Types of arcs in a fuzzy graph

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The concept of connectivity plays an important role in both theory and applications of fuzzy graphs. Depending on the strength of an arc, this paper classifies arcs of a fuzzy graph into three types namely a-strong, b-strong and d-arcs. The advantage of this type of classification is that it helps in understanding the basic structure of a fuzzy graph completely. We analyze the relation between strong paths and strongest paths in a fuzzy graph and obtain characterizations for fuzzy bridges, fuzzy trees and fuzzy cycles using the concept of a-strong, b-strong and d-arcs. An arc of a fuzzy tree is a-strong if and only if it is an arc of its unique maximum spanning tree. Also we identify different types of arcs in complete fuzzy graphs.

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1. Introduction

Fuzzy graphs were introduced by Rosenfeld [13], ten years after Zadeh’s landmark paper “Fuzzy Sets” [23]. Fuzzy graph theory is now finding numerous applications in modern science and technology especially in the fields of information theory, neural networks, expert systems, cluster analysis, medical diagnosis, control theory, etc. Fuzzy modeling is an essential tool in all branches of science, engineering and medicine. Fuzzy models give more precision, flexibility and compatibility to the system when compared to the classic models [24,25].

Rosenfeld has obtained the fuzzy analogues of several basic graph-theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties [13]. Bhattacharya [2] has established some connectivity concepts regarding fuzzy cutnodes and fuzzy bridges. The author has also introduced fuzzy groups and metric notion in fuzzy graphs. Bhutani [3] has studied automorphisms on fuzzy graphs and certain properties of complete fuzzy graphs. Bhattacharya and Suraweera have introduced an algorithm to find the connectivity of a pair of nodes in a fuzzy graph [1]. Also Saibal Banerjee [14] has obtained an optimal algorithm to find the degrees of connectedness in a fuzzy graph. Fuzzy intersection graphs were introduced by McAllister [8] and fuzzy line graphs by Mordeson [11]. The concept of domination in fuzzy graphs were studied by Somasundaram and Somasundaram [17,18], Fuzzy trees were characterized by Sunitha and Vijayakumar [19]. The authors have characterized fuzzy trees using its unique maximum spanning tree. A sufficient condition for a node to be a fuzzy cutnode is also established in [19]. Center problems in fuzzy graphs [21], blocks in fuzzy graphs [20] and properties of self complementary fuzzy graphs [22] were also studied by the same authors. They have obtained a characterization for blocks in fuzzy graphs using the concept of strongest paths [20]. Bhutani and Rosenfeld have introduced the concepts of strong arcs [4], fuzzy end nodes [5] and geodesics in fuzzy graphs [6]. In [4], the authors have defined the concepts of
strong arcs and strong paths. They have shown the existence of a strong path between any two nodes of a fuzzy graph and have studied the strong arcs of a fuzzy tree. In [5], the concepts of fuzzy end nodes and multimin and locamin cycles were studied. The concept of strong arc in maximum spanning trees [15] and its applications in cluster analysis and neural networks [16] were studied by Sameena and Sunitha.

The concept of strong fuzzy subgraphs in “Operations on fuzzy graphs” by Mordeson and Peng [12] and in “M-strong fuzzy graphs” by Bhutani and Battou [7] are the same. According to Mordeson and Peng a fuzzy subgraph (σ, μ) of \( G = (V, X) \) is called a strong fuzzy subgraph (or M-strong according to Bhutani and Battou) of \( G \) if \( \mu(u, v) = σ(u) ∧ σ(v) \) for all \( (u, v) ∈ X \). Also by Proposition 3.13 of [3], an arc \((x, y)\) in an M-strong fuzzy subgraph has the property that \( \mu(x, y) = CONN_C(x, y) \). The concept of strong arcs was introduced in “Strong arcs in fuzzy graphs” by Bhutani and Rosenfeld [4] in which an arc of a fuzzy graph is strong iff its weight is equal to the strength of connectedness of its end nodes. Thus every arc of an M-strong fuzzy graph is a strong arc in the sense of [4]. But the converse does not hold in general since the strength of a strong arc need not be equal to the minimum degree of its end nodes. Thus an M-strong arc is a strong arc and throughout this paper we use the notion of strong arcs in the sense of [4] which is more general.

In graph theory, arc analysis is not very important as all arcs are strong in the sense of [4]. But in fuzzy graphs it is very important to identify the nature of arcs and no such analysis on arcs is available in the literature except the division of arcs as strong and non strong in [4]. Depending on the strength of an arc, the authors classify strong arcs into two types namely \( α \)-strong and \( β \)-strong and introduce two other types of arcs in fuzzy graphs which are not strong and are termed as \( δ \)’- arcs. The analysis of types of arcs explores the structure of a fuzzy graph so that the concepts like fuzzy cutnode, fuzzy bridge, block, fuzzy trees, complete fuzzy graph etc. can be studied in detail. Also, as far as the applications are concerned, this classification highlights the importance of each arc, which will result in minimizing the cost and improving the efficiency of the system, especially in problems involving networks.

Section 2 contains preliminaries and in Section 3 we introduce the concept of \( α \)-strong, \( β \)-strong, \( δ \) and \( δ \)’- arcs (Definition 1–4). In this section we emphasis that the types of arcs cannot be determined simply by examining the weights of arcs (Examples 2 and 3). In Section 4 we examine the relationship between a strong path and a strongest path in a fuzzy graph (Proposition 1 and Remark 5). In Section 5 we show that an arc \((x, y)\) of \( G \) is a fuzzy bridge if and only if it is \( α \)-strong arc (Theorem 1). In Section 6 we analyze the types of arcs of a fuzzy tree and show that an arc of a fuzzy tree is \( α \)-strong if and only if it is an arc of its unique maximum spanning tree (Theorem 2). In this section we also show that a fuzzy graph is a fuzzy tree if and only if it has no \( β \)-strong arcs (Theorem 3). Also a fuzzy graph \( G \) is a fuzzy tree if and only if there exists a unique \( α \)-strong path between any two distinct nodes of \( G \) (Theorem 4). In Section 7 we discuss the types of arcs in a fuzzy cycle and is shown that a cycle is a fuzzy cycle if and only if it has at least two \( β \)-strong arcs (Theorem 6). Section 8 deals with the study of types of arcs in a complete fuzzy graph (CFG). It is seen that CFG has no \( δ \) arcs and has at most one \( α \)-strong arc (Lemma 1 and Lemma 2). The exact number of \( α \)-strong arcs in a CFG(Theorem 9) is given and the existence of a \( β \)-strong path between any two nodes of a CFG(Theorem 10) is established. We show that in a CFG without \( α \)-strong arcs the concepts of strong path and strongest path coincide (Theorem 11). In Section 9, an experiment based on the connectedness algorithm [14] is provided to identify the types of arcs in a fuzzy graph. Also an application of this experiment in clustering is mentioned.

2. Preliminaries

A fuzzy graph \((f\text{-graph})[13]\) is a pair \( G : (σ, μ) \) where \( σ \) is a fuzzy subset of a set \( S \) and \( μ \) is a fuzzy relation on \( σ \). We assume that \( S \) is finite and nonempty, \( μ \) is reflexive and symmetric [13]. In all the examples \( σ \) is chosen suitably. Also, we denote the underlying graph by \( G' : (σ', μ') \) where \( σ' = \{u ∈ S : σ(u) > 0\} \) and \( μ' = \{(u, v) ∈ S × S : μ(u, v) > 0\} \). A fuzzy graph \( H : (τ, ν) \) is called a partial fuzzy subgraph of \( G : (σ, μ) \) if \( τ(u) ≤ σ(u) \) for every \( u ∈ τ \) and \( ν(u, v) ≤ μ(u, v) \) for every \( (u, v) ∈ ν \). In particular we call a partial fuzzy sub graph \( H : (τ, ν) \) a fuzzy subgraph of \( G : (σ, μ) \) if \( τ(u) = σ(u) \) for every \( u ∈ τ \) and \( ν(u, v) = μ(u, v) \) for every \( (u, v) ∈ ν \). Now a fuzzy subgraph \( H : (τ, ν) \) spans the fuzzy graph \( G : (σ, μ) \) if \( τ = σ \). A connected \( f\text{-graph} \ G : (σ, μ) \) is a fuzzy tree (f-tree) if it has a fuzzy spanning subgraph \( F : (σ', ν) \), which is a tree, where for all arcs \((x, y)\) not in \( F \) there exists a path from \( x \) to \( y \) in \( F \) whose strength is more than \( μ(x, y) \) [13]. Note that here \( F \) is a tree which contains all nodes of \( G \) and hence is a spanning tree of \( G \). Also note that \( F \) is the unique maximum spanning(MST) of \( G \) [19]. A path \( P \) of length \( n \) is a sequence of distinct nodes \( u_0, u_1, \ldots, u_n \) such that \( μ(u_{i-1}, u_i) > 0, i = 1, 2, \ldots, n \) and the degree of membership of a weakest arc is defined as its strength. If \( u_0 = u_n \) and \( n ≥ 3 \), then \( P \) is called a cycle and a cycle \( P \) is called a fuzzy cycle (f-cycle) if it contains more than one weakest arc [10]. The strength of connectedness between two nodes \( x \) and \( y \) is defined as the maximum of the strengths of all paths between \( x \) and \( y \) and is denoted by \( CONN_C(x, y) \). An \( x \rightarrow y \) path \( P \) is called a strongest \( x \rightarrow y \) path if its strength equals \( CONN_C(x, y) \) [13]. An \( f\text{-graph} \ G : (σ, μ) \) is connected if for every \( x, y \) in \( σ \), \( CONN_C(x, y) > 0 \). Through out this, we assume that \( G \) is connected. An arc of a \( f\text{-graph} \) is called strong if its weight is at least as great as the connectedness of its end nodes when it is deleted and an \( x \rightarrow y \) path \( P \) is called a strong path if \( P \) contains only strong arcs [4].

An arc is called an \( f\text{-bridge} \) of \( G \) if its removal reduces the strength of connectedness between some pair of nodes in \( G \) [13]. Similarly an \( f\text{-cutnode} \) \( u \) is a node in \( G \) whose removal from \( G \) reduces the strength of connectedness between some pair of nodes other than \( u \). A complete fuzzy graph (CFG) is an \( f\text{-graph} \ G : (σ, μ) \) such that \( μ(x, y) = σ(x) ∧ σ(y) \) for all \( x \) and \( y \).
3. Types of arcs in a fuzzy graph

Depending on the $\text{CONN}_G(x, y)$ of an arc $(x, y)$ in a fuzzy graph $G$ we define the following three different types of arcs. Note that $\text{CONN}_G(x, y)$ is the strength of connectedness between $x$ and $y$ in the fuzzy graph obtained from $G$ by deleting the arc $(x, y)$.

**Definition 1.** An arc $(x, y)$ in $G$ is called $\alpha$-strong if $\mu(x, y) > \text{CONN}_G(x, y)$.

**Definition 2.** An arc $(x, y)$ in $G$ is called $\beta$-strong if $\mu(x, y) = \text{CONN}_G(x, y)$.

**Definition 3.** An arc $(x, y)$ in $G$ is called a $\delta$-arc if $\mu(x, y) < \text{CONN}_G(x, y)$.

**Remark 1.** A strong arc is either $\alpha$-strong or $\beta$-strong by Definition 1 and 2, respectively.

**Definition 4.** A $\delta$-arc $(x, y)$ is called a $\delta'$-arc if $\mu(x, y) > \mu(u, v)$ where $(u, v)$ is a weakest arc of $G$.

**Definition 5.** A path in an f-graph $G : (\sigma, \mu)$ is called an $\alpha$-strong path if all its arcs are $\alpha$-strong and is called a $\beta$-strong path if all its arcs are $\beta$-strong.

**Example 1.** Let $G : (\sigma, \mu)$ be with $\sigma^* = \{u, v, w, x\}$ and $\mu(u, v) = 0.2 = \mu(x, u), \mu(v, w) = 1 = \mu(w, x), \mu(v, x) = 0.3$. Here, $(\nu, w)$ and $(w, x)$ are $\alpha$-strong arcs, $(u, v)$ and $(x, u)$ are $\beta$-strong arcs and $(\nu, x)$ is a $\delta$-arc. Also $(\nu, x)$ is a $\delta'$-arc since $\mu(\nu, x) > \mu(u, v)$, where $(u, v)$ is a weakest arc of $G$. Here $P_1 = x, w, v$ is an $\alpha$-strong $x - v$ path whereas $P_2 = x, u, v$ is a $\beta$-strong $x - v$ path.

Note that in an f-graph $G$, the types of arcs cannot be determined by simply examining the arc weights; for, the membership value of a $\delta$-arc can exceed membership values of $\alpha$-strong and $\beta$-strong arcs. Also membership value of a $\beta$-strong arc can exceed that of an $\alpha$-strong arc as can be seen from the following examples.

(a) Membership value of $\delta$-arc exceeds membership value of $\beta$-strong arc.

In Example 1, $\mu(v, x) = 0.3 > 0.2 = \mu(u, v)$. Here, $(v, x)$ is a $\delta$-arc whereas $(u, v)$ is a $\beta$-strong.

(b) Membership value of $\delta$-arc exceeds membership value of $\alpha$-strong arc.

**Example 2.** Let $G : (\sigma, \mu)$ be with $\sigma^* = \{u, v, w, x\}$ and $\mu(u, v) = 1 = \mu(v, w), \mu(u, w) = 0.4, \mu(w, x) = 0.3, \mu(x, u) = 0.1$. Here, $(u, v), (v, w)$ and $(w, x)$ are $\alpha$-strong arcs whereas $(u, w)$ and $(x, u)$ are $\delta$-arcs with $\mu(u, w) = 0.4 > 0.3 = \mu(w, x)$.

(c) Membership value of $\beta$-arc exceeds membership value of $\alpha$-strong arc.

**Example 3.** Let $G : (\sigma, \mu)$ be with $\sigma^* = \{u, v, w, x\}$ and $\mu(u, v) = \mu(u, w) = \mu(v, w) = 1, \mu(w, x) = 0.5, \mu(x, u) = 0.1$. Here, $(u, v), (v, w), (u, w)$ are $\beta$-strong arcs whereas $(w, x)$ is $\alpha$-strong and $(x, u)$ is a $\delta$-arc with $\mu(u, w) = \mu(u, v) = \mu(v, w) = 1 > 0.5 = \mu(w, x)$.

4. Types of arcs in a strongest path

Now we shall discuss the types of arcs of a strongest path in $G$.

**Remark 2.** A strongest path may contain all types of arcs.

In Example 1, the strength of the path $P : u, v, x, w$ is 0.2, which is a strongest path from $u$ to $w$ and it contains all types of arcs, namely $(u, v)$ is $\beta$-strong, $(x, w)$ is $\alpha$-strong and $(v, x)$ is a $\delta$-arc.

**Remark 3.** As per Remark 1, a strong path contains only $\alpha$-strong and $\beta$-strong arcs but no $\delta$-arcs.

**Remark 4.** In a graph $G$, each path is strong as well as strongest. But in a fuzzy graph a strongest path need not be a strong path and a strong path need not be a strongest path[3]. In example 1, $P_1 : u, v, x, w$ is a strongest $u - w$ path, but not a strong $u - w$ path. Note that $P_2 : u, v, w$ and $P_3 : u, x, w$ are strong $u - w$ paths.

Conversely, $P_4 : v, u, x$ is a strong $v - x$ path which is not a strongest $v - x$ path and $P_5 : v, w, x$ is the strongest $v - x$ path.

We observe that these two concepts are equivalent in a CFG containing no $\alpha$-strong arcs (Theorem 11).

**Remark 5.** A strongest path without $\delta$-arcs is a strong path; for, it contains only $\alpha$-strong and $\beta$-strong arcs.
Proposition 1. A strong path $P$ from $x$ to $y$ is a strongest $x - y$ path in the following cases.

(i) If $P$ contains only $x$-strong arcs.
(ii) If $P$ is the unique strong $x - y$ path.
(iii) If all $x - y$ paths in $G$ are of equal strength.

Proof.

(i) Let $G : (\sigma, \mu)$ be an f-graph. Let $P$ be a strong $x - y$ path in $G$ containing only $x$-strong arcs. If possible suppose that $P$ is not a strongest $x - y$ path. Let $Q$ be a strongest $x - y$ path in $G$. Then $P \cup Q$ will contain at least one cycle $C$ in which every arc of $C - P$ will have strength greater than strength of $P$. Thus a weakest arc of $C$ is an arc of $P$ and let $(u, v)$ be such an arc of $C$. Let $C'$ be the $u - v$ path in $C$, not containing the arc $(u, v)$. Then,

$$\mu(u, v) \leq \text{strength of } C' \leq \text{CONN}_{C-(u,v)}(u, v)$$

which implies that $(u, v)$ is not $x$-strong, a contradiction. Thus $P$ is a strongest $x - y$ path.

(ii) Let $G : (\sigma, \mu)$ be an f-graph. Let $P$ be the unique strong $x - y$ path in $G$. If possible suppose that $P$ is not a strongest $x - y$ path. Let $Q$ be a strongest $x - y$ path in $G$. Then, strength of $Q >$ strength of $P$.

i.e. for every arc $(u, v)$ in $Q$, $\mu(u, v) > \mu(x', y')$ where $(x', y')$ is a weakest arc of $P$.

Claim. $Q$ is a strong $x - y$ path. For, otherwise, if there exists an arc $(u, v)$ in $Q$ which is a $\delta$-arc, then

$$\mu(u, v) < \text{CONN}_{C-(u,v)}(u, v) \leq \text{CONN}_{C}(u, v)$$

and hence $\mu(u, v) < \text{CONN}_{C}(u, v)$.

Then there exists a path from $u$ to $v$ in $G$ whose strength is greater than $\mu(u, v)$. Let it be $P'$. Let $w$ be the last node after $u$, common to $Q$ and $P'$ in the $u - w$ sub path of $P'$ and $w'$ be the first node before $v$, common to $Q$ and $P'$ in the $w' - v$ sub path of $P'$. (If $P'$ and $Q$ are disjoint $u - v$ paths then $w = u$ and $w' = v$). Then the path $P'$ consisting of the $x - w$ path of $Q$, $w - w'$ path of $P'$, and $w' - y$ path of $Q$ is an $x - y$ path in $G$ such that strength of $P'$ > strength of $Q$. Contradiction to the assumption that $Q$ is a strongest $x - y$ path in $G$. Thus $(u, v)$ cannot be a $\delta$-arc and hence $Q$ is a strong $x - y$ path in $G$.

Thus we have another strong path from $x$ to $y$, other than $P$, which is a contradiction to the assumption that $P$ is the unique strong $x - y$ path in $G$. Hence $P$ should be a strongest $x - y$ path in $G$.

(iii) If every path from $x$ to $y$ have the same strength, then each such path is strongest $x - y$ path. In particular a strongest $x - y$ path is a strongest $x - y$ path. □

We observe that if all arcs of an f-graph $G$ are $\beta$-strong, as in graphs without bridges, then each strongest path is a strong path but the converse need not be true. For, consider the f-graph $G : (\sigma, \mu)$ with $\sigma' = \{u, v, w, x, y\}$ and $\mu(u, v) = \mu(v, w) = \mu(w, x) = \mu(x, u) = 0.2, \mu(u, y) = \mu(y, w) = 0.1$. Here all arcs are $\beta$-strong and $u, y, w$ is a strong $u - w$ path but it is not a strongest $u - w$ path.

5. Characterization of fuzzy bridges in a fuzzy graph

Note that in a general f-graph, an f-bridge is strong and not conversely [4]. In the following theorem we present a necessary and sufficient condition for f-bridges.

Theorem 1 (Characterization of $f$-bridges in an $f$-graph). Let $G : (\sigma, \mu)$ be an $f$-graph. Then an arc $(x, y)$ of $G$ is an $f$-bridge if and only if it is $x$-strong.

Proof. Let $G : (\sigma, \mu)$ be an f-graph. Let $(x, y)$ be an f-bridge in $G$. Then by Theorem 9.1 of [13],

$$\text{CONN}_{G-(x,y)}(x, y) < \text{CONN}_{G}(x, y) \ldots \ldots (1)$$

By Theorem 4 of [19],

$$\text{CONN}_{G}(x, y) = \mu(x, y) \ldots \ldots (2)$$

From (1) and (2)

$$\mu(x, y) > \text{CONN}_{G-(x,y)}(x, y)$$

which shows that $(x, y)$ is $x$-strong.
Conversely suppose that \((x, y)\) is \(\alpha\)-strong. Then by definition, it follows that \((x, y)\) is the unique strongest path from \(x\) to \(y\) and the removal of \((x, y)\) will reduce the strength of connectedness between \(x\) and \(y\). Thus \((x, y)\) is a f-bridge. \(\square\)

Note that if an arc \((x, y)\) in \(G\) is an f-bridge, then \(\text{CONN}_C(x, y) = \mu(x, y)\). The converse need not be true \([19]\).

One of the characterizations for an f-bridge is that it is in every maximum spanning tree (MST) of \(G\) \([20]\). So we have,

**Corollary 1.** An arc \((x, y)\) in an f-graph \(G\) is \(\alpha\)-strong if and only if \((x, y)\) is in every MST of \(G\).

**Corollary 2.** Let \(G : (\sigma, \mu)\) be an f-graph with \(|\sigma'| = n\), then the number of \(\alpha\)-strong arcs in \(G\) is at most \(n - 1\).

**Remark 6.** If \(G\) is a fuzzy tree, then removal of any \(\alpha\)-strong arc reduces the strength of connectedness between its end nodes and also between some other pair of nodes (Theorem 9 of \([20]\)).

**Remark 7.** Note that the internal nodes of \(F\) are the f-cutnodes of f-tree \(G\) \([19]\) and hence in an f-tree, \(w\) is an f-cutnode if and only if \(w\) is a common node of at least two \(\alpha\)-strong arcs.

**Remark 8.** Also if \(w\) is a common node of at least two f-bridges then \(w\) is an f-cutnode \([19]\) and by Theorem 1 it follows that, if \(w\) is a common node of at least two \(\alpha\)-strong arcs in an f-graph \(G\), then \(w\) is an f-cutnode. But the converse is not true \([19]\). See Examples 4 and 5.

**Example 4.** Let \(G : (\sigma, \mu)\) be with \(\sigma' = \{u, v, w, x\}\) and \(\mu(u, v) = 0.4 = \mu(v, w) = \mu(u, w), \mu(x, u) = 0.3\) and \(\mu(w, x) = 0.7\). Then, \((w, x)\) is the only \(\alpha\)-strong arc, \((x, u)\) is a \(\delta\)-arc and \((u, v), (v, w)\) and \((u, w)\) are \(\beta\)-strong arcs. Also, \(w\) is an f-cutnode which is not a common node of at least two \(\alpha\)-strong arcs.

**Example 5.** Let \(G : (\sigma, \mu)\) be with \(\sigma' = \{u, v, w, x, y\}\) and \(\mu(u, v) = 0.1 = \mu(x, y), \mu(u, x) = \mu(v, y) = \mu(v, w) = \mu(u, w) = \mu(x, w) = \mu(y, w) = 0.3\). Here \(G\) has no \(\alpha\)-strong arcs. \((x, y)\) and \((u, v)\) are \(\delta\)-arcs and all other arcs are \(\beta\)-strong. Clearly, \(w\) is an f-cutnode and there are no \(\alpha\)-strong arcs incident on \(w\).

6. **Types of arcs in a fuzzy tree**

In an f-tree \(G\), an arc of \(G\) is strong if and only if it is an arc of \(F\) where \(F\) is the associated unique spanning tree of \(G\) \([4, 19]\). We identify that these strong arcs are \(\alpha\)-strong (Theorem 2) and there are no \(\beta\)-strong arcs in an f-tree (Theorem 3). Also compare Corollary 3 below and second corollary of proposition 4 of \([4]\).

**Theorem 2.** An arc \((x, y)\) in an f-tree \(G : (\sigma, \mu)\) is \(\alpha\)-strong if and only if \((x, y)\) is an arc of the spanning tree \(F : (\sigma, \nu)\) of \(G\).

**Proof.** Let \((x, y)\) be an \(\alpha\)-strong arc in \(G\). Then by definition 1,

\[
 \mu(x, y) > \text{CONN}_{C-\{x,y\}}(x, y) \ldots (1)
\]

Assume that \((x, y)\) is not in \(F\). Then by definition of an f-tree,

\[
 \text{CONN}_F(x, y) > \mu(x, y) \ldots (2)
\]

Now, By proposition 7.1 of \([13]\),

\[
 \text{CONN}_F(x, y) \leq \text{CONN}_{C-\{x,y\}}(x, y) \ldots (3)
\]

From (2) and (3) we get \(\mu(x, y) < \text{CONN}_{C-\{x,y\}}(x, y)\) which contradicts to (1). Hence \((x, y)\) is in \(F\).

Conversely, let \((x, y)\) be in \(F\). Then \((x, y)\) is an f-bridge \([13]\) and arc \((x, y)\) is the unique strongest \(x-y\) path \([19]\). Thus,

\[
 \text{CONN}_{C-\{x,y\}}(x, y) < \mu(x, y)
\]

which shows that \((x, y)\) is \(\alpha\)-strong. \(\square\)

**Corollary 3.** In an f-tree \(G : (\sigma, \mu)\), an arc \((x, y)\) is \(\alpha\)-strong if and only if \((x, y)\) is an f-bridge of \(G\).

Next we have a characterization of f-trees.

**Theorem 3.** An f-graph \(G\) is an f-tree if and only if it has no \(\beta\)-strong arcs.

**Proof.** Let \(G : (\sigma, \mu)\) be an f-tree and let \(F : (\sigma, \nu)\) be its maximum spanning tree. Now, all arcs in \(F\) are \(\alpha\)-strong by Theorem 2. Suppose \((x, y)\) is a \(\beta\)-strong arc in \(G\). Then \((x, y)\) is not in \(F\) and by definition of an f-tree,

\[
 \mu(x, y) < \text{CONN}_F(x, y) \ldots (1)
\]
Also by proposition 7.1 of [13],
\[ \text{CONN}_f(x, y) \subseteq \text{CONN}_{G-(x,y)}(x, y) \quad (2) \]

From (1) and (2),
\[ \mu(x, y) < \text{CONN}_{G-(x,y)}(x, y) \]
which implies that \((x, y)\) is a \(\delta\)-arc, which is a contradiction. Thus \(G\) contains no \(\beta\)-strong arcs.

Conversely, suppose that \(G\) has no \(\beta\)-strong arcs. If \(G\) has no cycles then \(G\) is an f-tree. Now assume that \(G\) has cycles. Let \(C\) be a cycle in \(G\). Then \(C\) will contain only \(\alpha\)-strong arcs and \(\delta\)-arcs. Also, all arcs of \(C\) cannot be \(\alpha\)-strong since otherwise it will contradict the definition of \(\alpha\)-strong arcs. Thus there exist at least one \(\delta\)-arc in \(C\). Then by Theorem 10.1 of [13] it follows that \(G\) is an f-tree.

\[ \square \]

Remark 9. An f-tree can have \(\delta\)-arcs as given in the following example.

Example 6. Let \(G : (\sigma, \mu)\) with \(\sigma = \{u, v, w, x\}\) and \(\mu(u, v) = 0.1, \mu(u, w) = 0.2, \mu(v, x) = 1 = \mu(w, x), \mu(v, x) = 0.3\). Then \(G\) is an f-tree with \((v, w), (w, x)\) and \((x, u)\) as \(\alpha\)-strong and \((v, x)\) and \((u, v)\) as \(\delta\)-arcs. Also \((v, x)\) is a \(\delta\)-arc, since \(\mu(v, x) > \mu(u, v)\), where \((u, v)\) is a weakest arc of \(G\).

Theorem 4. \(G\) is an f-tree if and only if there exists a unique \(\alpha\)-strong path between any two nodes in \(G\).

Proof. Assume \(G\) is an f-tree. Then \(G\) has a unique maximum spanning tree \(F\) [19]. By Theorem 2, an arc \((x, y)\) in \(G\) is \(\alpha\)-strong if and only if it is an arc of \(F\). Being a tree, \(F\) contains a unique path between any two nodes. But \(F\) contains all the nodes of \(G\). Thus there exists a unique \(\alpha\)-strong path between any two nodes of \(F\) and hence in \(G\).

Conversely, assume that there is a unique \(\alpha\)-strong path between any two nodes of \(G\). In an f-tree, a strong \(x - y\) path is a strongest \(x - y\) path[4] and hence it is the unique strongest \(x - y\) path in \(G\). Now by proposition 10.2 of [13], \(G\) is an f-tree.

Note that Theorem 4 also follows from proposition 5 of [4] and Theorem 3 above.

\(G\) is an f-tree iff \(G\) has a unique MST[19] and all arcs in the MST of an f-tree are \(\alpha\)-strong. In general, we have

Theorem 5. Let \(T\) be any spanning tree of an f-graph \(G\). Then \(T\) is an MST of \(G\) if and only if \(T\) contains no \(\delta\)-arcs. Further, an MST \(T\) is unique for \(G\) if and only if \(T\) contains no \(\beta\)-strong arcs.

Proof. First part follows from the definitions of \(\delta\) arc and MST[1] and the second part follows from the definition of \(\beta\)-strong arc, Theorem 3 above and Theorem 10 of [19].

Note that the strength of the unique \(x - y\) path in any MST of \(G\) gives \(\text{CONN}_G(x, y)\) [1] and it follows from the above theorem that there exists strong \(x - y\) path between any two nodes \(x\) and \(y\) in \(G\), as pointed out in the proposition 3 of [4].

7. Types of arcs in a fuzzy cycle

Next, the types of arcs in an f-cycle is discussed.

It is observed that there are no \(\delta\)-arcs in an f-cycle \(G\). For, if \((u, v)\) is a \(\delta\)-arc in \(G\), then it becomes the unique weakest arc of \(G\), which contradicts that \(G\) is an f-cycle. Also, an f-cycle cannot have all its arcs \(\alpha\)-strong, since the weakest arcs in the f-cycle cannot be \(\alpha\)-strong and note that these weakest arcs are \(\beta\)-strong arcs and all other arcs are \(\alpha\)-strong. This leads to the following theorem.

Theorem 6. Let \(G\) be an f-graph such that \(G^*\) is a cycle. Then \(G\) is an f-cycle if and only if \(G\) has at least two \(\beta\)-strong arcs.

Remark 10. Note that in an f-graph \(G\) such that \(G^*\) is a cycle, \(w\) is an f-cutnode if and only if it is a common node of at least two f-bridges [19] and using Theorem 1 we have,

Theorem 7. Let \(G\) be an f-graph such that \(G^*\) is a cycle. If \(G\) contains at most one \(\alpha\)-strong arc, then \(G\) has no f-cutnodes.

Remark 11. Converse of Theorem 7 is not true[4]. The condition for the converse to be true is discussed in the following theorem.

Theorem 8. If there is a unique strongest path between any two nodes \(x, y\) in an f-graph \(G\), then it is a strong \(x - y\) path.

Proof. By proposition 10.2 of [13] \(G\) becomes an f-tree and hence the unique strongest \(x - y\) path \(P\) is in \(F\). Now by Theorem 2, it follows that \(P\) is a strong \(x - y\) path. \(\square\)
8. Arcs in a complete fuzzy graph

**Lemma 1.** A complete fuzzy graph (CFG) has no $\delta$-arcs.

**Proof.** Let $G$ be a complete fuzzy graph. If possible assume that $G$ contains a $\delta$-arc, $(u, v)$. Then,

$$\mu(u, v) < \text{CONN}_{G-(u,v)}(u, v).$$

That is, there exists a stronger path $P$ other than the arc $(u, v)$ from $u$ to $v$ in $G$. Let $\mu(u, v) = p$ and the strength of the path $P$ be $q$. Then $p < q$. Let $w$ be the first node in $P$ after $u$. Then,

$$\mu(u, w) > p \ldots (1)$$

Similarly let $x$ be the last node in $P$ before $v$. Then,

$$\mu(x, v) > p \ldots (2)$$

Since $\mu(u, v) = p$, at least one of $\sigma(u)$ or $\sigma(v)$ should be $p$. Now $G$ being a CFG, (1) gives a contradiction if $\sigma(u) = p$ and (2) gives a contradiction if $\sigma(v) = p$; which completes the proof. $\square$

By Theorem 4 of [20], a CFG has at most one f-bridge and by Theorem 1, we have the lemma given below.

**Lemma 2.** There exists at most one $\alpha$-strong arc in a CFG.

Using Lemmas 1 and 2 we have the following two theorems.

**Theorem 9.** Let $G : (\sigma, \mu)$ be a CFG with $|\sigma^*| = n$. Then the number of $\beta$-strong arcs in $G$ is $^nC_2$ or $^nC_2 - 1$ where $^nC_2$ denotes the number of combinations of $n$ things taken two at a time given by the formula $^nC_2 = \frac{n!}{2!(n-2)!}$

**Theorem 10.** Let $G : (\sigma, \mu)$ be a CFG. Then there exist $\beta$-strong paths between any two nodes of $G$.

**Theorem 11.** Let $G$ be a CFG without $\alpha$-strong arcs and let $P$ be any $x - y$ path in $G$. Then the following two conditions are equivalent.

(i) $P$ is a strong $x - y$ path.
(ii) $P$ is a strongest $x - y$ path.

**Proof.** (i) $\Rightarrow$ (ii)

Let $G$ be a CFG without $\alpha$-strong arcs and let $P$ be any $x - y$ path in $G$. Assume that $P$ is a strong $x - y$ path. By Lemma 1, all arcs of $G$ are $\beta$-strong. Then by definition,

$$\text{CONN}_{G-(x,y)}(x,y) = \mu(x,y) = \text{strength of } P \ldots (1)$$

Now, since $G$ is complete,

$$\text{CONN}_G(x,y) = \mu(x,y) \ldots (2)$$

From (1) and (2),

$$\text{CONN}_G(x,y) = \text{strength of } P,$$

which implies that $P$ is a strongest $x - y$ path.

(ii) $\Rightarrow$ (i)

Let $P$ be a strongest $x - y$ path in $G$. By Lemma 1, $P$ contains only $\beta$-strong arcs and hence is a strong $x - y$ path, which completes the proof. $\square$

**Remark 12.** Converse of this result does not hold generally as illustrated in the following example.

**Example 7.** Let $G : (\sigma, \mu)$ with $\sigma^* = \{u, v, w, x\}$ and $\sigma(u) = 0.5, \sigma(v) = 0.4, \sigma(w) = 0.7, \sigma(x) = 0.5$ and $\mu(u,v) = \mu(u,w) = \mu(w,x) = \mu(x,u) = \mu(v,x) = \mu(u,w) = 0.1$. In $G$, all arcs are $\beta$-strong and all strong paths are strongest paths but $G$ is not a CFG.

9. Experimental part and application

In this section we present a procedure to identify the arcs and there by comprehend certain properties of fuzzy graphs. Also one of the applications of this method is discussed.
9.1. Arc analysis experiment

Banerjee [14] introduced an effective optimal algorithm to find the strength of connectedness between any two nodes of a fuzzy graph. This can be used to identify different types of arcs in an f-graph as follows.

Consider a fuzzy graph \( G : (\sigma, \mu) \). Let \( e = (u, v) \) be an arc in \( G \) such that \( \mu(u, v) = t > 0 \). Then do the following steps.

1. Obtain \( G - e \)
2. Apply the algorithm in [14] to find the strength of connectedness between \( u \) and \( v \). Let \( \text{CONNG}_{e-}(u, v) = t' \) and \( t'' \) be the strength of a weakest arc in \( G \).

3. (a) If \( t > t' \) then \( (u, v) \) is \( \alpha \)-strong.
    (b) If \( t = t' \) then \( (u, v) \) is \( \beta \)-strong.
    (c) If \( t < t' \) then \( (u, v) \) is a \( \delta \)-arc and \( (u, v) \) is a \( \delta^* \)-arc if \( t > t'' \).

4. Repeat steps 1–4 for all arcs in \( G \).

Using the above procedure, one may arrive at the following conclusions:

1. If \( G \) has no \( \beta \)-strong arcs, then \( G \) is a fuzzy tree.
2. Fuzzy bridges of \( G \) are precisely the \( \alpha \)-strong arcs in \( G \).
3. A path in \( G \) is a strong path if it contains no \( \delta \)-arcs.
4. A node common to two \( \alpha \)-strong arcs is a fuzzy cutnode.

**Illustration**

Consider a fuzzy graph \( G : (\sigma, \mu) \) where \( \sigma^* = \{u, v, w, x\} \) and \( \mu(u, v) = 1, \mu(v, w) = .1, \mu(w, x) = .5, \mu(x, u) = .2, \mu(u, w) = .8 \).

The output of the above experiment in \( G \) is:

1. Arcs \( (u, v), (w, x) \) and \( (u, w) \) are \( \alpha \)-strong.
2. Arc \( (v, w) \) is a \( \delta \)-arc, which is not a \( \delta^* \)-arc.
3. Arc \( (x, u) \) is a \( \delta^* \)-arc.

**Conclusions**

1. Given fuzzy graph is a fuzzy tree.
2. Arc \( (u, v), (w, x) \) and \( (u, w) \) are fuzzy bridges of \( G \).
3. The path \( vuwx \) is the longest strong path in \( G \).
4. \( u \) and \( w \) are fuzzy cutnodes in \( G \).

In [16] Sameena and Sunitha have discussed different clustering techniques using strong arcs. The first step in these techniques is to identify whether an arc is strong or not. When the given fuzzy graph is small with respect to the number of nodes and arcs, this is not difficult. But this is not the case when the number of nodes and arcs increase in which case arc analysis experiment is useful.

10. Conclusion

There exist no fuzzy graph theoretic generalizations for many of the fundamental results in graph theory. They include the characterizations of cutnodes, bridges and blocks. This arc analysis may help in studying the possibility of such generalizations. Even though it seems fundamental at a glance, the process of identification of arcs and hence that of fuzzy trees and complete fuzzy graphs are more demanding and time consuming as mentioned in Section 3. This approach does not seem to have been previously considered in the literature except in [4] to introduce strong arcs. Some of the results in Section 6 can be generalized to fuzzy forests also.

We have introduced four types of arcs in fuzzy graphs to analyze its structure. A complete analysis of types of arcs in fuzzy trees, complete fuzzy graphs, fuzzy cycles and strongest paths is presented. This analysis simplifies the problem of understanding the connectivity structure of a fuzzy graph and the problem of identification of fuzzy cutnodes and fuzzy bridges in fuzzy graphs. Also the concepts of \( \alpha \)-strong and \( \beta \)-strong paths characterize fuzzy trees and complete fuzzy graphs. The number of \( \beta \)-strong arcs in a CFG and the equivalence of strong and strongest paths in a CFG without \( \alpha \)-strong arcs are also discussed in the last section.

Currently, we are studying the properties of blocks in fuzzy graphs and hope that this arc analysis will be very crucial in the identification of blocks. Also this classification can be useful in the input–output analysis of networks especially in fuzzy
neural networks. We are looking forward for an effective algorithm which identifies fuzzy cutnodes, fuzzy bridges, fuzzy forests and blocks in fuzzy graphs.

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**References**