EXACT DOUBLE DOMINATION IN GRAPHS

MUSTAPHA CHELLALI

Department of Mathematics, University of Blida
B.P. 270, Blida, Algeria

e-mail: mchellali@hotmail.com

ABDELKADER KHELADI

Department of Operations Research
Faculty of Mathematics
University of Sciences and Technology Houari Boumediene
B.P. 32, El Alia, Bab Ezzouar, Algiers, Algeria

e-mail: kader.khelladi@yahoo.fr

AND

FRÉDÉRIC MAFFRAY

C.N.R.S., Laboratoire Leibniz-IMAG
46 Avenue Félix Viallet
38031 Grenoble Cedex, France

e-mail: frederic.maffray@imag.fr

Abstract

In a graph a vertex is said to dominate itself and all its neighbours. A doubly dominating set of a graph \(G\) is a subset of vertices that dominates every vertex of \(G\) at least twice. A doubly dominating set is exact if every vertex of \(G\) is dominated exactly twice. We prove that the existence of an exact doubly dominating set is an NP-complete problem. We show that if an exact double dominating set exists then all such sets have the same size, and we establish bounds on this size. We give a constructive characterization of those trees that admit a doubly dominating set, and we establish a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph.

Keywords: double domination, exact double domination.

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1. Introduction

In a graph $G = (V, E)$, a subset $S \subseteq V$ is a \textit{dominating set} of $G$ if every vertex $v$ of $V - S$ has a neighbour in $S$. The \textit{domination number} $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. For a comprehensive treatment of domination in graphs and its variations, see [8, 9].

Harary and Haynes [7] defined and studied the concept of double domination, which generalizes domination in graphs. In a graph $G = (V, E)$, a subset $S$ of $V$ is a \textit{doubly dominating set} of $G$ if, for every vertex $v \in V$, either $v$ is in $S$ and has at least one neighbour in $S$ or $v$ is in $V - S$ and has at least two neighbours in $S$. The \textit{double domination number} $\gamma \times_2(G)$ is the minimum cardinality of a doubly dominating set of $G$. Double domination was also studied in [2, 3, 4]. Analogously to exact (or perfect) domination introduced by Bange, Barkauskas and Slater [1], Harary and Haynes [7] defined an \textit{efficient doubly dominating set} as a subset $S$ of $V$ such that each vertex of $V$ is dominated by exactly two vertices of $S$. We will prefer here to use the phrase \textit{exact doubly dominating set}.

Every graph $G = (V, E)$ with no isolated vertex has a doubly dominating set; for example $V$ is such a set. In contrast, not all graphs with no isolated vertex admit an exact doubly dominating set; for example, the star $K_{1,p}$ ($p \geq 2$) does not. In Section 2 we prove that the existence of an exact doubly dominating set is an NP-complete problem. We then show in Section 3 that if a graph $G$ admits an exact doubly dominating set then all such sets have the same size, and we give some bounds on this number. Finally, we give in Section 4 a constructive characterization of those trees that admit an exact doubly dominating set, and we establish a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph.

Let us give some definitions and notation. In a graph $G = (V, E)$, the \textit{open neighbourhood} of a vertex $v \in V$ is the set $N(v) = \{u \in V \mid uv \in E\}$, the \textit{closed neighbourhood} is the set $N[v] = N(v) \cup \{v\}$, and the \textit{degree} of $v$ is the size of its open neighbourhood, denoted by $\deg_G(v)$. We denote respectively by $n$, $\delta$ and $\Delta$ the \textit{order} (number of vertices), \textit{minimum degree} and \textit{maximum degree} of a graph $G$.

2. NP-Completeness

In this section we consider the complexity of the problem of deciding whether
a graph admits an exact doubly dominating set.

**EXACT DOUBLY DOMINATING SET (X2D)**

Instance: A graph $G$;

Question: Does $G$ admit an exact doubly dominating set?

We show that this problem is NP-complete by reducing the following EXACT 3-COVER (X3C) problem to our problem.

**EXACT 3-COVER (X3C)**

Instance: A finite set $X$ with $|X| = 3q$ and a collection $C$ of 3-element subsets of $X$;

Question: Is there a subcollection $C'$ of $C$ such that every element of $X$ appears in exactly one element of $C'$?

EXACT 3-COVER is a well-known NP-complete problem [6].

**Theorem 1.** EXACT DOUBLY DOMINATING SET is NP-complete.

**Proof.** Clearly, X2D is in NP. Let us now show how to transform any instance $X, C$ of X3C into an instance $G$ of X2D so that one of them has a solution if and only if the other has a solution.

For each $x_i \in X$, we build a “gadget” graph with vertices $a_i, b_i, c_i$ and $d_i^1, \ldots, d_i^{k_i}$, where $k_i$ is the number of elements of $C$ that contain $x_i$, and with edges $a_i b_i, b_i c_i$ and $c_i d_{ij}$ ($j = 1, \ldots, k_i$). We view the $d_{ij}$’s as points of this gadget, each of them being associated with an element of $C$ that contains $x_i$. See Figure 1.

For each $C_t \in C$, we build a gadget graph with 15 vertices $y_{ij}^0, \ldots, y_{ij}^8$, $z_t, r_t, s_t, u_t, v_t, w_t$ and edges $y_{ij}^j y_{ij}^{j+1}$ ($j = 0, \ldots, 8 \mod 9$) (so that the $y_{ij}^j$’s induce a $C_9$) and $z_t y_{ij}^0, z_t y_{ij}^2, z_t y_{ij}^6, z_t r_t, z_t s_t, r_t s_t$ (so $z_t, r_t, s_t$ induce a triangle), and $u_t y_{ij}^1, u_t y_{ij}^2, v_t y_{ij}^4, v_t y_{ij}^5, w_t y_{ij}^7, w_t y_{ij}^8$. We view $u_t, v_t, w_t$ as the three points of this gadget, each of them being associated with an element of $C_t$. See Figure 1.

Now, for each $C_t$, if $C_t = \{x_i, x_j, x_k\}$ say, we identify the first, second and third point of the gadget of $C_t$ with the corresponding point in the gadget of $x_i, x_j, x_k$ respectively. We call $G$ the resulting graph. Clearly the size of $G$ is polynomial in the size of $X$ and $C$. 
1. Suppose that the instance \( X, C \) of X3C has a solution \( C' \). We build a set \( S \) of vertices of \( G \) as follows: for each \( C_t \in C' \), we put in \( S \) the vertices \( u_t, y^1_t, v_t, y^4_t, w_t, y^7_t, z_t, r_t \); for each \( C_t \in C - C' \), we put in \( S \) the vertices \( y^1_t, y^2_t, y^3_t, y^6_t, y^7_t, r_t, s_t \); for each \( x_i \in X \), we put in \( S \) the vertices \( a_i, b_i \) (note that exactly one of the \( d^j_i \)'s has been put in \( S \)). It is a routine matter to check that \( S \) is an exact doubly dominating set in \( G \).

2. Conversely, suppose that \( G \) has an exact doubly dominating set \( S \). Note the gadget of a given \( C_t \) is in exactly one of the following two possible states:

(a) \( z_t \in S \), and so exactly one of \( r_t, s_t \) is in \( S \), none of \( y^0_t, y^3_t, y^6_t \) is in \( S \), the other six \( y^j_t \)'s are in \( S \), and none of \( u_t, v_t, w_t \) is in \( S \); or

(b) \( z_t \notin S \), both \( r_t, s_t \) are in \( S \), none of \( y^0_t, y^3_t, y^6_t \) is in \( S \), exactly one of \( \{y^1_t, y^4_t, y^7_t\}, \{y^2_t, y^5_t, y^6_t\} \) is in \( S \) and the other is in \( V - S \), and each of \( u_t, v_t, w_t \) is in \( S \).

Clearly, for each \( x_i \in X \), we have \( a_i, b_i \in S \) (else \( a_i \) would not be doubly dominated), then \( c_i \notin S \) (else \( b_i \) would be dominated three times), and it follows that exactly one of the \( d^j_i \)'s is in \( S \). For each \( i = 1, \ldots, 3q \), let \( t(i) \) be the integer such that this special \( d^j_i \) is equal to one point of \( C_t(i) \subseteq C \), and let us say that \( C_t(i) \) is selected by \( x_i \). Thus the gadget of \( C_t(i) \) is in state (b), which means that \( C_t(i) \) is selected by each of its 3 elements. Therefore, the collection \( C' \) of all selected elements of \( C \) (i.e., those whose three points are in \( S \)) is an exact 3-cover.

3. Exact Doubly Dominating Sets

We begin by the following observation which is a straightforward property
of exact doubly dominating sets in graphs. A matching in a graph $G$ is a set of pairwise non-incident edges of $E$.

**Observation 2.** The vertex set of every exact doubly dominating set induces a matching.

Next, we show that all exact doubly dominating sets (if any) have the same size.

**Proposition 3.** If $G$ has an exact doubly dominating set then all such sets have the same size.

**Proof.** Let $D_1, D_2$ be two exact doubly dominating sets of $G$. Let us write $I = D_1 \cap D_2$, and let $X_0$ and $X_1$ be the subsets of $D_1 - I$ such that every vertex of $X_0$ has zero neighbours in $I$ and every vertex of $X_1$ has one neighbour in $I$. Clearly $D_1 - I = X_0 \cup X_1$. We define similarly subsets $Y_0$ and $Y_1$ of $D_2 - I$. We claim that $|X_1| = |Y_1|$. Indeed, let $x$ be any vertex of $X_1$, adjacent to a vertex $z \in I$. Since $D_2$ is an exact doubly dominating set, $z$ has a unique neighbour $y$ in $D_2$. We have $y \in D_2 - I$, for otherwise $z$ has two neighbours $x, y$ in $D_2$, a contradiction. Thus $y \in Y_1$. The symmetric argument holds for every vertex of $Y_1$, and so $|X_1| = |Y_1|$. Since $D_2$ is an exact doubly dominating set, every vertex of $X_1$ has exactly one neighbour in $Y_0 \cup Y_1$ and every vertex of $X_0$ has exactly two neighbours in $Y_0 \cup Y_1$. The same holds about the vertices of $Y_1$ and $Y_0$. This implies $|X_0| = |Y_0|$, and thus $|D_1| = |D_2|$.

The next result relates the size of an exact doubly dominating set with the order and minimum degree $\delta$ of a graph $G$.

**Proposition 4.** If $S$ is an exact doubly dominating set of a graph $G$, then $|S| \leq 2n/(\delta + 1)$.

**Proof.** Let $S$ be an exact doubly dominating set of a graph $G$ and let $t$ denote the number of edges joining the vertices of $S$ to the vertices of $V - S$. Then $t = 2|V - S|$ since $S$ is an exact doubly dominating set. By Observation 2, $S$ induces a matching of $G$, and so every vertex $v$ of $S$ has exactly $\deg_G(v) - 1$ neighbours in $V - S$. Thus $t = \sum_{v \in S}(\deg_G(v) - 1)$. So $|S|(\delta - 1) \leq t = 2|V - S|$. Hence $|S| \leq 2n/(\delta + 1)$.

In [7], Harary and Haynes gave a lower bound for the doubly domination number:
Theorem 5 ([7]). If $G$ has no isolated vertices, then $\gamma_x^2(G) \geq 2n/(\Delta + 1)$.

From Proposition 4 and Theorem 5, we have:

Corollary 6. If $S$ is an exact doubly dominating set of a regular graph $G$, then $|S| = 2n/(\Delta + 1)$.

Next, we establish a bound on the double domination number based on the neighbourhood packing number for any graph with no isolated vertices. Recall that a set $R \subseteq V(G)$ is a neighbourhood packing set of $G$ if $N[x] \cap N[y] = \emptyset$ holds for any two distinct vertices $x, y \in R$. The neighbourhood packing number $\rho(G)$ is the maximum cardinality of a neighbourhood packing in $G$. It is easy to see (see [8]) that every graph $G$ satisfies $\rho(G) \leq \gamma(G)$.

Theorem 7. If $G$ is a graph without isolated vertices, then $\gamma_x^2(G) \geq 2\rho(G)$.

Proof. Let $R$ be a maximum neighbourhood packing set of $G$. Then for every $v \in R$, every doubly dominating set of $G$ contains at least 2 vertices of $N[v]$ to doubly dominate $v$. Since $N[v] \cap N[u] = \emptyset$ holds for each pair of vertices $v, u$ of $R$, we have $\gamma_x^2(G) \geq 2|R|$. \qed

Corollary 8. If $S$ is an exact doubly dominating set of $G$, then $|S| \geq 2\rho(G)$.

Farber [5] proved that the domination number and neighbourhood packing number are equal for any strongly chordal graph. Thus we have the following corollary to Theorem 7 which extends the result of Blidia et al. [3] for trees.

Corollary 9. If $G$ is a strongly chordal graph without isolated vertices, then $\gamma_x^2(G) \geq 2\gamma(G)$.

4. Graphs with Exact Doubly Dominating Sets

We first consider paths and cycles. The double domination number for cycles $C_n$ and nontrivial paths $P_n$ were given in [7] and [3] respectively:

[7] $\gamma_x^2(C_n) = \lceil 2n/3 \rceil$.

[3] $\gamma_x^2(P_n) = 2\lceil n/3 \rceil + 1$ if $n \equiv 0 \pmod{3}$ and $\gamma_x^2(P_n) = 2\lceil n/3 \rceil$ otherwise.

Now we establish similar results for the exact doubly dominating sets in cycles and paths.
Proposition 10. A cycle $C_n$ has an exact doubly dominating set if and only if $n \equiv 0 \pmod{3}$. If this holds the size of any such set is $2n/3$.

Proof. Let $S$ be an exact doubly dominating set of a cycle $C_n$. By Corollary 6, we have $|S| = 2n/3$ and so $n \equiv 0 \pmod{3}$. Conversely, assume the vertices of $C_n$ are labelled $v_1, v_2, \ldots, v_n, v_1$. If $n \equiv 0 \pmod{3}$, then it is easy to check that the set $\{v_i, v_{i+1} \mid i \equiv 1 \pmod{3}, 1 \leq i \leq n-1\}$ is an exact doubly dominating set of $C_n$.

Proposition 11. A path $P_n$ has an exact doubly dominating set if and only if $n \equiv 2 \pmod{3}$. If this holds the size of any such set is $2(n+1)/3$.

Proof. If $n = 2$ the fact is obvious, so let us assume $n \geq 3$. Let $S$ be an exact doubly dominating set of a path $P_n$. Note that for every vertex $v$ of degree 2, either $v$ or its two neighbours are in $S$. So $V - S$ is an independent set, and $N(v) \cap N(w) = \emptyset$ for any two $v, w \in V - S$. By Observation 2, every vertex of $S$ has exactly one neighbour in $V - S$. Thus $|S| - 2 = 2|V - S|$ and so $n = |S| + |V - S| = 3|V - S| + 2$.

Conversely, assume that the vertices of $P_n$ are labelled $v_1, v_2, \ldots, v_n$. If $n \equiv 2 \pmod{3}$ then it is easy to check that the set $\{v_i, v_{i+1} \mid i \equiv 1 \pmod{3}, 1 \leq i \leq n-1\}$ is an exact doubly dominating set of $P_n$.

Chellali and Haynes [4] established the following upper bound for the double domination number:

Theorem 12 ([4]). Every graph $G$ without isolated vertices satisfies

$$
\gamma_{\times 2}(G) \leq n - \delta + 1.
$$

Theorem 13. Let $G$ be a graph that admits an exact doubly dominating set $S$. Then $|S| = n - \delta + 1$ if and only if either $G = tK_2$ with $t \geq 1$, if $\delta = 1$, or $G = K_n$ with $n \geq 3$ otherwise.

Proof. Let $S$ be an exact doubly dominating set of $G$ such that $|S| = n - \delta + 1$. If $\delta = 1$, then $|S| = n$. Since $S$ induces a 1-regular subgraph, $G$ itself is 1-regular, i.e., $G = tK_2$ with $t \geq 1$. Now assume that $\delta \geq 2$. Let $v$ be a vertex of $S$. Then $V - S$ contains all the neighbours of $v$ except one, and so $\deg_G(v) - 1 \leq |V - S| = n - (n - \delta + 1) = \delta - 1$. Thus all the vertices of $S$ have the same degree $\delta$, and $|V - S| = \delta - 1$. Let $u$ be a vertex of $N(v) \cap S$. Then $u$ is adjacent to all the vertices of $V - S$ and
hence at this point every vertex of \( V - S \) is doubly dominated by \( u \) and \( v \). Thus \( S = \{ u, v \} \) and all the vertices of \( V - S \) are mutually adjacent. So \( G \) is a complete graph.

Next, we consider nontrivial trees. A vertex of degree 1 is called a leaf, and a support vertex is any vertex adjacent to a leaf. It is easy to see that a star with at least three vertices is an example of a tree that does not admit an exact doubly dominating set. The following observation generalizes this remark.

**Observation 14.**
- If a graph \( G \) has a leaf, then any doubly dominating set of \( G \) contains this leaf and its neighbour.
- If a graph \( G \) has an exact doubly dominating set, then every support vertex is adjacent to exactly one leaf, and no two support vertices are adjacent.

We now define recursively a collection \( T \) of trees, where each tree \( T \in T \) has two distinguished subsets \( A(T) \), \( B(T) \) of vertices. First, \( T \) contains any tree \( T_1 \) with two vertices \( x, y \), and for such a tree we set \( A(T_1) = \{ x, y \} \) and \( B(T_1) = \{ y \} \). Next, if \( T' \) is any tree in \( T \), then we put in \( T \) any tree \( T \) that can be obtained from \( T' \) by any of the following two operations:

**Type-1 operation:** Attach a path \( P_3 = uvw \), with \( u, v, w \not\in V(T') \), by adding an edge from \( w \) to one vertex of \( A(T') \). Set \( A(T) = A(T') \cup \{ u, v \} \) and \( B(T) = B(T') \cup \{ u \} \).

**Type-2 operation:** Attach a path \( P_5 = a_1a_2a_3a_4a_5 \), with \( a_1, a_2, a_3, a_4, a_5 \not\in V(T') \), by adding an edge from \( a_3 \) to one vertex of \( V(T') \) - \( A(T') \). Set \( A(T) = A(T') \cup \{ a_1, a_2, a_4, a_5 \} \) and \( B(T) = B(T') \cup \{ a_1, a_5 \} \).

**Lemma 15.** If \( T \in T \), then:

(a) \( A(T) \) is the unique exact doubly dominating set of \( T \).

(b) \( B(T) \) is a neighbourhood packing set of \( T \).

(c) \( |A(T)| = 2\gamma(T) \).

**Proof.** Consider any \( T \in T \). So \( T \) can be obtained from a sequence \( T_1, T_2, \ldots, T_k \) \( (k \geq 1) \) of trees of \( T \), where \( T_1 \) is the tree with two vertices, \( T = T_k \), and, if \( 1 \leq i \leq k - 1 \), the tree \( T_{i+1} \) is obtained from \( T_i \) by one of the two operations. We prove (a) by induction on \( k \). If \( k = 1 \), then \( A(T) \) is
obviously the unique exact doubly dominating set of $T$. Assume now that $k \geq 2$ holds for $T$ and that the result holds for all trees in $T$ that can be constructed by a sequence of length at most $k - 1$. Let $T' = T_{k-1}$. We distinguish between two cases.

**Case 1.** $T$ is obtained from $T'$ by using the Type-1 operation. Note that $A(T)$ is an exact doubly dominating set of $T$ since, by the induction hypothesis, $A(T')$ is an exact doubly dominating set of $T'$ and $u, v$ and the neighbour of $w$ in $T'$ are in $A(T)$. Now let $S$ be any exact doubly dominating set of $T'$. By Observation 14, we have $\{u, v\} \subseteq S$, and consequently $w \notin S$ (for otherwise $v$ would be dominated three times by $S$). If $x$ is any vertex in $V(T')$, then $x$ is not dominated by any of $u, v$, so $S - \{u, v\}$ is an exact doubly dominating set of $T'$. By the inductive hypothesis $A(T')$ is the unique such set, so $S - \{u, v\} = A(T')$, and so $S = A(T)$, which shows the unicity announced in (a).

**Case 2.** $T$ is obtained from $T'$ by using the Type-2 operation. Note that $A(T)$ is an exact doubly dominating set of $T$ since, by the induction hypothesis, $A(T')$ is an exact doubly dominating set of $T'$ and the neighbour of $a_3$ in $T'$ is not in $A(T')$ while $a_1, a_2, a_4, a_5$ are in $A(T)$. Now let $S$ be any exact doubly dominating set of $T$. By Observation 14, we have $\{a_1, a_2, a_4, a_5\} \subseteq S$, and consequently $a_3 \notin S$. If $x$ is any vertex in $V(T')$, then $x$ is not dominated by any of $a_1, a_2, a_4, a_5$, so $S - \{a_1, a_2, a_4, a_5\}$ is an exact doubly dominating set of $T'$. By the inductive hypothesis we have $S - \{a_1, a_2, a_4, a_5\} = A(T')$, and so $S = A(T)$. So (a) is proved.

It is a routine matter to check item (b). Note that the tree $T_1$ with two vertices has $|A(T_1)| = 2$ and $|B(T_1)| = 1$; moreover, each operation adds twice as many vertices to $A(T)$ as to $B(T)$, so $|A(T)| = 2|B(T)|$ holds for every tree $T \in T$. It follows from this and from (a) and (b) that $\gamma_{x,2}(T) \leq |A(T)| = 2|B(T)| \leq 2\gamma(T)$, and we have equality throughout by Corollary 9.

This proves part (c) and concludes the proof of the lemma.

We now are ready to give a constructive characterization of trees with an exact doubly dominating sets.

**Theorem 16.** Let $T$ be a tree. Then $T$ has an exact doubly dominating set if and only if $T \in T$.

**Proof.** First suppose that $T \in T$. Then Lemma 15 implies that $T$ has an exact doubly dominating set. Conversely, assume that $T$ is a tree that has
an exact doubly dominating set $S$, and let $n$ be the order of $T$. Clearly, $n \geq 2$. If $n = 2$, then $T$ is in $\mathcal{T}$. Observation 14 implies that $n \in \{3, 4\}$ is impossible and that $n = 5$ implies that $T$ is a path on 5 vertices, which is in $\mathcal{T}$ since it can be obtained from $T_1$ by the Type-1 operation.

Now assume that $n \geq 6$ and that every tree $T'$ of order $n'$ with $2 \leq n' < n$ such that $T'$ has an exact doubly dominating set is in $\mathcal{T}$. Root $T$ at a vertex $r$. Let $u$ be a leaf at maximum distance from $r$, let $v$ be the parent of $u$ in the rooted tree, and let $w$ be the parent of $v$. By Observation 14, $u$ is the unique child of $v$, $\{u, v\} \subseteq S$, $w \notin S$, and $w$ is neither a support vertex nor a leaf. This implies that every child of $w$ is a support vertex. Furthermore, $w$ has at most two children, for otherwise $w$ would be dominated at least 3 times by $S$, a contradiction. So $w \neq r$. Let $z$ be the parent of $w$ in the rooted tree.

Suppose that $w$ has exactly one child in the rooted tree. Let $T' = T - \{u, v, w\}$. Since $\{u, v\} \subseteq S$ and $w \notin S$, we have $z \in S$ so that $w$ is dominated twice by $S$. Moreover, $S - \{u, v\}$ is an exact doubly dominating set of $T'$. By the inductive hypothesis, we have $T' \in \mathcal{T}$ and, by Lemma 15, $S - \{u, v\} = A(T')$ is the unique exact doubly dominating set of $T'$. Thus $T$ can be obtained from $T'$ by using Type-1 operation (with the path $uvw$ and since $z \in A(T')$), so $T \in \mathcal{T}$.

Now suppose that $w$ has exactly two children $v, v'$ in the rooted tree. Let $T_w$ be the subtree of $T$ induced by $w$ and its descendants, rooted at $w$. By Observation 14, each child of $w$ has exactly one child, and we call $u'$ the child of $v'$, so $T_w$ is a path on five vertices $uvwv'u'$ with central vertex $w$. Moreover, by Observation 14, we have $\{u, v, u', v'\} \subseteq S$, $w \notin S$, and $z \notin S$ since $w$ is dominated twice in $S$ by $v, v'$. Thus $z$ is doubly dominated by $S \cap V(T')$ and consequently $S \cap V(T')$ is an exact doubly dominating set of $T'$. By the inductive hypothesis, we have $T' \in \mathcal{T}$ and, by Lemma 15, $S \cap V(T') = A(T')$ is the unique exact doubly dominating set of $T'$. Thus $T$ can be obtained from $T'$ by using Type-2 operation (with the path $uvwv'u'$ and since $z \notin A(T')$), so $T \in \mathcal{T}$. This completes the proof of the theorem. □

The proof of the theorem suggests a polynomial-time algorithm which, given a tree $T$ with $n$ vertices, decides whether $T$ is in $\mathcal{T}$ and, if it is, returns the set $A(T)$. Here is an outline of the algorithm. If $T$ is a path on 2 or 5 vertices, answer $T \notin \mathcal{T}$, return the obvious set $A(T)$, and stop. Else, if either $n \leq 5$ or $T$ is a star, answer $T \notin \mathcal{T}$ and stop. Now suppose $n \geq 6$. Pick a vertex $r$, root the tree $T$ at $r$, and pick a vertex $u$ at maximum distance from $r$. Let $v$ be the parent of $u$ in the rooted tree and $w$ be the
parent of $v$. If either $v$ has at least two children, or $w$ has at least three children, or $w$ has exactly two children and its second child has either zero or at least two children, then return the answer $T \notin T$ and stop. Else, let $z$ be the parent of $w$. If $w$ has exactly one child, call the algorithm recursively on the tree $T' = T - \{u, v, w\}$; if the answer to the recursive call is $T' \in T$ and $z \notin A(T')$, then answer $T \notin T$ and stop, else answer $T \in T$, return $A(T) = A(T') \cup \{u, v\}$, and stop.

Next, we give a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph. Recall that a matching in a graph $G = (V, E)$ is perfect if its size is $|V|/2$. With any perfect matching $M = \{e_1, e_2, \ldots, e_{n/2}\}$ of a graph $G$ we associate a simple graph denoted by $G_M = (V_M, E_M)$ where each edge $e_i \in M$ is represented by a vertex in $V_M$ and two vertices of $V_M$ are adjacent if the corresponding edges in $M$ are joined by an edge in $G$. A graph is an equitable bipartite graph if its vertex set can be partitioned into two independent sets $S_1$ and $S_2$ such that $|S_1| = |S_2|$, and in this case $(S_1, S_2)$ is called an equitable bipartition of $G$.

**Theorem 17.** Let $G$ be a connected cubic graph. Then $G$ has an exact doubly dominating set if and only if $G$ has a perfect matching $M$ such that the associated graph $G_M$ is an equitable bipartite graph.

**Proof.** Let $G$ be a connected cubic graph with an exact doubly dominating set $S$. So $S$ induces a 1-regular graph, whose edges form a matching $M_1$, and every vertex of $S$ has two neighbours in $V - S$. Since every vertex of $V - S$ has exactly two neighbours in $S$, the subgraph induced by $V - S$ is 1-regular, and its edges form a matching $M_2$. Thus $G$ admits a perfect matching $M = M_1 \cup M_2$. Each edge of $E - M$ joins a vertex of $S$ with a vertex of $V - S$, and the bipartite subgraph $(S, V - S; E - M)$ is 2-regular, so $|S| = |V - S|$, and so $|M_1| = |M_2|$. It follows that the graph $G_M$ associated with $M$ is an equitable bipartite graph with equitable bipartition $(M_1, M_2)$.

Conversely, let $M$ be a perfect matching of a connected cubic graph $G$ such that the associated graph $G_M$ is equitable bipartite, with equitable bipartition $(A, B)$. Let $A_M$ (resp. $B_M$) be the vertices of $G$ that are contained in the edges corresponding to the vertices of $A$ (resp. $B$). Since $A$ (resp. $B$)
is independent in $G_M$, the subgraph of $G$ induced by $A_M$ (resp. by $B_M$) is 1-regular. This also implies that every vertex of $A_M$ (resp. of $B_M$) has two neighbours in $B_M$ (resp. in $A_M$) since $G$ is a cubic graph. Consequently, $A_M$ and $B_M$ are two disjoint exact doubly dominating sets of $G$. This completes the proof.

References


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