SOME PROPERTIES OF CENTROSYMMETRIC MATRICES AND ITS APPLICATIONS*

Liu Zhongyun (刘仲云)

Abstract In this paper, some properties of centrosymmetric matrices, which often appear in the construction of orthonormal wavelet basis in wavelet analysis, are investigated. As an application, an algorithm which is tightly related to a so-called Lawton matrix is presented. In this algorithm, about only half of memory units are required and quarter of computational cost is needed by exploiting the property of the Lawton matrix and using a compression technique, it is compared to one for the original Lawton matrix.

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1 Introduction

Wavelet analysis is a rapidly developing area in the mathematical science which is emerging as a brisk and important field of investigation. Moreover, it has already created a tight link between mathematic scientists and electrical engineers, and has attracted a great deal of attention from scientists and engineers in other disciplines.

Historically, the study of wavelet analysis was based on the standard approach of functional analysis. However, the basic aspects of wavelet analysis can be derived using fairly elementary means of matrix algebra, and matrix methods play an important role in the study of wavelets, see, for example, the references [2, 5-8, 10].

In this paper, we first investigate some properties of a kind of special matrices, which are called centrosymmetric matrices, they appear frequently in the construction of orthonormal wavelet basis. Then we develop an algorithm tightly related to a so-called Lawton matrix—a

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special centrosymmetric matrix, which is used to justify the orthogonality of the compactly supported wavelet bases. By exploiting the special property and using a compression technique, in particular, our algorithm needs only about half of memory units and quarter of computational costs. It is compared to one for the original Lawton matrix.

2 Some Properties of Centrosymmetric Matrices

In this section we’ll introduce some basic concept and present some useful results used in the sequel. We begin with some basic notation which refers to the definition of a symmetric matrix, skew symmetric matrix, Hermitian matrix, skew Hermitian matrix, respectively.

Let a matrix $A = (a_{i,j})_{p \times q}$ be a $p \times q$ complex matrix, then the matrix $A$ is called:

1. a centrosymmetric matrix, if the elements of $A$ satisfy the relation
   $$a_{i,j} = a_{p-i+1,q-j+1};$$

2. a skew centrosymmetric matrix, if the elements of $A$ satisfy the relation
   $$a_{i,j} = -a_{p-i+1,q-j+1};$$

3. a centrohermitian matrix, if the elements of $A$ satisfy the relation
   $$a_{i,j} = a_{p-i+1,q-j+1};$$

4. a skew centrohermitian matrix, if the elements of $A$ satisfy the relation
   $$a_{i,j} = -a_{p-i+1,q-j+1},$$

respectively, for all $1 \leq i \leq p$ and $1 \leq j \leq q$.

For the centrosymmetric matrices, one can easily check the following properties.

**Theorem 1**

1. (a) Let the matrix $C \in \mathbb{C}^{n \times n}$ be centrosymmetric, skew centrosymmetric, centrohermitian, skew centrohermitian, respectively. If $C$ is nonsingular, then $C^{-1}$ is centrosymmetric, skew centrosymmetric, centrosymmetric, skew centrosymmetric, centrohermitian, skew centrohermitian, respectively; (b) If the matrix $H \in \mathbb{C}^{p \times q}$ be centrosymmetric, skew centrosymmetric, centrohermitian, skew centrohermitian, respectively, then $H^T$ is $q \times p$ centrosymmetric, skew centrosymmetric, centrohermitian, skew centrohermitian, respectively; (c) If both the matrices $E$ and $F \in \mathbb{C}^{n \times l}$ be centrosymmetric, skew centrosymmetric, skew centrohermitian respectively, then $E \pm F$ are centrosymmetric, skew centrosymmetric, centrosymmetric, skew centrohermitian respectively.

2. If both the matrices $A \in \mathbb{C}^{n \times k}$ and $B \in \mathbb{C}^{k \times l}$ be centrosymmetric, skew centrosymmetric, respectively, then $AB \in \mathbb{C}^{n \times l}$ is centrosymmetric.
3. Let $A$ be an $n \times k$ complex matrix and $B$ be a $k \times l$ complex matrix. If one of the matrices $A$ and $B$ is centrosymmetric and the other is skew centrosymmetric, then $AB \in \mathbb{C}^{n \times l}$ is skew centrosymmetric.

4. If both the matrices $A \in \mathbb{C}^{n \times k}$ and $B \in \mathbb{C}^{k \times l}$ be centrohermitian, skew centrohermitian, respectively, then $AB \in \mathbb{C}^{n \times l}$ is centrohermitian.

5. Let $A$ be an $n \times k$ complex matrix and $B$ be a $k \times l$ complex matrix. If one of the matrices $A$ and $B$ is centrohermitian and another is skew centrohermitian, then $AB \in \mathbb{C}^{n \times l}$ is skew centrohermitian.

Let $J_n = (e_n, e_{n-1}, \cdots, e_1)$, where $e_i$ denotes the unit vector with $i$th entry 1. According to the definition of the centrosymmetric matrix, skew centrosymmetric matrix, centrohermitian matrix and skew centrohermitian matrix, a matrix $A \in \mathbb{C}^{p \times q}$ being centrosymmetric, skew centrosymmetric, centrohermitian and skew centrohermitian, respectively, is equivalent to that $J_pAJ_q = A$, $J_pAJ_q = -A$, $J_pAJ_q = \bar{A}$ and $J_pAJ_q = -\bar{A}$, respectively.

From the above equivalent definition, we easily obtain the following results:

**Theorem 2** Let $A$ be an $n \times n$ complex matrix.

1. If $A$ is an odd order skew centrosymmetric matrix, then $\det(A) = 0$;

2. If $A$ is a centrohermitian matrix, then $\det(A)$ is a real number;

3. If $A$ is an even order skew centrohermitian matrix, then $\det(A)$ is a real number and if $n$ is odd, then $\det(A)$ is a pure image number.

Our main purpose is to investigate some properties of square centrosymmetric matrices from the standpoint of computation. Except for special mention, henceforth, we only consider the square matrices. To my knowledge, less attention was focused on such an aspect of research.

We now focus our attention on the $n \times n$ centrosymmetric and skew centrosymmetric matrices. Using the partition of matrix, the central symmetric character of an $n \times n$ centrosymmetric matrix can be described as follows:

1. For the case of $n = 2m$, a centrosymmetric matrix can be written in the form

$$A = \begin{bmatrix} B & J_mCJ_m \\ C & J_mBJ_m \end{bmatrix},$$

where each of the block matrices $B$ and $C$ is an $m \times m$ complex matrix.

2. For the case of $n = 2m + 1$, a centrosymmetric matrix can be partitioned into the following form

$$A = \begin{bmatrix} B & J_mb & J_mCJ_m \\ a^T & a^TJ_m & \alpha \\ C & b & J_mBJ_m \end{bmatrix},$$

with $B, C \in \mathbb{C}^{m \times m}$, $a, b \in \mathbb{C}^{m \times 1}$ and $\alpha$ an complex scalar.

Similarly, the skew central symmetric character of a $n \times n$ skew centrosymmetric matrix can
be described as follows:

(1) If \( n = 2m \), then a skew centrosymmetric matrix can be written in the form

\[
A = \begin{bmatrix}
B & -J_mC
\end{bmatrix}
\begin{bmatrix}
C & -J_mB
\end{bmatrix},
\]  

with \( B, C \in \mathbb{C}^{m \times m} \).

(2) If \( n = 2m+1 \), then a skew centrosymmetric matrix can be partitioned into the following form

\[
A = \begin{bmatrix}
B & -J_m b & -J_mC
\end{bmatrix}
\begin{bmatrix}
\alpha & \sqrt{2} a^T
\end{bmatrix}
\begin{bmatrix}
\sqrt{2} J_m b & B + J_mC
\end{bmatrix}
\]

with \( B, C \in \mathbb{C}^{m \times m} \), \( a, b \in \mathbb{C}^{m \times 1} \).

Having the above preparation, we can immediately show the following important results for a centrosymmetric matrix and skew centrosymmetric matrix, respectively.

**Theorem 3**  Let \( A \) be an \( n \times n \) complex matrix.

1. If \( A \) is a centrosymmetric matrix, then \( A \) is orthogonal similar to one of the following forms:

\[
\begin{bmatrix}
B - J_mC & B + J_mC
\end{bmatrix}
\]

for \( n = 2m \),

and

\[
\begin{bmatrix}
B - J_mC & \sqrt{2} \alpha a^T
\end{bmatrix}
\begin{bmatrix}
\sqrt{2} J_m b & B + J_mC
\end{bmatrix}
\]

for \( n = 2m+1 \).

2. If \( A \) is a skew centrosymmetric matrix, then \( A \) is orthogonal similar to one of the following forms:

\[
\begin{bmatrix}
B + J_mC & B - J_mC
\end{bmatrix}
\]

for \( n = 2m \),

and

\[
\begin{bmatrix}
B + J_mC & \sqrt{2} \alpha a^T
\end{bmatrix}
\begin{bmatrix}
\sqrt{2} J_m b & B - J_mC
\end{bmatrix}
\]

for \( n = 2m+1 \).

**Proof**  If \( A \) is an \( n \times n \) centrosymmetric matrix, then, from (1) in the case of \( n = 2m \) and (2) in the case of \( n = 2m+1 \), choosing the orthogonal matrices

\[
Q = \frac{\sqrt{2}}{2}
\begin{bmatrix}
I_m & I_m
\end{bmatrix}
\begin{bmatrix}
-I_m & J_m
\end{bmatrix}
\]

and

\[
Q = \frac{\sqrt{2}}{2}
\begin{bmatrix}
I_m & \sqrt{2} I_m
\end{bmatrix}
\begin{bmatrix}
-I_m & \sqrt{2} J_m
\end{bmatrix}
\]
respectively, we can directly obtain (5) and (6), respectively.

If $A$ is an $n \times n$ skew centrosymmetric matrix, then, from (3) in the case of $n = 2m$ and (4) in the case of $n = 2m + 1$, selecting the orthogonal matrices

$$Q = \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & I_m \\ J_m & -J_m \end{bmatrix}$$

and

$$Q = \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & \sqrt{2} I_m \\ J_m & -J_m \end{bmatrix}$$

respectively, we straightforwardly obtain (7) and (8), respectively.

For a centrohermitian matrix, we have the following results:

**Theorem 4** Let $A = A_1 + iA_2 \in \mathbb{C}^{n \times n}$ be a centrohermitian matrix. Then the real representation matrix of the form

$$\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}$$

((denoted by $A_R$) of $A$ is a $2n \times 2n$ real centrosymmetric matrix.

**Proof** From the hypothesis, we have $J_n A_1 J_n = A_1$ and $J_n A_2 J_n = -A_2$. Note that

$$A_R = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}.$$

Then

$$J_{2n} A_R J_{2n} = \begin{bmatrix} J_n & A_1 & -A_2 \\ J_n & A_2 & A_1 \end{bmatrix} \begin{bmatrix} J_n \\ A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} = A_R.$$

Therefore, $A_R$ is a $2n \times 2n$ real centrosymmetric matrix.

For a skew centrohermitian matrix, we have the following results:

**Theorem 5** Let $A = A_1 + iA_2 \in \mathbb{C}^{n \times n}$ be a skew centrohermitian matrix. Then the real representation matrix of the form

$$\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}$$

((denoted by $A_R$) of $A$ is a $2n \times 2n$ real skew centrosymmetric matrix. If we consider another real representation of the form

$$\begin{bmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{bmatrix}$$

(denoted by $\tilde{A}_R$) of $A$, then $\tilde{A}_R$ is a $2n \times 2n$ real centrosymmetric matrix.

**Proof** From the hypothesis, we have $J_n A_1 J_n = -A_1$ and $J_n A_2 J_n = A_2$.

If we choose the real representation matrix of the form

$$A_R = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \in \mathbb{R}^{2n \times 2n},$$
then
\[
J_{2n} A_R J_{2n} = \begin{bmatrix} J_n & J_n \\ J_n & J_n \end{bmatrix} \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} J_n \\ J_n \end{bmatrix} = -A_R.
\]
Therefore, \( A_R \) is a \( 2n \times 2n \) real skew centrosymmetric matrix.

If we select the real representation matrix of the form
\[
\tilde{A}_R = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \in \mathbb{R}^{2n \times 2n},
\]
then
\[
J_{2n} \tilde{A}_R J_{2n} = \begin{bmatrix} J_n & J_n \\ J_n & J_n \end{bmatrix} \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} J_n \\ J_n \end{bmatrix} = A_R.
\]
Hence, \( \tilde{A}_R \) is a \( 2n \times 2n \) real centrosymmetric matrix.

3 Lawton Matrix and Its Application

In this section we’ll first show the definition of a so-called Lawton matrix and then consider an application of a Lawton matrix from the standpoint of computation.

Consider matrix
\[
L = (L_{j,k}) \in \mathbb{C}^{(2N-1) \times (2N-1)}
\]
which is justified the orthogonality of a compactly supported wavelet bases, see e.g. [2, 8], where

\[
L_{j,k} = \sum_{n=0}^{N} h_n \overline{h_{k-2j+n}}, \quad -N + 1 \leq j, k \leq N - 1
\]
with \( h_n = 0 \) for \( n < 0, n > N \). W. M. Lawton is credited with the discovery of this matrix (see, for example, [2, 8]), and we thus use the name of Lawton to denominate such a matrix.

Consider the facts that a compactly supported wavelet bases implies \( \sum_{n} h_n \overline{h_{n+2k}} = \delta_{k,0} \);
\[
\sum_{n} h_n = \sqrt{2} \quad \text{and} \quad \sum_{n} (-1)^n h_n = 0.
\]
Therefore, Lawton matrix \( L \) is a special matrix. It is easy to check that
\[
\sum_{j=-N+1}^{N-1} L_{j,k} = 1, \quad k = -N + 1, \ldots, N - 1.
\]
According to the above characters, by a somewhat tedious computation, we can characterize the Lawton matrix as follow:
1. For $N = 2P$

$$L = \begin{bmatrix}
L_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
L_2 & L_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
L_P & 0 & L_{P-1} & 0 & \cdots & 0 & L_1 & 0 & 0 & \cdots & 0 & 0 \\
L_{P-1} & 1 & L_P & 0 & \cdots & 0 & L_2 & 0 & L_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
L_1 & 0 & L_2 & 0 & \cdots & 0 & L_3 & 0 & L_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
L_1 & 0 & L_2 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}$$

where $L_p = \sum_{n=0}^{2P} h_n h_{n+N-2p+1}$ for $p = 1, \cdots, P$.

2. For $N = 2P - 1$

$$L = \begin{bmatrix}
0 & L_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & L_2 & L_1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & L_{P-1} & 0 & L_{P-2} & \cdots & 0 & L_1 & 0 & 0 & \cdots & 0 & 0 \\
1 & L_P & 0 & L_{P-1} & \cdots & 0 & L_2 & 0 & L_1 & \cdots & 0 & 0 \\
0 & L_P & 1 & L_P & 0 & \cdots & 0 & L_3 & 0 & L_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & L_3 & 0 & L_4 & \cdots & 1 & L_P & 0 & L_{P-1} & \cdots & L_1 & 0 \\
0 & L_2 & 0 & L_3 & \cdots & 0 & L_P & 1 & L_P & \cdots & L_2 & 0 \\
0 & L_1 & 0 & L_2 & \cdots & 0 & L_{P-1} & 0 & L_P & \cdots & L_3 & 0 \\
0 & 0 & 0 & L_1 & \cdots & 0 & L_{P-2} & 0 & L_{P-1} & \cdots & L_4 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}$$

where $L_p = \sum_{n=0}^{2P-1} h_n h_{n+N-2(p-1)}$ for $p = 1, \cdots, P$.

From the concrete representation of a Lawton matrix, we easily find that a Lawton matrix must be a centrosymmetric matrix. Based on Theorem 4, We now consider an application of Lawton matrix from the standpoint of computation. From the Lawton condition in [1, 8]: “A compactly supported wavelet bases being an orthonormal basis if and only if the eigenvalue 1 of the corresponding Lawton matrix being nondegenerate”, we find that this condition is
equivalent to that $\det(I_{2N-2} - \tilde{L}) \neq 0$, where $\tilde{L}$ denotes a $(2N-2) \times (2N-2)$ submatrix of $L \in \mathbb{C}^{(2N-1) \times (2N-1)}$ obtained by deleting the $N$th row and $N$th column of $L$. Apparently, $\tilde{L}$ is a $(2N-2) \times (2N-2)$ centrohermitian matrix.

Let $\tilde{L} = \tilde{L}_1 + i\tilde{L}_2$, where $\tilde{L}_1$ and $\tilde{L}_2 \in \mathbb{R}^{(2N-2) \times (2N-2)}$ denote the real and image parts of $\tilde{L}$, respectively. By Theorem 4, the real representation matrix of $\tilde{L}$

$$\tilde{L}_R = \begin{bmatrix} \tilde{L}_1 & -\tilde{L}_2 \\ \tilde{L}_2 & \tilde{L}_1 \end{bmatrix} \in \mathbb{R}^{(4N-4) \times (4N-4)}$$

is a centrosymmetric matrix.

It is well-known that $\det(I_{2N-2} - \tilde{L}) \neq 0$ if and only if $\det(I_{4N-4} - \tilde{L}_R) \neq 0$. By Theorem 3, we can immediately get the following result:

**Theorem 6** The eigenvalue 1 of a Lawton matrix $L \in \mathbb{C}^{(2N-1) \times (2N-1)}$ is nondegenerate if and only if $\det[(I_{2N-2} - \tilde{L}_1) \pm J_{2N-2} \tilde{L}_2] \neq 0$, where $\tilde{L}_1$ and $\tilde{L}_2$ are the real and image parts of the submatrix $\tilde{L} \in \mathbb{C}^{(2N-2) \times (2N-2)}$ obtained by deleting $N$th row and $N$th column of the matrix $L$, respectively.

If $L \in \mathbb{R}^{(2N-1) \times (2N-1)}$, then $\tilde{L}_1 = \tilde{L}$ and $\tilde{L}_2 = 0$. Therefore the matrix $\tilde{L}$ itself is a $2N-2$ order real centrosymmetric matrix and thus can be represented as the following block structure:

$$\tilde{L} = \begin{bmatrix} L_{1,1} & J_{N-1} L_{1,2} J_{N-1} \\ J_{N-1} L_{1,2} J_{N-1} & L_{1,2} \end{bmatrix}.$$

Using Theorem 3 again, we get the following conclusion:

**Theorem 7** The eigenvalue 1 of a Lawton matrix $L \in \mathbb{R}^{(2N-1) \times (2N-1)}$ is nondegenerate if and only if $\det[(I_{N-1} - L_{1,1}) \pm J_{N-1} L_{1,2}] \neq 0$, where $L_{1,1}$ and $L_{1,2}$ are the above block structure matrix of the submatrix $\tilde{L} \in \mathbb{R}^{(2N-2) \times (2N-2)}$ obtained by deleting $N$th row and $N$th column of the matrix $L$, respectively.

### 4 An Applied Algorithm for Lawton Matrix

In this section we consider a compression algorithm to compute $\det(I - \tilde{L})$, which are based on Theorem 6 and Theorem 7. To state this algorithm, some further notation is needed: If $u$ is an $n$-dimension vector, then for $j < k$, the vector $u(j : k)$ denotes the $(k - j)$-dimension vector of $u$ contained in the rows of $u$ indexed by $j, j + 1, \ldots, k$, i.e., $u(j : k) = (u_j, u_{j+1}, \ldots, u_k)^T$, where $u = (u_1, u_2, \ldots, u_n)^T$.

**Algorithm** (For $N = 2P - 1$)

For $p = 1, 2, \ldots, P - 1$

$$L_{p+p} = L_{p+(p-1)}$$

if $P = 2q$
to form:
\[
\hat{L}_{2p} = (L_{2p}, L_{2p-1}, \cdots, L_{p+1} - 1, L_p, \cdots, L_2, L_1, 0, \cdots, 0)^T
\]
and
\[
\hat{L}_{2p-1} = (L_{2p-1}, L_{2p-2}, \cdots, L_p, L_{p-1}, \cdots, L_2, L_1, 0, \cdots, 0)^T
\]
\((\hat{L}_{2p-1}, \hat{L}_{2p} \text{ are both } 2P - 2 \text{-dimension vectors})\)
to compute:
(1) \(\check{L}_k = \hat{L}_k\), for \(k = 1, \cdots, P - 1\);
(2) \(\check{L}_k = \hat{L}_k + \hat{L}_{2(k-p)+1}\), for \(k = P, \cdots, P + q - 1\);
(3) \(\check{L}_k = \hat{L}_k + \hat{L}_{2(k-p-q)+p+1}\), for \(k = P + q, \cdots, 2(P - 1)\).

else if \(P = 2q + 1\)
to form:
\[
\hat{L}_{2p-1} = (L_{2p-1}, L_{2p-2}, \cdots, L_p, L_{p-1}, \cdots, L_2, L_1, 0, \cdots, 0)^T
\]
and
\[
\hat{L}_{2p} = (L_{2p}, L_{2p-1}, \cdots, L_{p+1}, L_p, \cdots, L_2, L_1, 0, \cdots, 0)^T
\]
\((\hat{L}_{2p-1}, \hat{L}_{2p} \text{ are both } 2P - 2 \text{-dimension vectors})\)
to compute:
(1) \(\check{L}_k = \hat{L}_k\), for \(k = 1, \cdots, P - 1\);
(2) \(\check{L}_k = \hat{L}_k + \hat{L}_{2(k-p)+1}\), for \(k = P, \cdots, P + q\);
(3) \(\check{L}_k = \hat{L}_k + \hat{L}_{2(k-p-q)+p+1}\), for \(k = P + q + 1, \cdots, 2(P - 1)\).

to set \(\tilde{L}_p = \check{L}_{2p}\)
to form matrices
\[
B = \begin{bmatrix}
\tilde{L}_1^T (1 : P - 1) \\
\vdots \\
\tilde{L}_{P-1}^T (1 : P - 1)
\end{bmatrix}
\]
and
\[
C = \begin{bmatrix}
\tilde{L}_1^T (P : 2P - 2) \\
\vdots \\
\tilde{L}_{P-1}^T (P : 2P - 2)
\end{bmatrix}
\]
if \(B, C \in \mathbb{R}^{(P-1) \times (P-1)}\),
then to compute \(\det(B \pm CJ_{P-1})\)
if \(\det(B \pm CJ_{P-1}) \neq 0\),
then the eigenvalue 1 of \(L\) is nondegenerate.
else
the eigenvalue 1 of \( L \) is degenerate.
end if
else
Let \( B = B_1 + iB_2 \) and \( C = C_1 + iC_2 \)

\[
\Delta = \det \begin{bmatrix}
B_1 - C_2J_{P-1} & C_1 - B_2J_{P-1} \\
J_{P-1}C_1J_{P-1} + J_{P-1}B_2 & J_{P-1}B_1J_{P-1} + J_{P-1}C_2
\end{bmatrix}
\]
if \( \Delta \neq 0 \),
then the eigenvalue 1 of \( L \) is nondegenerate;
else
the eigenvalue 1 of \( L \) is degenerate;
end if
end if

For the case of \( N = 2P \), we note that \( \det(I - \tilde{L}) = (1 - L_1)(1 - \tilde{L}_1) \det(I - \hat{L}) \), where \( \tilde{L} \) denotes a \((2N - 2) \times (2N - 2)\) submatrix of \( L \) obtained by deleting the \( N \)th row and \( N \)th column of \( L \) and \( \hat{L} \) is a \( 2N - 4 \) order submatrix of \( \tilde{L} \) obtained by deleting the first row and column, the last row and column of \( \tilde{L} \). Apparently, both \( \hat{L} \) and \( \tilde{L} \) are even order centrohermitian matrices.

We also note that the matrix \( \hat{L} \) has the same structure as the matrix \( \tilde{L} \) in the case of \( N = 2P - 1 \). Therefore, the above algorithm can be copied and pasted to the case of \( N = 2P \), and only one extra determinant \((1 - L_1 \neq 0 ?)\) are needed. So we omit the algorithm for \( N = 2P \).

5 Conclusion

In our algorithm, to store vector \( \hat{L}_p, p = 1, 2, \cdots, 2P - 2 \), for \( N = 2P - 1 \) only need \((P - 1)(2P - 1)\) memory units and for \( N = 2P \) only require \( P(2P - 1) \) memory units. Compared with the storage of Lawton matrix, in which \( 2(P - 1)(2P + 1) \) memory units are required, our algorithm merely needs about half of its.

The main computation is to do \( \det(B \pm C J_{P-1}) \) for \( B, C \in \mathbb{R}^{(P-1) \times (P-1)} \) or to do \( \Delta \) for \( B, C \in \mathbb{C}^{(P-1) \times (P-1)} \). We remark that we omit the computation to do

\[
\hat{\Delta} = \det \begin{bmatrix}
B_1 + C_2J_{P-1} & C_1 + B_2J_{P-1} \\
J_{P-1}C_1J_{P-1} - J_{P-1}B_2 & J_{P-1}B_1J_{P-1} - J_{P-1}C_2
\end{bmatrix}
\]
in our algorithm. Because we find that \( \hat{\Delta} = \Delta \), according to the properties of centrohermitian matrices.
We consider the computation of \( \det(\hat{L} - I_{2N-2}) \), where \( \hat{L} \) denotes a \((2N - 2) \times (2N - 2)\) submatrix of \( L \) obtained by deleting the \(N\)th row and \(N\)th column of Lawton matrix \( L \), where \( L \) is a \((2N - 1) \times (2N - 1)\) Lawton matrix, by using LU factorization method (see, e.g., [1, 3, 4]). We note that the compression form of the matrix \( \hat{L} - I_{2N-2} \) is

\[
\begin{bmatrix}
B & C \\
J_{P-1} \bar{C} J_{P-1} & J_{P-1} \bar{B} J_{P-1}
\end{bmatrix},
\]

where the matrices \( B \) and \( C \) are defined in (9) and (10), respectively, in our algorithm.

We now compare the computational cost of LU factorization (without pivoting strategy) of the matrices \( B \pm CJ_{P-1} \) and (11), ignoring the working amount of compression.

For the case of \( B, C \in \mathbb{R}^{(P-1) \times (P-1)} \), we observe that the differences among the matrices \( B + CJ_{P-1}, B - CJ_{P-1} \) and \( B \) are in the positions of nonzero entries of the matrix \( CJ_{P-1} \). Therefore, when we decompose \( B + CJ_{P-1} \) and \( B - CJ_{P-1} \) by using LU factorization method, the same element need only do once. Thus the computational cost of LU factorization of the matrices \( B + CJ_{P-1} \) and \( B - CJ_{P-1} \) is tantamount to one of the matrix

\[
\begin{bmatrix}
B \\
J_{P-1} \bar{C} J_{P-1}
\end{bmatrix}.
\]

For the computational cost of LU factorization of the matrix (11), however, the computational cost of LU factorization of the remaining part of (11) and the contribution of the matrix \( C \) must be included. By and large, the computational cost for doing \( \det(B \pm CJ_{P-1}) \) is half of that for computing the determinant of (11).

It is well-known that the operation count of LU factorization of a general matrix with order \( n \) is \( \frac{2}{3} n^3 + O(n^2) \). Therefore, for a real centrosymmetric matrix with order \( n \), due to Theorem 3, we need only to factor two matrices with order \( \frac{n}{2} \) for even \( n \) or two matrices (one with order \( \lfloor \frac{n}{2} \rfloor \) and another with order \( \lceil \frac{n}{2} \rceil + 1 \)) for odd \( n \). For a sufficient large \( n \), it is obvious that the operation count of LU factorization of (1) or (2) is about 4 times one of (5) or (6), respectively.

For \( B, C \in \mathbb{C}^{(P-1) \times (P-1)} \), the real representation matrix of (11) is in the following form

\[
\begin{bmatrix}
B_1 & C_1 & -B_2 & -C_2 \\
J_{P-1} C_1 J_{P-1} & J_{P-1} B_1 J_{P-1} & J_{P-1} C_2 J_{P-1} & J_{P-1} B_2 J_{P-1} \\
B_2 & C_2 & B_1 & C_1 \\
-J_{P-1} C_2 J_{P-1} & -J_{P-1} B_2 J_{P-1} & J_{P-1} C_1 J_{P-1} & J_{P-1} B_1 J_{P-1}
\end{bmatrix},
\]

where the matrices \( B_1 + iB_2 = B \) and \( C_1 + iC_2 = C \). An analogous analysis can show that the computational cost for doing \( \Delta \) is about quarter of that for computing the determinant of (12), because we need not to do \( \bar{\Delta} \). For a general complex centrohermitian matrix with order \( n \), due to Theorem 4, we need only to factor a real matrix with order \( n \). For a sufficient large \( n \), it
is easy to show that the real operation count of LU factorization of $A$ is about 8 times that of $A_1 + A_2 J_n$, where $A = A_1 + i A_2$.

We end this section with some low order Lawton matrices.

**Example 1** (N=2) According to the coefficient formula of Lawton matrix, we get

$$L = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} \\
0 & 0 & \frac{1}{2}
\end{bmatrix}.$$  

It’s a $3 \times 3$ centrosymmetric matrix. Note that

$$\tilde{L} = \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{bmatrix},$$

and $\det(I_2 - \tilde{L}) = \frac{1}{4} \neq 0$. By Theorem 3 and the above Algorithm, we immediately know that the eigenvalue 1 of $L$ is nondegenerate. In fact, the eigenvalues of $L$ are $\frac{1}{2}, \frac{1}{2}$ and 1, respectively.

**Example 2** (N=3) We have the corresponding Lawton matrix as follows:

$$L = \begin{bmatrix}
0 & L_1 & 0 & 0 & 0 \\
1 & L_2 & 0 & L_1 & 0 \\
0 & \tilde{L}_1 & L_2 & 0 & 0 \\
0 & 0 & \tilde{L}_2 & 1 & 0 \\
0 & 0 & 0 & \tilde{L}_1 & 0
\end{bmatrix},$$

where $L_1 = h_0 \overline{h}_3$, $L_2 = h_0 \overline{h}_1 + h_1 \overline{h}_2 + h_2 \overline{h}_3$.

From $N = 3$, $P = 2$ and $q = 1$ comparing with our algorithm,

$$\hat{L}_2 = (L_2 - 1, L_1)^T$$
$$\hat{L}_1 = (L_1, 0)^T$$

we have

$$\tilde{L}_1 = \hat{L}_1 = (L_1, 0)^T,$$

and

$$\tilde{L}_2 = \hat{L}_2 + \hat{L}_1 = (L_2 + L_1 - 1, L_1)^T,$$

$$\tilde{L}_1 = \hat{L}_2; B = L_2 + L_1 - 1, C = L_1.$$

If $B, C \in \mathbb{R}$, then $\det(B + C) = L_1 - \frac{1}{2}$ and $\det(B - C) = L_2 - 1$. Therefore, the eigenvalue 1 of $L$ is nondegenerate if and only if $L_1 \neq \frac{1}{2}$ and $L_2 \neq 1$. 

If \( B, C \in \mathbb{C} \), let \( B = B_1 + iB_2 \) and \( C = C_1 + iC_2 \), then we obtain

\[
\Delta_1 = \det \begin{bmatrix}
B_1 - C_2 & C_1 + B_2 \\
C_1 - B_2 & B_1 + C_2
\end{bmatrix} = \Delta_2
\]

Therefore, the eigenvalue 1 of \( L \) is nondegenerate if and only if \( \Delta_1 \neq 0 \).

References

2. Daubechies I. Ten lectures on wavelets. CBS, 61, SIAM, Philadelphia, 1992