Affine shape analysis and image analysis

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1 Summary

A study of affine shape is needed in certain problems that arise in bioinformatics and pattern recognition. In particular affine shape analysis is useful in the analysis of 2D electrophoresis images and in the reconstruction of a larger area from a number aerial images. Two aerial images taken from different distances of a ground scene are given in Figure 1.; the problem asks to reconstruct a larger contiguous image that contains the information in both these images.

Figure 1. Aerial photographs of two sections of a ground scene.

Another classical example consists in matching two labelled electrophoresis gels (see Figure 2, based on data from www.lecb.ncifcrf.gov/flicker/).

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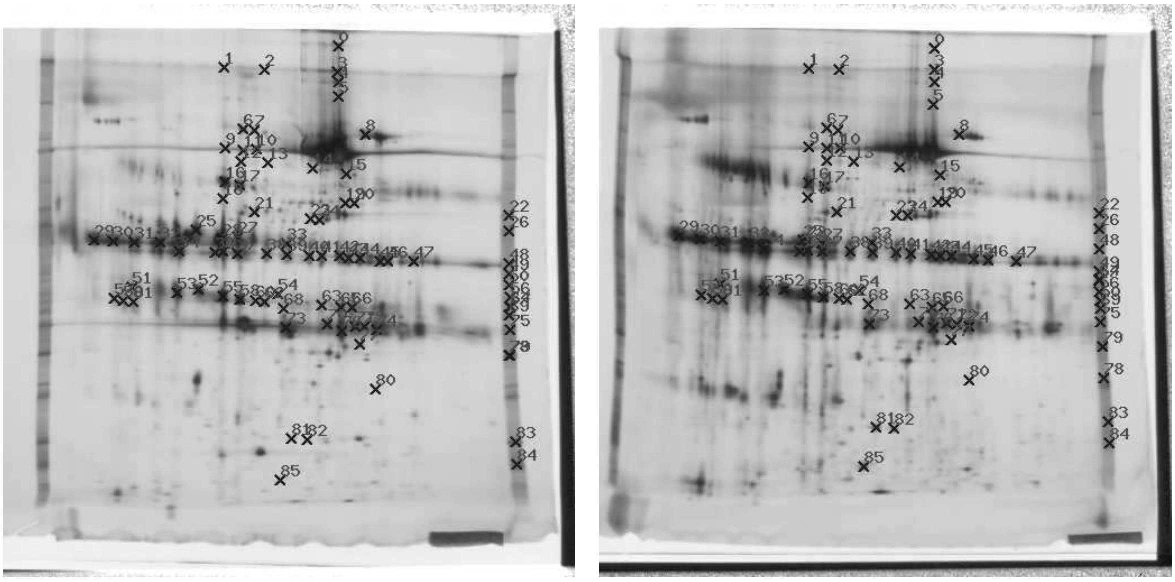

Figure 2. Labelled electrophoresis gels.

Such simple problems involve image warping based on mean affine shapes of conFigurations of convenient landmarks. When it comes to affine shape, researchers from different areas are apparently using two different definitions for affine shape. In this paper we give two such definitions, and show that they are are equivalent.

As a consequence we show that the *affine shape space* of configurations of points in general position is a real Grassmann manifold, therefore statistical analysis of affine shape is statistical analysis on a real Grasmann manifold. Grassmann manifolds are not apriori submanifolds of a numerical space. For an extrinsic statistical analysis of probability distributions, such equivariant embeddings are sought, and such an equivariant embedding is defined here. This preliminary report ends with a discussion.

2 The affine shape space

In this section we will assume that $k \geq m+1$. An *affine transformation* γ of $\mathbb{R}^m$ is a linear automorphism of $\mathbb{R}^m$, followed by a translation. $Aff(m)$ denotes the group of affine transformations. In affine shape analysis, for $k \geq m$ we consider the set $(\mathbb{R}^m)^k$ of all systems $(x_1, ..., x_k,)$ of points $x_j \in \mathbb{R}^m$. The group of affine transformations acts diagonally on the left on $(\mathbb{R}^m)^k$; the action A is given by

$$A(\gamma, (x_1, ..., x_k)) = (\gamma(x_1), ..., \gamma(x_k)). \tag{1}$$

DEFINITION 2.1.a. The affine shape in statistical shape analysis $\sigma[x_1, ..., x_k]$ of $(x_1, ..., x_k)$, is the orbit of $(x_1, ..., x_k)$ via the action (1).

Another definition of shape is given by A. Heyden (1995) and G. Sparr (1996):

DEFINITION 2.1.b. The affine shape in computer vision $\mathcal{S}[x_1, ..., x_k]$ of $(x_1, \ldots, x_k)$, is the linear subspace of $\mathbb{R}^k$ given by

$$S[x_1, ..., x_k] = \{\xi = (\xi^1, ..., \xi^k), \sum_{j=1}^{j=k} \xi^j x_j = 0, \sum_{j=1}^{j=k} \xi^j = 0\} \tag{2}$$

We can prove the following:

PROPOSITION 2.1. *There is a natural one to one correspondence between affine shapes in statistical shape analysis and affine shapes in computer vision.*

DEFINITION 2.2. The affine shape space $A\Sigma_m^k$, or space of *affine k-ads* in $\mathbb{R}^m$ is the quotient $(\mathbb{R}^m)^k/Aff(m)$. The Grassmann manifold of r-dimensional vector subspaces of $\mathbb{R}^d$ will be denoted by $G_p(d)$. Namely, $G_r(d) = \{Y : Y = Y^T = Y^TY,\ \text{rank}(Y) = r\}$ where Y is a $d \times d$ matrix.

As a consequence of Proposition 2.1 one may show that

THEOREM 2.1. *The affine shape space has a stratification, $A\Sigma_m^k = \sqcup_{j=0,m} A_j\Sigma_m^k$, where the j-stratum $A_j\Sigma_m^k$ is diffeomorphic to the Grassmann manifold $G_{m-j}(k-1)$ of $m-j$-dimensional vector subspaces of $\mathbb{R}^{k-1}$. In particular, the stratum $A_0\Sigma_m^k$ of affine shapes of $k-$ads in general position is diffeomorphic to the Grassmann manifold $G_m(k-1)$.*

3 Extrinsic means of affine shapes and reconstruction of larger planar scenes

Affine shape distributions have been considered by Goodall and Mardia (1993), Leung, Burl and Perona(1998), Berthilsson and Heyden (1999) et al. In view of Theorem 2.1, extrinsic means of distributions of affine shapes can be determined using the general approach in Bhattacharya and Patrangenaru (2003), for certain convenient equivariant embeddings of $G_{m-j}(k-1)$ in an Euclidean space. Such an embedding can be defined as follows: let $Sym(k-1)$ be the set of $(k-1)\times(k-1)$ symmetric matrices endowed with the canonical Euclidean square norm $\|A\|^2 = Tr(AA^t)$; a natural embedding of $G_{m-j}(k-1)$ into $Sym(k-1)$ is obtained by identifying each $m-j$-dimensional vector subspace L with the matrix p_L of orthogonal projection into L. Dimitric (1996) proved that this embedding is equivariant, has parallel second fundamental form and embeds the Grassmannian minimally into a hypersphere. This is an extension of the Veronese-Whitney embedding of projective spaces (see Bhattacharya and Patrangenaru(2003)), that is commonly used for axial data (see Mardia and Jupp, 1999), or for multivariate axial data (Mardia and Patrangenaru, 2002).

Assume a probability distribution Q of affine shapes of conFigurations in general position is *nonfocal* w.r.t. this embedding j. In this case, the mean $\mu_{j(Q)}$ of the corresponding distribution $j(Q)$ of $(k-1)\times(k-1)$ symmetric matrices of rank $m-j$, has the eigenvalues $\lambda_1 \geq ... \geq \lambda_{k-1}$ such that $\lambda_{m-j} > \lambda_{m-j+1}$. The *extrinsic mean* of Q is the vector subspace spanned by unit eigenvectors corresponding to the first $m-j$ eigenvalues of $\mu_{j(Q)}$. Assume $(\pi_1, ...\pi_n)$ is a sample of size n of $m-j$-vector subspaces $\pi_1, ...\pi_n$ in $\mathbb{R}^{k-1}$, and the subspace π_r is spanned by the orthonormal unit vectors $\{x_{r,a}\}_{a=1,...,m-j}$ and set $x_r = (x_{r,a})_{a=1,...,m-j}$. The extrinsic sample mean is of this sample, when it exists, is the $m-j$-vector subspace $\overline{\pi}$ generated by the unit eigenvectors corresponding to the first largest $m-j$ eigenvalues of $\sum_r x_x x_r^t$ (to be compared with the Procrustes sample mean in the sense of Chikuse (1999, 2003)). There are exceptional distributions, such as the uniform distributions (Chikuse and Watson (1995), for which the extrinsic mean does not exist, and therefore the Procrustes sample mean is inconsistent).

Nevertheless, the condition of existence of the mean of a distribution on a Grassmann manifold is generic and for a given random sample the extrinsic mean exists almost with no exception. The extrinsic sample mean is useful in averaging images of remote planar scenes, by adapting the standard method of image averaging of Dryden and Mardia (1998) as shown in Mardia et al. (2001). This method can be used in reconstruction of larger planar scenes as in Faugeras and Luong(2001), as shown in Figure 3.

Figure 3. Reconstruction of a larger view of the scene in images in Figure 1, based on an extrinsic mean affine shape.

4 Discussion

In summary, in affine shape analysis, the real Grasmann manifolds play the key rôle in the same way as that of the complex projective spaces in similarity shape analysis (Kendall, 1984). While the first large sample results on Grassmann manifolds are about distributions without an extrinsic mean, data driven analysis is needed for concentrated distributions on such manifolds. Therefore large sample and nonparametric bootstrap methods should be developed for extrinsic sample means on Grassmann manifolds and applied in practice. The method of image warping was used to align microscope images by Glasbey and Mardia (2001). It would be interesting to use such results or the methods briefly described in Section 3 when applied to electrophoresis gels, in matching of a new gel image with an image average based on mean affine shapes of conFigurations of marked spots in existent samples of such images, such as those in Figure 2 (see http://www.cmis.csiro.au/iap/RecentProjects/gelregistration.htm). In this paper we showed that Grassmann manifolds are useful in image analysis; other applications of statistical analysis on such manifolds are in signal subspace estimation (Srivastava and Klassen, 2002). It should be noted that affine shape is applicable only for analysis of remote scenes, otherwise projective shape analysis should be used (see Mardia and Patrangenaru, 2002)

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