Constructive points of Powerlocales

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October 1995

Abstract

Results of Bunge and Funk and of Johnstone, providing constructively sound descriptions of the global points of the lower and upper powerlocales, are extended here to describe the generalized points and proved in a way that displays in a symmetric fashion two complementary treatments of frames: as suplattices and as preframes. We then also describe the points of the Vietoris powerlocale.

In each of two special cases, an exponential $\$^D$ (\$ being the Sierpinsky locale) is shown to be homeomorphic to a powerlocale: to the lower powerlocale when D is discrete, and to the upper powerlocale when D is compact regular.

1 Introduction

The aim of this paper is to investigate the points of the lower, upper and Vietoris powerlocales $P_L D$, $P_U D$ and VD, for a locale D. (We shall use the term "point" in a generalized sense: a point of a locale D is a map targeted on D. A point in the narrower sense, a map from 1 to D, we shall call a *global* point.)

The paper falls into two somewhat separate parts. The first describes the points of the three powerlocales of D in complete generality, as sublocales, and the second describes the points in two special cases — of $P_L D$ when D is discrete and of $P_U D$ when D is compact regular — by showing a homeomorphism with $\$^D$.

Let us first briefly summarize our basic language.

Definition 1.1

1. *A* suplattice *(Joyal and Tierney [10]) is a complete join semilattice. A homomorphism between suplattices is a function that preserves all joins.*

2. *A* preframe *(Banaschewski [2]; but see mainly Johnstone and Vickers [9]) is a directed complete partial order (dcpo) with all finite meets, such that binary meet distributes over directed join. A homomorphism between preframes is a function that preserves directed joins and finite meets. We shall use the notation "$\bigvee^{\uparrow}$" to indicate that a join is directed.*

3. *A* frame *is a complete lattice in which binary meet distributes over arbitrary joins. (The main reference is Johnstone [5], but we shall largely follow the notation of Vickers [13] and*

in particular we shall make heavy use of the technique of presenting frames by generators and relations.) A homomorphism between frames is a function that preserves arbitrary joins and finite meets — so it is both a suplattice homomorphism and a preframe homomorphism.

4. *A* locale *D is equipped with a frame ΩD. A* map *(or* continuous *map) between locales ($f : D \to E$) is a frame homomorphism $\Omega f : \Omega E \to \Omega D$. A* point *of D is a map targeted on D; the source of the map is the* stage of definition *of the point, and the point is* global *iff its stage of definition is the terminal locale 1 (which is defined by $\Omega 1 = \Omega = \wp 1$, the subobject classifier), so a global point of D is a frame homomorphism from ΩD to Ω. An* open *of D is an element of ΩD. These are equivalent to maps from D to the* Sierpinsky locale *\$, defined by $\Omega\$ =$ free frame on one generator.*

5. *A* sublocale *of D is a regular subobject. Various ways of describing sublocales are discussed in the standard texts; we shall frequently view them as sets of extra relations imposed on ΩD as constraints narrowing the range of points. If E is a sublocale of D, given by regular monic $i : E \to D$, and a and b are opens of D, then we shall write "$a \leq b (\mathrm{mod}\ E)$" to mean that $\Omega i(a) \leq \Omega i(b)$.*

6. *We shall normally identify an open a (element of ΩD) with its corresponding open sublocale (often written $u(a)$), which is presented by the extra relation $\mathbf{true} \leq a$. We have $b \leq c(\mathrm{mod}\ a)$ iff $b \wedge a \leq c$. For a general sublocale E, we have that $a \leq b(\mathrm{mod}\ E)$ iff, in the lattice of sublocales, $a \wedge E \leq b$.*

7. *We shall write $D - a$ for the corresponding closed sublocale, often written $c(a)$, which is presented by $a \leq \mathbf{false}$. We have $b \leq c(\mathrm{mod}\ D - a)$ iff $b \leq c \vee a$.*

8. *Some miscellaneous notation: We write $! : D \to 1$ for the unique locale map, so that $\Omega! : \Omega \to \Omega D$ is the unique frame homomorphism. If $f : D \to E$ is a locale map, then we write $\forall_f$ for the right adjoint of Ωf and $\exists_f$ for the left adjoint (if it exists). Note that if $\exists_f$ exists, then it is necessarily a suplattice homomorphism. $\forall_f$ always exists, and preserves all meets, but need not preserve directed joins. When it does then it is a preframe homomorphism.*

Definition 1.2 *Let D be a locale. The* lower *powerlocale, $P_L D$ is defined by:*

$$\Omega P_L D = Fr < \Diamond a (a \in \Omega D) \mid \Diamond \text{ preserves all joins} >$$

In other words, $\Omega P_L D$ is the free frame over ΩD qua suplattice.

The upper *powerlocale, $P_U D$, is defined by:*

$$\Omega P_U D = Fr < \Box a (a \in \Omega D) \mid \Box \text{ preserves finite meets and directed joins} >$$

In other words, $\Omega P_U D$ is the free frame over ΩD qua preframe.

The Vietoris *powerlocale, VD, (Johnstone [7]) is defined by:*

$$\Omega VD = Fr < \Diamond a, \Box a (a \in \Omega D) \mid \begin{array}{l} \Diamond \text{ preserves all joins,} \\ \Box \text{ preserves finite meets and directed joins} \\ \Box a \wedge \Diamond b \leq \Diamond (a \wedge b) \\ \Box (a \vee b) \leq \Box a \vee \Diamond b > \end{array}$$

The lower and upper powerlocales are known in computer science as the localic forms of — respectively — the Hoare and Smyth powerdomains. The Vietoris powerlocale VD was introduced and comprehensively treated in Johnstone [7], and it corresponds to the convex, or Plotkin powerdomain. Notice that we do not follow the widespread computer science practice of "excluding the empty set" by imposing relations $\Box\mathbf{false} \leq \mathbf{false}$ or $\mathbf{true} \leq \Diamond\mathbf{true}$.

In the second part of the paper we show how in some special cases we have results relating the powerlocale points to opens. If D is discrete, then the exponential $\$^D$ exists (of course, so much is very well-known already, for a discrete locale is locally compact and hence exponentiable) and is homeomorphic to $P_L D$. This essentially says that the points of $P_L D$ are the opens of D, for the opens of D are the maps from D to \$. Similarly, if D is compact regular then $\$^D$ is homeomorphic to $P_U D$. Meanwhile, we shall continue this Introduction by summarizing the established results on global points of powerlocales.

1.1 The lower powerlocale P_L

Turning first to the lower powerlocale $P_L D$, a global point (which by definition is equivalent to a suplattice homomorphism X from ΩD to Ω) is *classically* equivalent to an open of D, namely $a = \bigvee\{b : X(b) = \mathbf{false}\}$. Then $X(b) = \mathbf{true} \iff b \not\leq a$. This sets up the bijection. A large point X (*i.e.,* lots of b's with $X(b) = \mathbf{true}$) corresponds to a small a, so the bijection is order-reversing and to counter this it is usual to identify the point with the closed sublocale $D - a$. Then

$$X(b) = \mathbf{true} \iff (D - a) \wedge b \neq \varnothing \text{ (in the lattice of sublocales)}$$
$$\iff b \neq \mathbf{false}(\text{mod } D - a)$$

Unfortunately, this simple argument relies heavily on classical principles and is constructively (*i.e.,* in a general topos) unsound. Bunge and Funk [3] identified the global points constructively with the *weakly closed* sublocales *with open domain.* To motivate this result, let us quickly point out some constructive flaws in the classical reasoning.

First, given a suplattice homomorphism $X : \Omega D \to \Omega$, consider the relations $b \leq \Omega! \circ X(b)(b \in \Omega D)$. Classically, each of these is either trivial (when $X(b) = \mathbf{true}$) or presents the closed sublocale $D - b$ (when $X(b) = \mathbf{false}$). Either way, it presents a closed sublocale, so taking all the relations together gives a meet of the closed sublocales, which is again closed — it's $D - \bigvee\{b : X(b) = \mathbf{false}\}$.

Constructively, however, we may have other values for $X(b)$, giving non-closed sublocales. It therefore seems that the suplattice homomorphism X in general contain more information than is naturally expressed through closed sublocales: so we generalize to *weakly closed* sublocales, presented by relations $b \leq \Omega!(p)(p \in \Omega)$.

Next, from a closed sublocale $D - a$, the function $X : \Omega D \to \Omega$ corresponds to a subset of ΩD, namely $\{b : b \not\leq a\}$. But constructively, this subset does not in general give an X that is a suplattice homomorphism — because from $\bigvee_i b_i \not\leq a$ we can't deduce that $b_i \not\leq a$ for some i. The problem here is with the non-classical negation. Since $b \not\leq a$ iff $b \neq \mathbf{false}(\text{mod } D - a)$, one can address the problem through a more careful analysis of "non-emptiness" of opens.

Definition 1.3 *A sublocale E of a locale D is* weakly closed *iff ΩE can be presented over ΩD by a set of relations of the form $a \leq \Omega!(p)(a \in \Omega D, p \in \Omega)$.*

Classically, the relations described in Definition 1.3 are either $a \leq \mathbf{true}$, which can be omitted, or $a \leq \mathbf{false}$, which presents the closed sublocale $D - a$. It follows that classically weak closedness is equivalent to closedness.

This is not actually the definition to which Bunge and Funk work. Instead, following Johnstone [8], they make a definition out of what appears below as Proposition 1.5 (2).

Definition 1.4 *A continuous map $f : D \to E$ is* strongly dense *(Johnstone [8]) iff f is dense under pullback along every closed sublocale of 1.*

Proposition 1.5

1. *A continuous map $f : D \to E$ is strongly dense iff*
   $\forall p \in \Omega.\forall a \in \Omega E.(\Omega f(a) \leq \Omega!(p) \Rightarrow a \leq \Omega!(p))$

2. *A sublocale E of a locale D is weakly closed iff every strongly dense inclusion $E \to E'$, E' also a sublocale of D, is a homeomorphism.*

Proof

1. The closed sublocales of 1 are those of the form $1 - p, p \in \Omega$, and the pullbacks referred to in the definition are the sublocales $D - \Omega!(p)$ and $E - \Omega!(p)$ of D and E. The pulled back map is dense iff $\forall a \in \Omega E.(\Omega f(a) \leq \Omega!(p) \Rightarrow a \leq \Omega!(p))$.

2. $\Rightarrow$: Let $E \to E' \to D$ be sublocale inclusions, with $E \to E'$ strongly dense. By strong density, every presenting relation for ΩE also holds in $\Omega E'$, and it follows that $E \to E'$ is a homeomorphism.

   $\Leftarrow$: Let E', a sublocale of D, have $\Omega E'$ presented over ΩD by all relations of the form $a \leq \Omega!(p)$ where $a \in \Omega D, p \in \Omega$ and $a \leq \Omega!(p)(\mathrm{mod}\ E)$. Then $E \to E'$ is strongly dense and hence a homeomorphism, so ΩE can be presented by those used for $\Omega E'$, which are all of the required form.

□

We now address the other point, about "non-emptiness" of opens.

Definition 1.6 *(Johnstone [6]) Let D be a locale.*

1. *An open $a \in \Omega D$ is* positive *iff whenever $a \leq \bigvee S (S \subseteq \Omega D)$ then S is inhabited.*
2. *D is* open *iff every open $a \in \Omega D$ is a join of positive opens.*

Classically, $S \subseteq \Omega D$ is either empty or inhabited, and it follows that a is positive iff $a \neq$ **false**. If $a =$ **false** then $a = \bigvee \varnothing$, and otherwise $a = \bigvee\{a\}$ is a join of positives. Hence classically *every* locale is open.

Proposition 1.7 *A locale D is open iff $\Omega! : \Omega \to \Omega D$ has a left adjoint $\exists_!$. Moreover, if these hold we then have for each $a \in \Omega D, \exists_! a$ is the truth value of "a is positive".*

Proof Johnstone [6]. □

(Following Joyal and Tierney [10], Proposition 1.7 is normally taken as the definition of openness. Then Definition 1.6 (2) appears in Johnstone [6] as a Proposition. Johnstone also proves the useful result that if D is open, then for any $S \subseteq \Omega D$ we have $\bigvee S = \bigvee\{a \in S : a \text{ positive}\}$.)

If D' is a sublocale of D, then we say that D' *has open domain* iff D' is an open locale. It is important to realize that this is quite different from saying that D' is an open sublocale of D.

Theorem 1.8 *(Bunge and Funk [3])*

Let D be a locale. Then there is a bijection between the global points of $P_L D$ and weakly closed sublocales of D with open domain.

Proof Bunge and Funk prove this in the course of a more general result describing arbitrary weakly closed sublocales. A simple direct proof can be derived from our proof of 2.3. □

1.2 The upper powerlocale P_U

The global points of $P_U D$ are, immediately from the definition, equivalent to preframe homomorphisms from ΩD to Ω, or to Scott open filters in ΩD. The Hofmann-Mislove [4] theorem then shows an order reversing bijection between these and the *compact saturated* sets of global points of D by which a Scott open filter F corresponds to $\bigcap\{\text{extent}(b) : b \in F\}$. (A *saturated* subset is one that is upper closed in the specialization preorder, and that is equivalent — classically — to being an intersection of open subsets.) Actually, Hofmann and Mislove assumed spatiality for D, but their proof is localic and it is not hard to see (Vickers [13]) that it holds for arbitrary locales — indeed, it can be naturally used in proving a number of standard spatiality results.

The Hofmann-Mislove result relies on the axiom of choice (or, more precisely, the prime ideal theorem) and is constructively unsound. Johnstone [7] proved a different result, showing an order reversing bijection between global points of $P_U D$ and compact fitted sublocales of D. (*Fitted* is the localic analogue of *saturated*: a sublocale is fitted iff it is a meet of opens, in other words it can be presented by a set of relations of the form $\mathbf{true} \leq b$.)

Recall that we are writing $\forall_!$ for the right adjoint of $\Omega!$, the unique frame homomorphism from the initial frame Ω. Clearly $\forall_!(a) = \mathbf{true}$ iff $a = \mathbf{true}$, so $\forall_!(a)$ is the truth value $[a = \mathbf{true}]$. It follows that the locale D is compact iff $\forall_!$ preserves directed joins, and hence is a preframe homomorphism.

Note that a sublocale is fitted iff it can be presented by a set of relations of the form $\Omega!(p) \leq a (a \in \Omega D, p \in \Omega)$. For $\Omega!(p) = \bigvee\{\mathbf{true} : p\}$, so the relation $\Omega!(p) \leq a$ is equivalent to the set of relations $\{\mathbf{true} \leq a : p\}$.

Theorem 1.9 *(Johnstone [7])*

Let D be a locale. Then there is a bijection between global points of $P_U D$ and compact fitted sublocales of D.

Proof Johnstone's proof uses ordinal-indexed chains, but the result is constructively sound. A rather simpler proof based on the preframe methods of Johnstone and Vickers [9] is easily derived from our proof of Theorem 3.3. □

1.3 The Vietoris powerlocale V

For the Vietoris powerlocale VD, there are classical results identifying the points with subspaces of D only in special cases — for instance (Johnstone [7]) if D is stably locally compact. Johnstone [7] identified them for a general locale with the compact, semifitted sublocales of D. (A sublocale is *semifitted* iff it is the meet of a fitted sublocale and a closed sublocale.) Johnstone himself pointed out that the result is constructively unsound and that more careful consideration had to be given to the openness of locales (as in Joyal and Tierney [10]). In fact the problems are essentially those that arise with the lower powerlocale, and we shall prove that there is a bijection between points of VD and *weakly semifitted* (meet of fitted with weakly closed) sublocales with compact open domain.

1.4 Coverage Theorems

Throughout the paper we shall make heavy use of two "coverage theorems", so called because they arise out of considerations of Johnstone's [5] original coverage theorem. This states that if a meet semilattice S is equipped with a *coverage* C, a relation between $\wp S$ and S satisfying certain conditions, then the so-called *C-ideals* of S form a frame with a certain universal property in the category of frames that can be conveniently described as a frame presentation by generators and relations.

A more refined analysis in Abramsky and Vickers [1] — which we refer to as the *suplattice coverage theorem* — shows that the *suplattice* of C-ideals also has a universal property in the category of suplattices that can be conveniently described as a *suplattice* presentation. Hence it shows how to translate frame presentations into suplattice presentations. The preframe coverage theorem (Johnstone and Vickers [9]) is an analogous result for getting preframe presentations.

The importance of these results lies in the fact that maps from D to $P_L E$ or $P_U E$ are equivalent to suplattice or preframe homomorphisms from ΩE to ΩD.

Theorem 1.10 The Suplattice Coverage Theorem.

Let S be a $\wedge$-semilattice, and let C ("covers") be a relation from $\wp S$ to S such that if X covers u then (the "coverage condition"):

- *if $x \in X$ then $x \leq u$*
- *if $a \in S$ then $\{x \wedge a : x \in X\}$ covers $u \wedge a$*

Then

$Fr < S(qua\ \wedge\text{-}semilattice) \mid u \leq \bigvee X \quad (X\ covers\ u) >$

$\cong SupLat < S(qua\ poset) \mid u \leq \bigvee X \quad (X\ covers\ u) >$

("Qua $\wedge$-semilattice" means that the injection of generators, a function from S to the frame, is to be a $\wedge$-semilattice homomorphism. This can be achieved by adding extra relations to the presentation. "Qua poset" is similar.)

Proof The proof is given in Abramsky and Vickers [1] and what we must note here is that it is constructive. The first step to show that

$\text{SupLat} < S(\text{qua poset}) \mid u \leq \bigvee X \quad (X \text{ covers } u) >$

exists. This can be done by explicit construction as Johnstone's [5] $C - Idl(S)$, or by the standard methods of universal algebra that first construct the free suplattice over S (which is $\wp S$ by Joyal and Tierney [10]) and then factor out an appropriate suplattice congruence. Both methods are constructive. Once we have this suplattice, it is simple and constructive to use its universal properties to describe its frame structure and to prove that it is $\text{Fr} < S(\text{qua } \wedge\text{-semilattice}) \mid u \leq \bigvee X \quad (X \text{ covers } u) >$. □

Theorem 1.11 The Preframe Coverage Theorem.

Let S be a $\vee$-semilattice, and let C ("covers") be a relation from $\wp \mathcal{F} S$ to $\mathcal{F} S$ such that if X covers G then:

- *if $F \in X$ then $F \leq_U G$*
- *X is inhabited*
- *if $F_1, F_2 \in X$ then there is some $F \in X$ with $F_i \leq_U F (i = 1, 2)$*
- *if $a \in S$ then $\{\{x \vee a : x \in F\} : F \in X\}$ covers $\{y \vee a : y \in G\}$*

(Here $\mathcal{F} S$ is the finite powerset of S, and if $F, G \in \mathcal{F} S$ then we write $F \leq_U G$ to mean $\forall y \in G. \exists x \in F. x \leq y$.)

Then

$Fr < S(qua\ \vee\text{-}semilattice) \mid \bigwedge G \leq \bigvee^{\uparrow}_{F \in X} \bigwedge F \quad (X\ covers\ G) >$

$\cong PreFr < S(qua\ poset) \mid \bigwedge G \leq \bigvee^{\uparrow}_{F \in X} \bigwedge F \quad (X\ covers\ G) >$

Note that the middle two coverage conditions ensure that the joins "$\bigvee^{\uparrow}$" are directed.

Proof The proof is given in Johnstone and Vickers [9], and again we must note its constructive validity. Once the preframe presented in the statement is known to exist, it is fairly easy to show — much as in the suplattice case — that it is a frame and that it can be presented as stated. However, the existence of the preframe is by no means easy. We do not have a concrete description analogous to Johnstone's $C - Idl(S)$, and the standard techniques of universal algebra cannot be applied straightforwardly. In fact, the main result of "Preframe Presentations Present" (its Proposition 3.2) is that any presentation of a preframe by generators and relations does indeed present a preframe. The proof as given is constructive; it also relies on a constructively valid result of Banaschewski [2]. □

Though the frame presentations described in these two theorems look special, in fact any frame presentation can be manipulated into equivalent ones in these two forms. Hence, the theorems provide general techniques for converting frame presentations into suplattice presentations and preframe presentations, and hence for defining suplattice homomorphisms and preframe homomorphisms out of frames. Moreover, though we shan't use this here, it can be used to show how the powerlocales can be constructed by geometric (*i.e.,* stable under inverse image parts of geometric morphisms) manipulation of frame presentations.

2 The lower powerlocale

We are interested in the general points of $P_L D$, at stage of definition E — *i.e.,* the maps from E to $P_L D$. By analogy with functions from X to $\wp Y$, which are equivalent to subsets of $X \times Y$, we might hope for such points to be equivalent to certain sublocales D' of $E \times D$, and indeed this is the case. However, we also need D' to be "weakly closed with open domain" *over* E and so we must first define these notions.

Definition 2.1 *Let $f : D \to E$ be a map of locales.*

1. *D is* open over *E iff f is an open map. In other words (Joyal and Tierney [10]), Ωf has a left adjoint $\exists_f$ that satisfies the* Frobenius identity

   $\exists_f(a \wedge \Omega f(b)) = \exists_f a \wedge b$

2. *A sublocale D' of D is* weakly closed (in D) over E *iff its frame can be presented over ΩD by relations of the form $a \leq \Omega f(b)(a \in \Omega D, b \in \Omega E)$. We also say that D'* has open domain over E *if it is open over E.*

3. *The* weak closure *of D' in D over E is the sublocale of D presented by all the relations $a \leq \Omega f(b)(a \in \Omega D, b \in \Omega E)$ that hold modulo D'.*

Lemma 2.2 *Let D and E be locales. We write $p : E \times D \to E$ and $q : E \times D \to D$ for the projections. Let $i : D' \to E \times D$ be a sublocale that is open over E, and let X be the suplattice homomorphism $\Omega(i;q); \exists_{i;p} : \Omega D \to \Omega E$. Then the weak closure of D' in $E \times D$ over E can be equivalently be presented by the relations $\mathbf{true} \otimes b \leq X(b) \otimes \mathbf{true} \quad (b \in \Omega D)$.*

Proof Because the elements $c \otimes b$ form a base of opens of $E \times D$, the weak closure is presented by the relations $c \otimes b \leq a \otimes \mathbf{true}$ that hold modulo D'. But

$c \otimes b \leq a \otimes \mathbf{true}(\text{mod } D') \iff \Omega(i;p)(c) \wedge \Omega(i;q)(b) \leq \Omega(i;p)(a) \iff \exists_{i;p}(\Omega(i;p)(c) \wedge \Omega(i;q)(b)) = c \wedge \exists_{i;p} \circ \Omega(i;q)(b) = c \wedge X(b) \leq a$

The relations $\mathbf{true} \otimes b \leq X(b) \otimes \mathbf{true}$ are the special case when $c = \mathbf{true}$ and $a = X(b)$. Now if $c \wedge X(b) \leq a$, then from $\mathbf{true} \otimes b \leq X(b) \otimes \mathbf{true}$ we deduce that $c \otimes b \leq (c \wedge X(b)) \otimes \mathbf{true} \leq a \otimes \mathbf{true}$. □

Theorem 2.3 *Let D and E be locales. Then there is an order isomorphism, natural in E, between the points of $P_L D$ at stage E and the sublocales of $E \times D$ that, over E, are weakly closed with open domain.*

Proof As in 2.2, we write p and q for the projections from $E \times D$.

If $i : D' \to E \times D$ is, over E, a weakly closed sublocale with open domain (indeed, any sublocale with open domain over E), then the function X as in 2.2 is a point of $P_L D$ at stage E. D' is its own weak closure in $E \times D$ over E, and hence by Lemma 2.2 is presented by the relations $\mathbf{true} \otimes b \leq X(b) \otimes \mathbf{true}$.

On the other hand, if we have a point of $P_L D$, given by a suplattice homomorphism $X : \Omega D \to \Omega E$, then we can present a sublocale $i : D' \to E \times D$, weakly closed over E, by the relations $\mathbf{true} \otimes b \leq X(b) \otimes \mathbf{true}$. To show that D' has open domain over E, we define a suplattice homomorphism $\exists_{i;p} : \Omega D' \to \Omega E$ by $\exists_{i;p} \circ \Omega i(a \otimes b) = a \wedge X(b)$. *If* this is possible, then $\exists_{i;p}$ is the left adjoint to $\Omega(i;p)$ with the Frobenius condition holding. For

$\Omega i(a \otimes b) \leq \Omega i(a \wedge X(b) \otimes \mathbf{true})$ by definition of D'

$= \Omega(i;p) \circ \exists_{i;p} \circ \Omega i(a \otimes b)$

$\exists_{i;p} \circ \Omega i(a \otimes \mathbf{true}) = a \wedge X(\mathbf{true}) \leq a$

For the Frobenius identity, we have

$\exists_{i;p}(\Omega(i;p)(a) \wedge \Omega i(c \otimes b)) = \exists_{i;p} \circ \Omega i((a \wedge c) \otimes b) = a \wedge c \wedge X(b) = a \wedge \exists_{i;p} \circ \Omega i(c \otimes b)$

and this suffices because the elements $\Omega i(c \otimes b)$ generate $\Omega D'$ as a suplattice. We also get that $X = \Omega(i;q); \exists_{i;p}$, the X described in Lemma 2.2.

To show that $\exists_{i;p}$ is well-defined, we use the suplattice coverage theorem.

$\Omega D' = \mathrm{Fr} < \Omega E \times \Omega D$ (qua $\wedge$-semilattice) (we shall write $a \otimes b$ for the pair (a, b)) |

$\otimes$ is bilinear with respect to $\bigvee$,

$a \otimes b \leq (a \wedge X(b')) \otimes b \quad (a \in \Omega E, b \leq b' \in \Omega D) >$

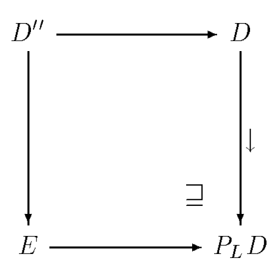
$\cong$ SupLat $< \Omega E \times \Omega D$ (qua poset) | same relations $>$ (by Theorem 1.10)

Therefore, to check that $\exists_{i;p}$ is well-defined, we just need to check that the function from $\Omega E \times \Omega D$ to ΩE given by $a \otimes b \mapsto a \wedge X(b)$ respects the relations. This is obvious.

We have now established the bijection. It is clear that it preserves order — if $X \sqsubseteq X'$ (*i.e.*, $X(b) \leq X'(b)$ for all b) then the relations $\mathbf{true} \otimes b \leq X'(b) \otimes \mathbf{true}$ are less constraining than those for X, and so present a larger sublocale. As for naturality in E, suppose $f : E' \to E$ is a map of locales. This acts on points by taking $X : \Omega D \to \Omega E$ to $X; \Omega f$, and on sublocales by pullback along $f \times \mathrm{id}_D$. These match, because the pullback of the sublocale presented by relations $\mathbf{true} \otimes b \leq X(b) \otimes \mathbf{true}$ is presented by relations $\mathbf{true} \otimes b \leq \Omega f \circ X(b) \otimes \mathbf{true}$. □

It is worth mentioning that some of this manipulation of generators and relations can be expressed more categorically. Lemma 2.2 says that the weak closure of D' is constructed by a comma square. For the openness of $i;p$ tells us that $P_L(i;p)$ has a right adjoint $(i;p)^{-1} : P_L E \to P_L D'$ (see Vickers [12]), and then X corresponds to the map

$\downarrow; (i;p)^{-1}; P_L(i;q) : E \to P_L E \to P_L D' \to P_L D$ ($\downarrow : E \to P_L E$ is given by $\Omega \downarrow (\Diamond a) = a$). If we construct the comma square



then D'' is the sublocale of $E \times D$ presented by the relations $\mathbf{true} \otimes b \leq X(b) \otimes \mathbf{true}$.

Theorem 2.3 shows in some detail that in a comma square like the one above the left hand side is open. However, once one knows this in the generic case where $E = P_L D$ and the bottom map is the identity — and this is shown in Vickers [12] — then the more general case follows from the fact that pullbacks of open maps are open: for the general comma square is a pullback of the generic one.

3 The upper powerlocale

Again, we must first relativize some of our notions.

Definition 3.1 *Let $f : D \to E$ be a map of locales.*

1. *D is* compact over E *iff f is a proper map. In other words (Vermeulen [11]), Ωf has a right adjoint $\forall_f$ that preserves directed joins and satisfies the* Frobenius identity

   $\forall_f(a \vee \Omega f(b)) = \forall_f a \vee b$

2. *A sublocale D' of D is* fitted (in D) over E *iff its frame can be presented over ΩD by relations of the form $\Omega f(b) \leq a (a \in \Omega D, b \in \Omega E)$.*

   We also say that D' has compact domain over E *if it is compact over E.*

3. *If D' is a sublocale of D, then the* fitted hull *of D' in D over E is the sublocale of D presented by all the relations $\Omega f(b) \leq a (a \in \Omega D, b \in \Omega E)$ that hold modulo D'. It is the least sublocale of D, fitted over E, that contains D'.*

Lemma 3.2 *Let D and E be locales. We write $p : E \times D \to E$ and $q : E \times D \to D$ for the projections. Let $i : D' \to E \times D$ be a sublocale that is compact over E, and let X be the preframe homomorphism $\Omega(i;q); \forall_{i;p} : \Omega D \to \Omega E$. Then the fitted hull of D' in $E \times D$ over E is presented by the relations $X(b) \otimes \mathbf{true} \leq \mathbf{true} \otimes b (b \in \Omega D)$.*

Proof If $u \in \Omega E \otimes \Omega D$, then $u = \bigvee^{\uparrow}\{\bigwedge_j (c_j \parr b_j) : \bigwedge_j (c_j \parr b_j) \leq u\}$. (We follow Johnstone and Vickers [9] in writing $c \parr b$ for $c \otimes \mathbf{true} \vee \mathbf{true} \otimes b$ — these elements generate $\Omega E \otimes \Omega D$ as a preframe. The meets mentioned are finite, of course.) Now

$a \otimes \mathbf{true} \leq u (\text{mod } D') \iff \Omega(i;p)(a) \leq \Omega i(u)$

$\iff a \leq \forall_{i;p} \circ \Omega i(u) = \bigvee^{\uparrow}\{\bigwedge_j \forall_{i;p}(\Omega(i;p)(c_j) \vee \Omega(i;q)(b_j)) : \bigwedge_j (c_j \parr b_j) \leq u\} = \bigvee^{\uparrow}\{\bigwedge_j (c_j \vee \forall_{i;p} \circ \Omega(i;q)(b_j)) : \bigwedge_j (c_j \parr b_j) \leq u\} = \bigvee^{\uparrow}\{\bigwedge_j (c_j \vee X(b_j)) : \bigwedge_j (c_j \parr b_j) \leq u\}$

By considering the meet $\bigwedge_j (c_j \parr b_j)$ where j ranges over a singleton, $c_j = \mathbf{false}$ and $b_j = b$, we see that $X(b) \otimes \mathbf{true} \leq \mathbf{true} \otimes b (\text{mod } D')$ and hence that relation holds modulo the fitted hull. On the other hand, if $a \otimes \mathbf{true} \leq u (\text{mod } D')$, then it is deducible from the relations $X(b) \otimes \mathbf{true} \leq \mathbf{true} \otimes b$:

$$a \otimes \mathbf{true} \leq \bigvee^{\uparrow}\{\bigwedge_j (c_j \vee X(b_j)) \otimes \mathbf{true} : \bigwedge_j (c_j \parr b_j) \leq u\}$$

$$\leq \bigvee^{\uparrow}\{\bigwedge_j (c_j \parr b_j) : \bigwedge_j (c_j \parr b_j) \leq u\} \leq u$$

□

Theorem 3.3 *Let D and E be locales. Then there is an order antiisomorphism, natural in E, between the points of $P_U D$ at stage E and the sublocales of $E \times D$ that, over E, are fitted with compact domain.*

Proof We write $p : E \times D \to E$ and $q : E \times D \to D$ for the projections.

If $i : D' \to E \times D$ is, over E, a fitted sublocale with compact domain (indeed, any sublocale with compact domain over E), then the function

$$X = \Omega(i;q);\forall_{i;p} : \Omega D \to \Omega(E \times D) \to \Omega D' \to \Omega E$$

is a preframe homomorphism, and hence a point of $P_U D$ at stage E. By fittedness over E, and using Lemma 3.2, D' is presented by the relations $X(b) \otimes \mathbf{true} \leq \mathbf{true} \otimes b$.

On the other hand, if we have a point of $P_U D$, given by a preframe homomorphism $X : \Omega D \to \Omega E$, then we can present a sublocale $i : D' \to E \times D$, fitted over E, by the relations $X(b) \otimes \mathbf{true} \leq \mathbf{true} \otimes b$. To show that D' has compact domain over E, we define a preframe homomorphism $\forall_{i;p} : \Omega D' \to \Omega E$ by $\forall_{i;p} \circ \Omega i(a \parr b) = a \vee X(b)$. If this is possible, then $\forall_{i;p}$ is right adjoint to $\Omega(i;p)$ with the Frobenius condition holding:

$\Omega(i;p) \circ \forall_{i;p} \circ \Omega i(c \parr b) = \Omega(i;p)(c \vee X(b)) = \Omega i(c \parr \mathbf{false} \vee X(b) \parr \mathbf{false}) \leq \Omega i(c \parr b)$ by presentation of D'

$\forall_{i;p} \circ \Omega(i;p)(c) = \forall_{i;p} \circ \Omega i(c \parr \mathbf{false}) = c \vee X(\mathbf{false}) \geq c$

$\forall_{i;p}(\Omega(i;p)(a) \vee \Omega i(c \parr b)) = \forall_{i;p} \circ \Omega i(a \vee c \parr b) = a \vee c \vee X(b) = a \vee \forall_{i;p} \circ \Omega i(c \parr b)$

To show that $\forall_{i;p}$ is well-defined, we use the preframe coverage theorem.

$\Omega D' = \mathrm{Fr} < \Omega E \times \Omega D$ (qua $\vee$-semilattice) (we shall write $a \parr b$ for the pair (a, b)) |

$\parr$ is bilinear with respect to $\wedge$ and $\bigvee^{\uparrow}$,

$(a \vee X(b')) \parr b \leq a \parr b \quad (a \in \Omega E, b' \leq b \in \Omega D) >$

$\cong \mathrm{PreFr} < \Omega E \times \Omega D$(qua poset) | same relations $>$ (by Theorem 1.11)

Therefore, to check that $\forall_{i;p}$ is well-defined, we just need to check that the function from $\Omega E \times \Omega D$ to ΩE given by $a \parr b \mapsto a \vee X(b)$ respects the relations. This is obvious.

We have now established the bijection. It is clear that it reverses order — if $X \sqsubseteq X'$ (*i.e.*, $X(b) \leq X'(b)$ for all b) then the relations $X'(b) \otimes \mathbf{true} \leq \mathbf{true} \otimes b$ are more constraining than those for X, and so present a smaller sublocale. As for naturality in E, the proof is similar to 2.3.
$\square$

4 The Vietoris powerlocale

We write $\Downarrow : VD \to P_L D$ and $\Uparrow : VD \to P_U D$ for the obvious maps ($\Omega \Downarrow (\Diamond a) = \Diamond a$ and $\Omega \Uparrow (\Box a) = \Box a$). $(\Downarrow, \Uparrow) : VD \to P_L D \times P_U D$ is an inclusion of locales.

Definition 4.1

1. *A sublocale D' of D is* weakly semifitted *iff it is the meet of a fitted sublocale and a weakly closed sublocale. (Johnstone [7] defines a sublocale to be semifitted iff it is the meet of a fitted sublocale and a closed sublocale.) If $f : D \to E$ is a map, then D' is weakly semifitted over E iff it is the meet of a fitted sublocale and a weakly closed sublocale over E: in other words, it can be presented by relations of the form $b \leq \Omega f(a)$ and $\Omega f(a') \leq b'$ $(a, a' \in \Omega E, b, b' \in \Omega D)$.*

2. *If $f : D \to E$ is open, then we define $L : E \to P_L D$ by $\Omega L(\Diamond a) = \exists_f a$. (This is the left Kan extension of $\downarrow : D \to P_L D$ along f, and when $E = 1$ it is the greatest point of $P_L D$, right adjoint to $! : P_L D \to 1$.) In terms of Vickers [12], this can also be described as $\downarrow; f^{-1}$ where f^{-1} is the right adjoint of $P_L f$ that exists by openness of f.*

3. *If $f : D \to E$ is proper, then we define $R : E \to P_U D$ by $\Omega R(\Box a) = \forall_f a$. (This is the right Kan extension of $\uparrow : D \to P_U D$ along f, and when $E = 1$ it is the least point of $P_U D$, left adjoint to $! : P_U D \to 1$.) Again in terms of Vickers [12], $R = \uparrow; f^{-1}$ where this time f^{-1} is the left adjoint to $P_U f$.*

The following result is proved when $E = 1$ by Johnstone [7] in the classical case, referring to Johnstone [6] for the constructive version (although the result is not stated explicitly there, it is certainly very easy using the techniques presented).

Lemma 4.2 *Let $f : D \to E$ be open and proper. Then $< L, R >: E \to P_L D \times P_U D$ factors via a map $\Xi : E \to VD$.*

Proof $\Omega\Xi$ must map $\Diamond a$ to $\exists_f a, \Box a$ to $\forall_f a$, so it remains to check that this respects the mixed relations:

$$\forall_f a \wedge \exists_f b = \exists_f (\Omega f \circ \forall_f a \wedge b) \leq \exists_f (a \wedge b)$$
$$\forall_f (a \vee b) \leq \forall_f (a \vee \Omega f \circ \exists_f b) = \forall_f a \vee \exists_f b$$
□

Theorem 4.3 *Let D and E be locales. Then there is a bijective correspondence, natural in E, between points of VD at stage E and sublocales of $E \times D$ that, over E, are weakly semifitted with compact, open domain.*

Proof Though more intricate, the proof is structured along similar lines to those of Theorems 2.3 and 3.3 for the lower and upper sublocales.

Suppose $i : D' \to E \times D$ is, over E, a weakly semifitted sublocale of $E \times D$ with compact, open domain. Then Lemma 4.2 gives us a point $\Xi : E \to VD'$, so $X = \Xi; V(i; q) : E \to VD$ is a point of VD at stage E.

We now show that D' is presented by the following relations:

$$(1) \qquad \mathbf{true} \otimes b \leq \Omega X(\Diamond b) \otimes \mathbf{true}$$
$$(2) \qquad \Omega X(\Box b) \otimes \mathbf{true} \leq \mathbf{true} \otimes b$$

We have

$\Omega X(\Diamond b) = \Omega\Xi \circ \Omega V i \circ \Omega V q(\Diamond b) = \Omega\Xi \circ \Omega V i(\Diamond(\mathbf{true} \otimes b)) = \Omega\Xi \circ \Omega V i \circ \Omega \Downarrow (\Diamond(\mathbf{true} \otimes b)) = \Omega\Xi \circ \Omega \Downarrow \circ \Omega P_L i(\Diamond(\mathbf{true} \otimes b)) = \Omega L(\Diamond \Omega i(\mathbf{true} \otimes b)) = \exists_{i;p} \circ \Omega(i; q)(b)$

It follows from Lemma 2.2 that the relations (1) present the weak closure of D' over E. Similarly, by Lemma 3.2, the relations (2) present the fitted hull over E, and it follows from weak semifittedness that together the relations present D'.

We have now shown, much as before, how from a weakly semifitted sublocale with compact, open domain we can construct a point of VD, and then how we can recover the original sublocale from it.

Now from a point $X : E \to VD$, we can present a sublocale $i : D' \to E \times D$, weakly semifitted over E, by the relations (1) and (2). We shall define suplattice and preframe homomorphisms $\exists_{i;p}$ and $\forall_{i;p} : \Omega D' \to \Omega E$ such that $\exists_{i;p} \circ \Omega i(c \otimes b) = c \wedge \Omega X(\Diamond b)$ and $\forall_{i;p} \circ \Omega i(c \otimes b) = c \vee \Omega X(\Box b)$.

If we can do this, then we have $\exists_{i;p} \dashv \Omega(i; p) \dashv \forall_{i;p}$ with the Frobenius identities — the calculations are virtually identical to those in Theorems 2.3 and 3.3.

It would follow that D' is compact and open over E, giving, at stage E, points L and R of $P_L D'$ and $P_U D'$, and Ξ of VD'. Now $\Xi; V(i; q); \Downarrow = \Xi; \Downarrow; P_L(i; q) = L; P_L(i; q) = X; \Downarrow$ by definition of $\exists_{i;p}$: for

$$\Omega L \circ \Omega P_L(i; q)(\Diamond b) = \Omega L(\Diamond \Omega(i; q)(b)) = \exists_{i;p} \circ \Omega i(\mathbf{true} \otimes b) = \Omega X(\Diamond b)$$

Similarly $\Xi; V(i; q); \Uparrow = X; \Uparrow$, and hence, because $<\Downarrow, \Uparrow>$ makes VD a sublocale of $P_L D \times P_U D$, we get that $\Xi; V(i; q) = X$ and so X is recovered by the construction first described. All that remains is to justify the definition of $\exists_{i;p}$ and $\forall_{i;p}$.

For $\exists_{i;p}$,

$\Omega D' = \mathrm{Fr} < \Omega E \times \Omega D$ (qua $\wedge$-semilattice)(writing, as usual, $c \otimes b$ for (c, b)) |

$\otimes$ bilinear with respect to $\bigvee$

$c \otimes b \leq c \wedge \Omega X(\Diamond b') \otimes b$ if $b \leq b'$

$c \wedge \Omega X(\Box b') \otimes b \leq c \wedge \Omega X(\Box b') \otimes b \wedge b' >$

$\cong \mathrm{SupLat} < \Omega E \times \Omega D$ (qua poset) $|$ same relations $>$

Hence we must show that $c \otimes b \mapsto c \wedge \Omega X(\Diamond b)$ respects the relations. The first two are just as for the lower powerlocale. For the third,

$$c \wedge \Omega X(\Box b') \wedge \Omega X(\Diamond b) = c \wedge \Omega X(\Box b' \wedge \Diamond b) \leq c \wedge \Omega X(\Box b') \wedge \Omega X(\Diamond(b \wedge b'))$$

Next, for $\forall_{i;p}$,

$\Omega D' = \mathrm{Fr} < \Omega E \times \Omega D$ (qua $\vee$-semilattice)(writing $c \parr b$ for (c, b)) $|$

$\parr$ bilinear with respect to $\wedge$ and $\bigvee^{\uparrow}$

$c \vee \Omega X(\Box b') \parr b \leq c \parr b$ if $b' \leq b$

$c \vee \Omega X(\Diamond b') \parr b \vee b' \leq c \vee \Omega X(\Diamond b') \parr b >$

$\cong \mathrm{PreFr} < \Omega E \times \Omega D$ (qua poset) $|$ same relations $>$

Just as for $\exists_{i;p}$, the first two relations are similar to the upper powerlocale, and the third corresponds to the other mixed relation in the Vietoris powerlocale. This completes the proof. $\Box$

5 Open sublocales of discrete locales

We now turn to two results that seem somewhat perverse in the light of the preceding sections, in that they identify points of powerlocales as *open* sublocales. Specifically, let D be a locale, and let \$ be the Sierpinsky locale. If the exponential $\$^D$ exists, then its global points are the continuous maps from D to \$, *i.e.,* the opens of D. We show that in the special cases where D is discrete or compact regular, $\$^D$ (exists and) is homeomorphic to $P_L D$ or $P_U D$ respectively. (The existence of $\$^D$ is well-known in these cases, for discrete locales and compact regular locales are locally compact and hence exponentiable.)

In this section we consider discrete D. For general D, of course, it is absolutely out of the question to expect the sublocales described in the Bunge-Funk Theorem, the weakly closed sublocales with open domain, to be the same as the open sublocales — for instance, the former are lower closed under the specialization order, whereas the latter are upper closed. However, this does not apply to discrete locales, because they are T_1. If D is discrete, then a simple argument shows that the global points of $P_L D$ are equivalent to the opens of D. For a global point of $P_L D$ is a suplattice homomorphism from ΩD (*i.e.,* $\wp D$) to Ω. By Joyal and Tierney [10] $\wp D$ is the free suplattice over D, so these are equivalent to functions from D to Ω, *i.e.,* subsets of D, *i.e.,* opens of D. We argue more carefully to give a result holding for *all* points of $P_L D$, not just the global ones.

Theorem 5.1 *Let E and D be locales, with D discrete, and let D' be a sublocale of $E \times D$. Then the following are equivalent:*

1. *D' is open in $E \times D$.*
2. *Over E, D' is weakly closed in $E \times D$ with open domain.*

Proof As in Lemma 2.2, we write p and q for the projections from $E \times D$ and i for the sublocale inclusion of D' in $E \times D$. If D' is open in $E \times D$, then i and p are both open maps, and hence D' is open over E. Hence for both directions of the proof we can take it that D' has open domain over E. Let $X = \Omega(i;q) \circ \exists_{i;p}$ as in Lemma 2.2.

By Lemma 2.2, the weak closure of D' (in $E \times D$ over E) is presented by the relations $\mathbf{true} \otimes \{x\} \leq X(\{x\}) \otimes \mathbf{true}$ and from these we quickly deduce that $\mathbf{true} \leq \bigvee_{x \in D} X(\{x\}) \otimes \{x\}$, using the fact that in ΩD we have $\mathbf{true} = \bigvee_{x \in D} \{x\}$. However, the converse also holds, for

$\mathbf{true} \otimes \{y\} \leq \bigvee_{x \in D} X(\{x\}) \otimes (\{x\} \wedge \{y\}) \leq X(\{y\}) \otimes \mathbf{true}$

It follows that the weak closure of D' is the open sublocale $\bigvee_{x \in D} X(\{x\}) \otimes \{x\}$.

Imediately, if D' is weakly closed then it is open. Conversely, if D' is open then it can be expressed in the form $\bigvee_{x \in D} b_x \otimes \{x\}$. Then

$X(\{y\}) = \exists_p \circ \exists_i \circ \Omega i \circ (\mathbf{true} \otimes \{y\})$

$= \exists_p(\bigvee_{x \in D} b_x \otimes \{x\} \wedge \mathbf{true} \otimes \{y\}) = \exists_p(b_y \otimes \{y\}) = b_y \wedge \Omega ! \circ \exists_! \{y\}$

$= b_y$ since $\{y\}$ is positive.

Hence D' is its own weak closure. □

Note from the case $E = 1$ that if, as classically, "weakly closed with open domain" is equivalent to "closed", then in any discrete locale all opens are clopen — in other words, the topos we are working in is Boolean.

Theorem 5.2 *Let D be a discrete locale. Then $P_L D \cong \$^D$.*

Proof Define a map $e : P_L D \times D \to \$$ by the open $\bigvee_{x \in D} \Diamond\{x\} \otimes \{x\}$. Then if $f : E \to P_L D$ corresponds to $X : \Omega D \to \Omega E$, we have $(f \times Id); e$ given by the open $\bigvee_{x \in D} X(\{x\}) \otimes \{x\}$ and from the proof of Theorem 5.1 this is the corresponding weakly closed sublocale with open domain (over E). Hence the function $\mathbf{Loc}(E, P_L D) \to \mathbf{Loc}(E \times D, \$)$, $f \mapsto (f \times Id); e$, is a composite of isomorphisms

$\mathbf{Loc}(E, P_L D) \cong \{$ sublocales of $E \times D$ as in Theorem 2.3 $\}$

$= \{$ open sublocales of $E \times D\}$

$\cong \mathbf{Loc}(E \times D, \$)$ □

6 Open sublocales of compact regular locales

We now prove an analogous result: when D is compact regular, then $\$^D \cong P_U D$. Of course, it is not the case that the open sublocales of D are the same as the compact fitted sublocales; rather, the result comes by two oddities canceling out. First, for a compact regular locale D the compact sublocales (all sublocales of D are fitted, because the relation $a \leq b$ is equivalent to the set of relations $\mathbf{true} \leq b \vee \neg a'$ for a' well inside a) are the same as the *closed* sublocales; and next, because the specialization order on $P_U D$ is reversed (big sublocales are low in the ordering), we get that the global points of $P_U D$ are in order isomorphism with the open sublocales. Again, we prove the result for generalized points.

Theorem 6.1 *Let E and D be locales, with D compact regular, and let D' be a sublocale of $E \times D$. Then the following are equivalent:*

1. *D' is closed in $E \times D$.*
2. *Over E, D' is fitted in $E \times D$ with compact domain.*

Proof As in Lemma 3.2, we write p and q for the projections from $E \times D$ and i for the sublocale inclusion of D' in $E \times D$. If D' is closed in $E \times D$, then i and p are both proper maps, and hence D' is compact over E. Hence for both directions of the proof we can take it that D' is compact over E; let $X = \Omega(i;q);\forall_{i;p}$, so that by Lemma 3.2 its fitted hull in $E \times D$ over E is presented by relations (1) $X(b) \otimes \mathbf{true} \leq \mathbf{true} \otimes b$. We show that this is closed, and in fact is presented by the relations (2) $X(b) \otimes c \leq \mathbf{false}$ (whenever $b \wedge c \leq \mathbf{false}$). From (1) we have that $X(b) \otimes c \leq \mathbf{true} \otimes b \wedge c \leq \mathbf{false}$ if $b \wedge c \leq \mathbf{false}$. Conversely, assume (2). Then by regularity,

$$X(b) \otimes \mathbf{true} = \bigvee^{\uparrow}\{X(b') \otimes \mathbf{true} : \exists c.(b \vee c = \mathbf{true}, b' \wedge c = \mathbf{false})\}$$

But given such b' and c, we have $X(b') \otimes \mathbf{true} = X(b') \otimes b \vee X(b') \otimes c = X(b') \otimes b$ (modulo (2)) $\leq \mathbf{true} \otimes b$.

Immediately, we see that if D' is fitted then it is closed. Conversely, suppose that D' is closed. It suffices to show that any relation $a \otimes b \leq \mathbf{false}$ holding modulo D' also holds modulo the fitted hull D'' (say). $a \otimes b = \bigvee^{\uparrow}\{a \otimes b' : \exists c.(b \vee c = \mathbf{true}, b' \wedge c = \mathbf{false})\}$. Given such b' and c, we have $a \otimes \mathbf{true} = a \otimes b \vee a \otimes c \leq a \otimes c$ modulo D' and hence also modulo the fitted hull D''. Hence $a \otimes b' \leq a \otimes b' \wedge c$ (mod D'') $= \mathbf{false}$. □

Theorem 6.2 *Let D be a compact regular locale. Then $P_U D \cong \$^D$.*

Proof Define a map $e : P_U D \times D \to \$$ by the open $\bigvee\{\Box b \otimes c : b \wedge c = \mathbf{false}\}$. Then if $f : E \to P_U D$ corresponds to $X : \Omega D \to \Omega E$, we have $(f \times Id); e$ given by the open $\bigvee\{X(b) \otimes c : b \wedge c = \mathbf{false}\}$ and from the proof of Theorem 6.1 this is the complement of the corresponding compact fitted sublocale (over E). Hence the function $\mathbf{Loc}(E, P_U D) \to \mathbf{Loc}(E \times D, \$)$, $f \mapsto (f \times Id); e$, is a composite of isomorphisms

$\mathbf{Loc}(E, P_U D) \cong \{$ sublocales of $E \times D$ as in Theorem 3.3 $\}^{op}$

$= \{$ closed sublocales of $E \times D\}^{op}$

$\cong \mathbf{Loc}(E \times D, \$)$ □

As a corollary, consider relations in the category of compact regular locales. A relation from X to Y, a subobject of $X \times Y$ in that category, is just a closed sublocale of $X \times Y$. These correspond (reversing the order) to maps $X \to P_U Y$ and hence to preframe homomorphisms $\Omega Y \to \Omega X$.

7 Conclusions

Though the constructive treatment of the global points is more complicated that the classical treatment — we have had to generalize "closed" to "weakly closed" and restrict at a certain point to open locales — the complications display a remarkable symmetry that is quite invisible classically and which leads to proofs of Theorems 2.3 and 3.3 that are virtually identical in structure. On the one hand we have the lower powerlocale, suplattices, open locales and weakly closed sublocales, while on the other we have the upper powerlocale, preframes, compact locales and fitted sublocales.

Much of the argument here is amenable to the "synthetic" reasoning of Vickers [12], and it would be interesting to push this further. I would hope that the manipulations of generators and relations could thereby be packaged up into a few axioms of the synthetic point-based logic.

Acknowledgements

This work was assisted by the UK Science and Engineering Research Council through the "Foundational Structures in Computer Science" project.

I should also like to give warmest thanks to Ian Mackie for converting this paper from its non-electronic form into a Latex document.

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