider the following linear time-invariant system:

\[ \Sigma \left\{ \begin{array}{l}
\dot{x}(t) = Ax(t) + E_1 d(t) + F_1 f(t) \\
y(t) = C x(t) + E_2 d(t) + F_2 f(t)
\end{array} \right. \] (1)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( d(t) \in \mathbb{R}^q \) the ungovernable input (or disturbance), \( f(t) \in \mathbb{R}^r \) the fault vector and \( y(t) \in \mathbb{R}^p \) the measured output vector, and \( A, C, E_1, E_2, F_1 \) and \( F_2 \) are matrices of appropriate dimensions.

Note that that the control input effects are not considered here as, for any observer-based FDI problem, it is well known that these can be taken into account in the observer structure.

In this paper, we are interested in providing solvability results for two diagnosis schemes: the single observer-based scheme and the one using a bank of observers.
A. The single observer scheme

Consider first a full state observer given as follows:

\[ \dot{\hat{x}}(t) = A\hat{x}(t) + K(y(t) - C\hat{x}(t)) \]  

(2)

where \( \hat{x}(t) \in \mathbb{R}^n \) is the estimated state, \( K \) is a matrix to be designed such that \( \hat{x}(t) \) asymptotically converges to \( x(t) \), when no fault and no disturbance are considered.

Using (1) and (2) the estimation error \( e(t) = x(t) - \hat{x}(t) \) satisfies the equation:

\[ \dot{e}(t) = (A - KC)e(t) + (E_1 - KE_2)d(t) + (F_1 - KF_2)f(t) \]

The residual is designed using the output estimation error derived from the observer:

\[ r(t) = Q(y(t) - C\hat{x}(t)) = QCe(t) + QE_2d(t) + QF_2f(t) \]

(3)

where \( Q \) is a \( r \times p \) matrix.

Therefore the transfer matrices from the disturbance to the residual and from the fault to the residual are given by:

\[ r(s) = \begin{bmatrix} T_{rd}(s) & T_{rf}(s) \end{bmatrix} \begin{bmatrix} d(s) \\ f(s) \end{bmatrix} \]

with \( T_{rd}(s) = QC(sI - A + KC)^{-1}(E_1 - KE_2) + QE_2 \) and \( T_{rf}(s) = QC(sI - A + KC)^{-1}(F_1 - KF_2) + QF_2 \).

Using the residual generation, let us consider the following FDI problem.

Definition 1: The single observer based triangular fault detection and isolation problem (with disturbances) consists in finding, whenever possible, matrices \( K \) and \( Q \) such that \( A - KC \) is stable, \( T_{rd}(s) = 0 \), and \( T_{rf}(s) \) is such that:

\[ T_{rf}(s) = \begin{bmatrix} t_{11}(s) & t_{12}(s) & \cdots & t_{1r}(s) \\ 0 & t_{22}(s) & \cdots & t_{2r}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{rr}(s) \end{bmatrix} \]

(4)

with \( t_{ii}(s) \neq 0 \) and \( t_{ij}(s) \) proper for \( i, j = 1, 2, \ldots, r \).  

♦  

Remark: The triangular structure is more general than the ”one residual - one fault” scheme where \( t_{ij} \equiv 0 \), for \( i \neq j \). However one should note that the diagonal structure allows to cope with simultaneous faults while the triangular one cannot.

B. The bank of observers scheme

The other FDI scheme is considered now. A dedicated residual set is designed using a bank of \( r \) observers for system (1), according to the dedicated observer scheme [1]. Each residual will be designed to be sensitive to a single fault while remaining insensitive to the other faults and disturbances.
For this purpose let us first consider the system $\Sigma^i$ obtained from the system $\Sigma$ (1), as follows for $i = 1, \ldots, r$:

$$
\Sigma^i \begin{cases}
\dot{x}(t) = Ax(t) + E_1^i d^i(t) + F_1 f_i(t) \\
y(t) = Cx(t) + E_2^i d^i(t) + F_2 f_i(t)
\end{cases}
$$

where $f_i(t)$ is the $i$th component of $f(t)$, i.e. the $i$th fault, and :

$$d^i(t) = \begin{bmatrix} d^i(t) & f_1(t) & \cdots & f_{i-1}(t) & f_{i+1}(t) & \cdots & f_r(t) \end{bmatrix}^T$$

Thus $F_{1i}$ (resp. $F_{2i}$) is the $i$th column of $F_1$ (resp. $F_2$) and $E_1^i$ (resp. $E_2^i$) is the composite matrix obtained from the matrices $E_1$ and $F_1$ (resp. $E_2$ and $F_2$) by deleting the $i$th column of $F_1$ (resp. $F_2$).

The $i$th observer of this bank of $r$ observers is designed for system $\Sigma^i$ (5) :

$$\dot{x}^i(t) = A\hat{x}^i(t) + K^i(y(t) - C\hat{x}^i(t))$$

where $\hat{x}^i(t) \in \mathbb{R}^n$ is the state of the $i$th observer, $K^i$ is the observer gain to be designed such that $\dot{x}^i(t)$ asymptotically converges to $x(t)$, when no fault and no disturbance are considered.

Using (5) and (6) the estimation error $e^i(t) = x(t) - \hat{x}^i(t)$ is defined as :

$$e^i(t) = (A - K^i C)e^i(t) + (E_1^i - K^i E_2^i)d^i(t) + (F_{1i} - K^i F_{2i})f_i(t)$$

The residual is :

$$r_i(t) = Q^i(y(t) - C\hat{x}^i(t)) = Q^iCe^i(t) + Q^iE_2^i d^i(t) + Q^iF_{2i} f_i(t)$$

where $Q^i$ is a $1 \times p$ matrix.

Therefore the transfer matrices from the disturbance to the residual and from the fault to the residual are given, for $i = 1, \ldots, r$, by :

$$r_i(s) = \begin{bmatrix} T_{rd}^i(s) & T_{rf}^i(s) \end{bmatrix} \begin{bmatrix} d_i(s) \\ f_i(s) \end{bmatrix}$$

with $T_{rd}^i(s) = Q^i C(sI - A + K^i C)^{-1} (E_1^i - K^i E_2^i)$ and $T_{rf}^i(s) = Q^i C(sI - A + K^i C)^{-1} (F_{1i} - K^i F_{2i}) + Q^i F_{2i}$.

**Definition 2:** The bank of observers FDI problem (with disturbances) consists in finding, if possible, matrices $K^i$ and $Q^i$, such that, for $i = 1, 2, \ldots, r$, $A - K^i C$ is stable, $T_{rd}^i(s) = 0$, and the fault to residual transfer matrix is non zero, proper and diagonal, i.e. :

$$T_{rf}^i(s) = t_{ii}(s) \neq 0$$
which gives a complete FDI structure:

\[
\begin{pmatrix}
0 & \cdots & 0 & t_{11}(s) & \cdots & 0 \\
0 & \cdots & 0 & t_{22}(s) & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & t_{rr}(s)
\end{pmatrix}
\begin{pmatrix}
d(s) \\
f(s)
\end{pmatrix}
\]

(10)

where \( r_i(s) \) is the \( i \)th component of \( r(s) \) for \( i = 1, 2, \ldots, r \).

\[\diamondsuit\]

III. LINEAR STRUCTURED SYSTEMS

In this part we recall some definitions and results on linear structured systems. More details can be found in [9], [15].

We consider linear systems described by (1), denoted by \( \Sigma_\Lambda \) and called linear structured systems if the entries of the composite matrix \( J = \begin{bmatrix} A & E_1 & F_1 \\ C & E_2 & F_2 \end{bmatrix} \) are either fixed zeros or independent parameters (not related by algebraic equations). \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) denotes the set of independent parameters of the composite matrix \( J \). For sake of simplicity the dependence of the system matrices on \( \Lambda \) will not be explicit in the notation. A structured system represents a large class of parameter dependent linear systems. The structure is given by the location of the fixed zero entries of \( J \). The internal structure of the realization often comes from physical particularities of the system (i.e. interconnection of subsystems); thus the only exact knowledge on the system is the absence of direct relations between variables as state variables for example (see [16] for a detailed discussion on internal structure representation).

For such systems one can study generic properties i.e. properties which are true for almost all values of the parameters collected in \( \Lambda \) [17], [18]. More precisely a property is said to be generic (or structural) if it is true for all values of the parameters (i.e. any \( \Lambda \in \mathbb{R}^k \)) outside a proper algebraic variety of the parameter space, i.e. the zero set of a finite number of nontrivial polynomials in the parameters.

A directed graph \( G(\Sigma_\Lambda) = (Z, W) \) can be easily associated to the structured system \( \Sigma_\Lambda \) (1) as follows:

- the vertex set is \( Z = D \cup F \cup X \cup Y \) where \( D, F, X \) and \( Y \) are the disturbance, fault, state and output sets given by \( \{d_1, d_2, \ldots, d_q\}, \{f_1, f_2, \ldots, f_r\}, \{x_1, x_2, \ldots, x_n\} \) and \( \{y_1, y_2, \ldots, y_p\} \) respectively,
- the arc set is \( W = \{(d_i, x_j) | e_{1ji} \neq 0\} \cup \{(f_i, x_j) | f_{1ji} \neq 0\} \cup \{(x_i, x_j) | a_{ji} \neq 0\} \cup \{(x_i, y_j) | c_{ji} \neq 0\} \cup \{(d_i, y_j) | e_{2ji} \neq 0\} \cup \{(f_i, y_j) | f_{2ji} \neq 0\} \) where \( a_{ji} \) (resp. \( c_{ji}, e_{1ji}, e_{2ji}, f_{1ji}, f_{2ji} \)) denotes the entry \( (j, i) \) of the matrix \( A \) (resp. \( C, E_1, E_2, F_1, F_2 \)).

Moreover, recall that a directed path in \( G(\Sigma_\Lambda) \) from a vertex \( i_{\mu_0} \) to a vertex \( i_{\mu q} \) is a sequence of arcs \( (i_{\mu_0}, i_{\mu 1}), (i_{\mu 1}, i_{\mu 2}), \ldots, (i_{\mu q-2}, i_{\mu q-1}), (i_{\mu q-1}, i_{\mu q}) \) such that \( i_{\mu t} \in Z \) for \( t = 0, 1, \ldots, q \) and \( (i_{\mu t-1}, i_{\mu t}) \in W \) for \( t = 1, 2, \ldots, q \). The length of a path is the number of its arcs, each arc being counted the number of times it appears in the sequence. For the last sequence, the path has length \( q \). Occasionally, we denote the path \( P \) by the sequence of vertices it consists of, i.e. by:

\[ P = (i_{\mu 0}, i_{\mu 1}, \ldots, i_{\mu q-1}, i_{\mu q}) \]
Moreover, if $i_{\mu 0} \in D$ (resp. $F, X$) and, $i_{\mu q} \in Y$, $P$ is called a disturbance-output (resp. fault-output, state-output) path. A path which is such that $i_{\mu 0} = i_{\mu q}$ is called a circuit.

A set of paths with no common vertex is said to be a vertex disjoint. Using their associated graphs many important results have been obtained for these systems on structural controllability, decoupling, disturbance rejection, ... [9], [15], [8]. As a first example of these results, recall the graph characterization of the structural observability, which will be useful later [8], [18].

**Proposition 1:** Let $\Sigma_\Lambda$ be the linear structured system defined by (1) with its associated graph $G(\Sigma_\Lambda)$. The system (in fact the pair $(C, A)$) is structurally observable if and only if:

- there exists a state-output path starting from any state vertex in $X$,
- there exists a set of vertex disjoint circuits and state-output paths which cover all state vertices.

Consider now system $\Sigma_\Lambda$ (1) which transfer matrix is:

$$T(s) = \begin{bmatrix} T_{yd}(s) & T_{yf}(s) \end{bmatrix}$$

where $T_{yd}(s) = C(sI - A)^{-1}E_1 + E_2$ and $T_{yf}(s) = C(sI - A)^{-1}F_1 + F_2$.

We can calculate the generic rank of $T(s)$ and $T_{yd}(s)$ by using the following result [15], [19].

**Theorem 1:** Let $\Sigma_\Lambda$ be the linear structured system defined by (1) with its associated graph $G(\Sigma_\Lambda)$. The generic rank of $T(s)$ is equal to the maximum number of all vertex disjoint disturbance-output and fault-output paths in $G(\Sigma_\Lambda)$.

**Example 1:** Let us now present an example to illustrate the previous notions and results. Consider the following structured system $\Sigma_\Lambda$ which is of type (1) with one disturbance, two faults and three outputs:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 & 0 \end{bmatrix}, E_1 = \begin{bmatrix} \lambda_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \lambda_5 & \lambda_6 \end{bmatrix}, C = \begin{bmatrix} \lambda_7 & \lambda_8 & 0 & 0 \\ 0 & 0 & \lambda_9 & 0 \\ 0 & 0 & 0 & \lambda_{10} \end{bmatrix}$$

The entries of these matrices are the free parameters $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{10})$. The associated graph $G(\Sigma_\Lambda)$ is given in figure 1.

This system is structurally observable as can be seen from Proposition 1. Indeed there is a state-output path starting from any state vertex and the set of vertex disjoint paths $(x_1, y_1), (x_2, x_3, y_2)$ and $(x_4, y_3)$ covers all state vertices. In this example the generic rank of the transfer matrix $T(s)$ (for almost all values of the parameters collected in $\Lambda$) is 3 (vertex disjoint paths $(d_1, x_1, y_1)$, $(f_1, x_2, x_3, y_2)$ and $(f_2, x_4, y_3)$).

In this paper we look for structural conditions under which the previously mentioned FDI problems have solutions for almost any value of the parameters (which are the entries of $J$).
IV. Single observer-based FDI problem

Let us consider the observer-based triangular FDI problem (see Definition 1) that amounts to find a single observer for designing a residual vector such that the closed loop system is stable, the disturbance to residual transfer matrix is zero and the fault to residual transfer matrix is triangular with non zero diagonal entries (see (4)).

The solution to this single observer-based triangular FDI problem, was stated first in [13] but only a sketchy proof was given. We will assume in the sequel that the system $\Sigma_{\Lambda}$ is structurally observable. This weak condition will allow us to find stable solutions, i.e. satisfying $(A - KC)$ stable. This structural observability can be easily checked on the associated graph using Proposition 1.

Theorem 2: Consider the structurally observable system $\Sigma_{\Lambda}$ (1) and its associated graph $G(\Sigma_{\Lambda})$. The single observer-based triangular FDI problem of Definition 1 is generically solvable with stability if and only if :

$$k = k_q + r$$

where :

- $k$ is the maximum number of (disturbance / fault)-output vertex disjoint paths in $G(\Sigma_{\Lambda})$.
- $k_q$ is the maximum number of disturbance-output vertex disjoint paths in $G(\Sigma_{\Lambda})$.

Proof:

Let us recall

$$T(s) = C(sI - A)^{-1} \begin{bmatrix} E_1 & F_1 \\ \end{bmatrix} + \begin{bmatrix} E_2 & F_2 \end{bmatrix} = \begin{bmatrix} T_{yd}(s) & T_{yf}(s) \end{bmatrix}$$

By Theorem 1 the condition (12) is equivalent to :

$$\text{rank}_g(\begin{bmatrix} T_{yd}(s) & T_{yf}(s) \end{bmatrix}) - \text{rank}_g(T_{yd}(s)) = r$$

where $\text{rank}_g$ is used here for the generic rank of a matrix.

We will prove theorem 2 using (14) instead of (12).
Necessity: We suppose that the FDI problem generically has a solution i.e., for almost any parameter in $\Lambda$, there exist matrices $K$ and $Q$, $Q$ is a matrix of dimension $(r \times p)$, such that the conditions of Definition 1 are satisfied.

First, from the observer definition (2), it follows that, in Laplace variable:

$$\hat{x}(s) = (sI_n - A + KC)^{-1}Ky(s)$$

Using the residual definition (3) we get:

$$r(s) = Q(y(s) - C\hat{x}(s)) = Q(I_p - C(sI - A + KC)^{-1}K)y(s) = T_{ry}(s)y((s))$$

Then the (disturbance / fault) to residual transfer is:

$$\begin{bmatrix} T_{rd}(s) & T_{rf}(s) \end{bmatrix} = T_{ry}(s) \begin{bmatrix} T_{yd}(s) & T_{yf}(s) \end{bmatrix} \quad (15)$$

From Definition 1 it follows that:

$$T_{rd}(s) = T_{ry}(s)T_{yd}(s) = 0$$

From (4), $T_{rf}(s)$ has generic rank $r$, and it follows that the $r$ columns of $T_{yf}(s)$ are independent and not in the image of $T_{yd}(s)$. Therefore

$$\text{rank}_g \begin{bmatrix} T_{yd}(s) & T_{yf}(s) \end{bmatrix} - \text{rank}_g(T_{yd}(s)) = r.$$ and the condition (14) is satisfied.

Sufficiency: We will show in what follows that when condition (14) is satisfied there exist matrices $K$ and $Q$ such that the single observer based triangular FDI problem has a solution. To prove this result, we will use the solution of the triangular decoupling control problem, which is recalled in the Appendix.

Let us assume that condition (14) is true. First, let us decompose $\begin{bmatrix} T_{yd}(s) & T_{yf}(s) \end{bmatrix}$ according to the input partition:

$$\begin{bmatrix} T_{yd}(s) & T_{yf}(s) \end{bmatrix} = \begin{bmatrix} T_1(s) & T_2(s) & \cdots & T_{r+1}(s) \end{bmatrix} \quad (16)$$

where $T_1(s) = T_{yd}(s)$ and $T_{i+1}$ is the $i$th column of $T_{yf}(s)$, and note $p_1 = q, p_2 = 1, p_3 = 1, \ldots, p_{r+1} = 1$. Now as (14) is satisfied, then

$$\text{rank}_g \begin{bmatrix} T_{yd}(s) & T_{yf}(s) \end{bmatrix} - \text{rank}_g(T_{yd}(s)) = r,$$

which is equivalent to:

$$\text{rank}_g \begin{bmatrix} T_1(s) & T_2(s) & \cdots & T_{i+1}(s) \end{bmatrix} - \text{rank}_g \begin{bmatrix} T_1(s) & T_2(s) & \cdots & T_i(s) \end{bmatrix} = 1, \text{ for } i = 1, \ldots, r.$$  

(17)
Then,
\[ \text{rank} \left[ \begin{array}{cccc} T_1(s) & T_2(s) & \cdots & T_i(s) \end{array} \right] > \text{rank} \left[ \begin{array}{cccc} T_1(s) & T_2(s) & \cdots & T_{i-1}(s) \end{array} \right] \quad (18) \]
We will use the notations Im for the image of a rational matrix and Ker for its kernel. It follows from (18) that:
\[ \dim\left( \sum_{i=1}^{j} \text{Im} T_i(s) \right) > \dim\left( \sum_{i=1}^{j-1} \text{Im} T_i(s) \right) \]
Using the orthogonal form of the previous equation, we obtain:
\[ \dim\left( \bigcap_{i=1}^{j-1} \text{Ker} T_i^T(s) \right) > \dim\left( \bigcap_{i=1}^{j} \text{Ker} T_i^T(s) \right) \]
This corresponds to condition i) of Lemma 1 (see Appendix) applied on the dual partition \( H_i(s) = T_i^T(s) \).
This proves that the dual problem of Lemma 1, relative to the input partition (16), has a solution , i.e. there exists \( K \) and \( Q \) such that
\[ \overline{Q} C(sI - A + KC)^{-1} \left[ \begin{array}{cc} E_1 - KE_2 & F_1 - KF_2 \end{array} \right] + \overline{Q} \left[ \begin{array}{cc} E_2 & F_2 \end{array} \right] = \left[ \begin{array}{cc} M_{11}(s) & M_{12}(s) \\ 0 & M_{22}(s) \end{array} \right] \quad (19) \]
with \( M_{11}(s) \in \mathbb{R}^{k_q \times q}(s) \), \( M_{12}(s) \in \mathbb{R}^{k_q \times r}(s) \) and \( M_{22}(s) \in \mathbb{R}^{(p-k_q) \times r}(s) \) has the following form \( M_{22}(s) = \left[ \begin{array}{c} X^T \\ 0 \end{array} \right] \), where \( X \in \mathbb{R}^{r \times r}(s) \) has the same form as in (4), with \( A - KC \) stable (since \( G(\Sigma_{\Lambda}) \) is structurally observable). By choosing
\[ Q = \left[ \begin{array}{cc} 0_{r \times k_q} & I_r \\ 0_{r \times (p-k)} \end{array} \right] \overline{Q} \quad (20) \]
The pair \((K, Q)\) is a solution of the single observer based FDI problem of Definition 1. \( \square \)
The given condition is necessary and sufficient but we cannot detect and isolate simultaneous faults.
In the next section we give the necessary and sufficient condition for solving the diagonal FDI problem with stability using a bank of observers.

V. BANK OF OBSERVER-BASED FDI PROBLEM

In this section we will give our result concerning the diagonal FDI problem by using a bank of observers which was stated first in [14]. We will then compare this result with the one given by [7]

**Theorem 3:** Consider the structurally observable system \( \Sigma_{\Lambda} \) (1) and the associated graph \( G(\Sigma_{\Lambda}) \). The bank of observer-based diagonal FDI problem of Definition 2, is generically solvable if and only if:
\[ k = k_q + r \quad (21) \]
where :
• $k$ is the maximum number of (disturbance / fault)-output vertex disjoint paths in $G(\Sigma_{\Lambda})$
• $k_q$ is the maximum number of disturbance-output vertex disjoint paths in $G(\Sigma_{\Lambda})$

It is interesting to notice that the necessary and sufficient condition for this result is exactly the same as in theorem 2, although in the present case we are able to detect and isolate simultaneous faults.

Proof 1: Note first that in the case of the bank of observers FDI problem, the system $\Sigma^i$ (5) is also a structured system with the same set of independent parameters $\Lambda$ and will be denoted by $\Sigma^i_{\Lambda}$.

From Theorem 1, the condition (21) is equivalent to:

$$\text{rank}_g(T(s)) - \text{rank}_g(T_{yd}(s)) = r$$

(22)

where $T(s)$ and $T_{yd}(s)$ are defined by (13). The condition (22) is equivalent to:

$$\text{rank}_g(T(s)) - \text{rank}_g(T^i(s)) = 1,$$

(23)

for $i = 1, 2, \ldots, r$ where $T^i(s)$ is:

$$T^i(s) = \begin{bmatrix} T_{yd}(s) & t_1(s) & \cdots & t_{i-1}(s) & t_{i+1}(s) & \cdots & t_r(s) \end{bmatrix}$$

where $t_j(s)$ is the jth column of $T_{yf}(s)$ defined by (13). Using again Theorem 1, the condition (23) is equivalent to:

$$k = k^i + 1$$

(24)

where:
• $k$ is the maximum number of (disturbance / fault)-output vertex disjoint paths in $G(\Sigma_{\Lambda})$
• $k^i$ is the maximum number of (disturbance / fault)-output vertex disjoint paths in $G(\Sigma_{\Lambda})$ when deleting the $f_i(t)$ vertex and all arcs which have their origin in $f_i(t)$.

Now, consider the system (5) with a disturbance vector of dimension $q + r - 1$ and a unique fault.

Applying Theorem 2 on system $\Sigma^i_{\Lambda}$ under condition (24), there exist matrices $K^i$ and $Q^i$ such that $A - K^iC$ is stable and:

$$r_i(s) = \begin{bmatrix} 0 & 0 & \cdots & 0 & t_{ii}(s) \end{bmatrix} \begin{bmatrix} d^i(s) \\ f_i(s) \end{bmatrix}$$

where $t_{ii} \neq 0$.

Remember that the matrices $K^i$ and $Q^i$ correspond to the $i^{th}$ observer and $i^{th}$ residual generator, respectively. Using this technique for $i = 1, 2, \ldots, r$ one can design the bank of observers according to Definition 2.

In this paper we have addressed a similar problem to that tackled in [7]. In [7] generic FDI solvability conditions have been given when there are no disturbances and when the system matrices have no specific structure. In order to compare both papers, we will consider the specific case where there is no disturbance and where there is no direct feedthrough between faults and measured outputs (which
corresponds to $F_2 = 0$ in our case and to the absence of state space extension in [7]). We will show in this case that the condition given in [7] easily follows from theorem 3. In this particular case, the state space representation considered in [7] is such that all the system parameters are free. Hence, in our framework, the associated graph of the system is such that:

- there is an arc between any fault vertex $f_i$ to any state vertex $x_j$.
- there is an arc between any state vertex $x_i$ and any output vertex $y_j$.
- there is an arc between state vertices $x_i$ and $x_j$ for $i, j = 1, \ldots, n$

According to Proposition 1 the considered system is structurally observable since all state vertices are connected to output vertices, and there exists a circuit covering all the state vertices. In this case the conditions in [7] reduce to $r \leq n$ and $r \leq p$. Our condition (21) reduces in this case to $k = r$. Due to the specific structure of the graph, the maximal number of fault-output vertex disjoint paths equals to $k = \min(r, n, p)$. Hence the problem is solvable if and only if $r = k = \min(r, n, p)$ which is equivalent to $r \leq n$ and $r \leq p$.

VI. Illustrative Example

The following example illustrates the results presented in this paper. Consider again example 1. It has been shown before that this system is structurally observable.

It can be shown that no solution exists for the solvability of the single observer-based diagonal FDI problem [20]. Now, using Theorem 2 one has: $k = 3$ (paths $(d_1, x_1, y_1)$, $(f_1, x_2, x_3, y_2)$ and $(f_2, x_4, y_3)$) $k_q = 1$ (path $(d_1, x_1, y_1)$). Condition (12) is satisfied, $(k = k_q + r)$, the single observer-based triangular FDI problem is therefore generically solvable.

As the necessary and sufficient condition of Theorem 3 is exactly the same as in Theorem 2, the observer-based diagonal FDI problem is therefore generically solvable using two observers. In [20] the residual are proved to be such that:

$$r(s) = \begin{bmatrix} 0 & (1/\lambda_7)(s + a)^{-2} & 0 \\ 0 & 0 & (\lambda_9/\lambda_5\lambda_7)(s + b)^{-2} \end{bmatrix} \begin{bmatrix} d(s) \\ f(s) \end{bmatrix}$$

where $a$ and $b$ are arbitrary positive real numbers.

VII. Concluding remarks

In this paper we have presented a new analysis tool for solving Fault Detection and Isolation problems, which relies on the internal system structure knowledge. We have considered observer-based FDI problems for linear structured systems including faults and disturbances. We obtained intrinsic easy to check solvability conditions which do not depend on the parameters, but the solutions are of course parameter dependent. The necessary and sufficient conditions for generically solving these problems are expressed in terms of input-output paths in the system’s associated graph and are often satisfied in practical situations. Note that the graph theoretic solvability conditions can be checked using max flow min cost techniques [11].
Dedication: the first three authors would like to dedicate this paper to Reza Motyeian who deceased in August 2000.

References


Appendix

Lemma 1: [21] Let $H(s)$ be a $p \times m$ proper rational transfer matrix. Consider an output partition, that is a set of positive integers $p_1, \ldots, p_k$ such that $\sum_{i=1}^{k} p_i = p$, $H(s)$ is decomposed accordingly in $k$ row blocks:

$$
H(s) = \left[ \begin{array}{ccc}
H^T_1(s) & H^T_2(s) & \cdots & H^T_k(s)
\end{array} \right]^T
$$

Consider a controllable state space realization $(A, B, C, D)$ of $H(s)$ such that $H(s) = C(sI-A)^{-1}B+D$. There exists a state feedback control $u(t) = Fx(t) + Gv(t), G$ non singular, such that the closed loop
transfer $H_{F,G}(s) = (C + DF)(sI - A - BF)^{-1}BG + DG$ is stable and lower block-triangular decoupled in a non degenerate way relative to $p_1, \ldots, p_k$ if and only if:

i) $\dim(\cap_{i=1}^k \ker H_i(s)) > \dim(\cap_{i=1}^j \ker H_i(s)), j = 1, 2, \ldots, k$

ii) $\dim(\ker H_1(s)) < m$

This means that under condition i) and ii) there exists $(F, G)$ such that $H_{F,G}(s)$ is stable and has the following form:

$$H_{F,G}(s) = \begin{bmatrix}
H_{11}(s) & 0 & \cdots & 0 \\
H_{21}(s) & H_{22}(s) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
H_{k1}(s) & H_{k2}(s) & \cdots & H_{kk}(s)
\end{bmatrix}$$ (25)

where $H_{ij}(s)$ has dimension $(p_i \times m_j)$ with $m_j > 0$. Non degenerate simply means that $H_{ii}(s) \neq 0$ for $i = 1 \ldots k$.

This result has been proved in [21] for minimal realizations of strictly proper systems and for an upper block triangular structure. The result remains true for controllable realizations of proper systems. A lower block triangular structure can also be easily obtained via input permutations.

In the proof of theorem 2, we use the dual of lemma 1, the controllability assumption becomes an observability assumption and the lower block triangular structure becomes an upper-triangular one. Moreover the condition ii) of Lemma 1 ensures that the first block of the decoupled system, $H_{11}(s)$, is non zero. In our case this is not relevant as the disturbance block $M_{11}$ will no longer appear by the choice of $Q$ (20).