# ON THE SEMI-CLASSICAL LIMIT FOR THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We review some results concerning the semi-classical limit for the nonlinear Schrödinger equation, with or without an external potential. We consider initial data which are either of the WKB type, or very concentrated as the semi-classical parameter goes to zero. We sketch the techniques used according to various frameworks, and point out some open problems.

Contents



2000 *Mathematics Subject Classification.* 35A35; 35B05; 35B40; 35Q55; 81Q20.

## 1. Introduction

Consider the nonlinear Schrödinger equation (NLS):

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + \varepsilon^\kappa f\left(|u^\varepsilon|^2\right)u^\varepsilon, \quad (t,x)\in I\times\mathbb{R}^n, \tag{1.1}$$

where the potential $V = V(x)$ and the nonlinearity $f$ are real-valued. In some specified cases, we allow the potential to be time-dependent. To simplify the discussion, we assume that $\kappa \geqslant 0$ is an integer. More precise assumptions will be made according to the different cases we study. We assume $\varepsilon \in ]0,1]$, and we aim at describing the asymptotic behavior of $u^\varepsilon$ as $\varepsilon \to 0$, for the following two families of initial data:

*Monokinetic WKB initial data:*

$$u^\varepsilon(0,x) = a_0^\varepsilon(x)e^{i\phi_0(x)/\varepsilon}, \quad \text{with } a_0^\varepsilon(x) \underset{\varepsilon\to 0}{\sim} a_0(x) + \varepsilon a_1(x) + \varepsilon^2 a_2(x) + \dots, \tag{1.2}$$

in the sense of asymptotic expansion.

*Concentrated initial data:*

$$u^\varepsilon(0,x) = R\left(\frac{x-x_0}{\varepsilon}\right)e^{ix\cdot\xi_0/\varepsilon}, \tag{1.3}$$

for some point $(x_0,\xi_0)$ in the phase space $\mathbb{R}^{2n}$, independent of $\varepsilon$.

There are at least two motivations for such a study, referred to as *semi-classical analysis* or *geometrical optics*. We outline them here, and refer to the survey [47] for a broader discussion on this subject. The first one comes from the applied mathematics, and may find its origins in physics. In the case of (1.1), suppose that $\varepsilon$ represents the (rescaled) Planck constant. It may be small compared to the other parameters at stake. In this case, it is sensible to consider that the asymptotic behavior of $u^\varepsilon$ as $\varepsilon \to 0$ provides a reliable approximation of the exact solution. Hopefully, the asymptotic model is easier to describe than the initial one (1.1)–(1.2). If $V$ is a confining potential (e.g. harmonic potential), then (1.1) may be a model to describe Bose-Einstein condensation; see for instance [22, 46]. The value of $\kappa$ then depends on the asymptotic régime considered. Another motivation stems from the propagation of singularities for equations where the small parameter $\varepsilon$ is not necessarily present initially. Most of the studies in this direction concern hyperbolic equations. However, this field is applicable to Schrödinger equations as well (see e.g. [7, 41, 51]). The following illustration is a straightforward consequence of the analysis presented in §3.2:

**Theorem 1.1** ([13], Cor. 1.7)**.** *Let $n \geqslant 3$. Consider the cubic, defocusing NLS:*

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^2 u, \quad x\in\mathbb{R}^n \quad ; \quad u_{|t=0} = u_0\,. \tag{1.4}$$

*Denote $s_c = \frac{n}{2}-1$. Let $0 < s < s_c$. We can find a family $(u_0^\varepsilon)_{0<\varepsilon\leqslant 1}$ in $\mathcal{S}(\mathbb{R}^n)$ with*

$$\|u_0^\varepsilon\|_{H^s(\mathbb{R}^n)} \to 0 \text{ as } \varepsilon \to 0\,,$$

*and $0 < t^\varepsilon \to 0$ such that the solution $u^\varepsilon$ to (1.4) associated to $u_0^\varepsilon$ satisfies:*

$$\|u^\varepsilon(t^\varepsilon)\|_{H^k(\mathbb{R}^n)} \to +\infty \text{ as } \varepsilon\to 0\,,\ \forall k \in \left]\frac{s}{\frac{n}{2}-s}, s\right]\,.$$

This result was first established in [21] in the case $k = s$. The fact that one can consider a broader range for $k$, in the spirit of [42], relies on a fine analysis of the limit for (1.1)–(1.2), provided essentially in [32].

1.1. **Monokinetic WKB initial data.** In the case of initial data of the form (1.2), an approximation of the form

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \left(\mathtt{a}_0(t,x) + \varepsilon \mathtt{a}_1(t,x) + \varepsilon^2 \mathtt{a}_2(t,x) + \dots\right) e^{i\Phi(t,x)/\varepsilon} \tag{1.5}$$

is expected. Note that only one phase and one harmonic are sought: this is an important feature of Schrödinger equations with gauge invariant nonlinearity. In the case of wave equations for instance, the story is completely different (see e.g. [47] and references therein). Note also that such an approximation must be expected for bounded time only. Even in the linear case $f \equiv 0$, a caustic appears in finite time in general. Near a caustic, all the terms $\Phi$, $\mathtt{a}_0$, $\mathtt{a}_1$, ... become singular. Past the caustic, several phases are necessary in general to describe the asymptotic behavior of the solution (see e.g. [25] for a general theory in the linear case). However, we will see that the analogous phenomenon in the nonlinear setting (say, $f(y) = y$) with $\kappa = 0$ (highly nonlinear régime) might be very different.

Plug a formal expansion of the form (1.5) into (1.1). Ordering the terms in powers of $\varepsilon$, and canceling the cascade of equations thus obtained is aimed at yielding $\Phi$, $\mathtt{a}_0$, $\mathtt{a}_1$, ...

Assume for a while that $\kappa \geqslant 1$. To cancel the term of order $\mathcal{O}(\varepsilon^0)$, we find

$$\mathtt{a}_0 \left(\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + V\right) = 0 \quad ; \quad \Phi_{|t=0} = \phi_0 \,.$$

Since we seek a non-trivial profile $\mathtt{a}_0$, we impose a stronger condition: $\Phi$ must solve the *eikonal equation*

$$\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + V = 0 \quad ; \quad \Phi_{|t=0} = \phi_0 \,.$$

Canceling the term of order $\mathcal{O}(\varepsilon^1)$, we get:

$$\partial_t \mathtt{a}_0 + \nabla\Phi\cdot\nabla \mathtt{a}_0 + \frac{1}{2}\mathtt{a}_0 \Delta\Phi = \begin{cases} 0 & \text{if } \kappa > 1, \\ -if\left(|\mathtt{a}_0|^2\right)\mathtt{a}_0 & \text{if } \kappa = 1. \end{cases}$$

We see that the value $\kappa = 1$ is critical as far as nonlinear effects are concerned: if $\kappa > 1$, no nonlinear effect is expected at leading order, since formally, $u^\varepsilon \sim \mathtt{a}_0 e^{i\Phi/\varepsilon}$, where $\Phi$ and $\mathtt{a}_0$ do not depend on the nonlinearity $f$. If $\kappa = 1$, then $\mathtt{a}_0$ solves a nonlinear equation involving $f$.

We will see in Section 2 that when $\kappa = 1$, $\mathtt{a}_0$ solves a transport equation that turns out to be a ordinary differential equation along the rays of geometrical optics, as is usual in the hyperbolic case (see e.g. [47]). More typical of Schrödinger equation is the fact that this ordinary differential equation can be solved explicitly.

Assume now $\kappa = 0$, and proceed the same way. Plugging (1.5) into (1.1), we get:

$$\begin{cases} \mathcal{O}\left(\varepsilon^0\right): & \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + V + f\left(|\mathtt{a}_0|^2\right) = 0, \\ \mathcal{O}\left(\varepsilon^1\right): & \partial_t \mathtt{a}_0 + \nabla\Phi\cdot\nabla \mathtt{a}_0 + \frac{1}{2}\mathtt{a}_0\Delta\Phi = -2if'\left(|\mathtt{a}_0|^2\right)\mathrm{Re}\left(\mathtt{a}_0\overline{\mathtt{a}_1}\right)\mathtt{a}_0. \end{cases} \tag{1.6}$$

We see that there is a strong coupling between the phase and the main amplitude: $\mathtt{a}_0$ is present in the equation for $\Phi$. In addition, the above system is not closed: $\Phi$

is determined in function of $\mathtt{a}_0$, and $\mathtt{a}_0$ is determined in function of $\mathtt{a}_1$. Even if we pursued the cascade of equations, this phenomenon would remain: no matter how many terms are computed, the system is never closed (see [30]). This is a typical feature of supercritical cases in nonlinear geometrical optics (see [19, 20]). We shall call the study of this case *highly nonlinear WKB analysis.* We will see in §3 some ways to overcome the difficulties pointed out above, especially in the case $f' > 0$ (defocusing, cubic at the origin, nonlinearity).

*Remark* 1.2. We consider only *monokinetic* initial data. Studying the nonlinear effects relevant at leading order ($\kappa = 0$ or 1) when the datum is of the form

$$u^\varepsilon(0,x) = a_0(x)e^{i\phi_0(x)/\varepsilon} + b_0(x)e^{i\varphi_0(x)/\varepsilon} \quad (\phi_0 \neq \varphi_0),$$

for instance, seems to be an open problem.

1.2. **Concentrated initial data.** For data of the form (1.3), a formal analysis shows that the case $\kappa = 0$ is critical: if $\kappa > 0$ (not necessarily an integer), no nonlinear effect is expected at leading order. We shall therefore restrict our attention to the case $\kappa = 0$. We also consider the case of a pure power nonlinearity,

$$f\left(|u^\varepsilon|^2\right) = \lambda |u^\varepsilon|^{2\sigma},$$

for some $\sigma > 0$ and $\lambda \in \mathbb{R}$. In this case, setting $\mathtt{u}^\varepsilon = \varepsilon^{-n/2}u^\varepsilon$, (1.1)–(1.3) is equivalent to:

$$i\varepsilon\partial_t \mathtt{u}^\varepsilon + \frac{\varepsilon^2}{2}\Delta \mathtt{u}^\varepsilon = V\mathtt{u}^\varepsilon + \lambda\varepsilon^{n\sigma}|\mathtt{u}^\varepsilon|^{2\sigma}\mathtt{u}^\varepsilon \ ; \ \mathtt{u}^\varepsilon(0,x) = \frac{1}{\varepsilon^{n/2}}R\left(\frac{x-x_0}{\varepsilon}\right)e^{ix\cdot\xi_0/\varepsilon}. \tag{1.7}$$

In the case $\lambda > 0$ (defocusing nonlinearity), dispersive effects are expected to alter the concentrated form of the initial data. This is proved in [11, 16] when the external potential is a polynomial of degree at most two. It seems that proving a similar result in the more general (and fairly natural) framework of smooth, sub-quadratic, potentials, is still an open problem. Note also that the dispersive effect can be just the first step of the dynamics. It can be followed by a linear dynamics induced by the potential. In this régime, the potential may cause a refocusing phenomenon. This is the case for instance when $V$ is an isotropic harmonic potential [11]. We discuss more precisely these results in §4.1.

When $\lambda < 0$ (focusing nonlinearity), several papers have considered the case when the profile $R$ is the ground state associated to NLS without potential, that is when $R = Q$, where $Q$ is the unique positive, radially symmetric, solution of:

$$-\frac{1}{2}\Delta Q + Q + \lambda |Q|^{2\sigma}Q = 0.$$

When $\sigma < 2/n$ (sub-critical case at the $L^2$ level), orbital stability of the solitary wave suggests that the solution $u^\varepsilon$ evolves under the form

$$u^\varepsilon(t,x) = Q\left(\frac{x-x(t)}{\varepsilon}\right)e^{ix\cdot\xi(t)/\varepsilon}e^{i\varphi^\varepsilon(t)}.$$

We will see that this is the case, with $(x(t),\xi(t))$ given by the Hamiltonian flow associated to $-\frac{1}{2}\Delta + V$: the additional purely time dependent phase shift $\varphi^\varepsilon$ is known explicitly in the case without potential, but not in general. The first mathematical result on this problem is due to J. Bronski and R. Jerrard [6]. Refinements were then given by S. Keraani [36, 37, 38]. We outline the approach of [38] in §4.2.

Note also that the semi-classical limit $\varepsilon \to 0$ for (1.7) is analogous to the long time behavior for the solutions to (1.7) with $\varepsilon = 1$; see e.g. [27, 34].

## 2. WKB analysis for a weak nonlinearity

When $\kappa \geqslant 1$, the first step in the WKB analysis presented in §1.1 consists in solving the eikonal equation. This step relies on the Hamilton-Jacobi theory. It is well-known, at least when the potential $V$ and the initial phase $\phi_0$ are smooth, that the local inversion theorem yields a local in time, smooth solution in the neighborhood of $(t = 0, x)$, for all $x \in \mathbb{R}^n$ (see e.g. [23]). In order to have a local existence time which is uniform with respect to $x \in \mathbb{R}^n$, the following assumption is essentially necessary (see e.g. [14]):

**Assumption 2.1.** *The potential $V$ may depend on time: $V = V(t, x)$. We assume that the potential and the initial phase are smooth and sub-quadratic:*

- $V \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$, *and* $\partial^\alpha_x V \in C(\mathbb{R}_t; L^\infty(\mathbb{R}^n_x))$ *as soon as* $|\alpha| \geqslant 2$.
- $\phi_0 \in C^\infty(\mathbb{R}^n)$, *and* $\partial^\alpha_x \phi_0 \in L^\infty(\mathbb{R}^n)$ *as soon as* $|\alpha| \geqslant 2$.

*Remark* 2.2. Of course, if we worked on a compact set instead of $\mathbb{R}^n$, the above assumptions would not be necessary.

A global inversion result (see [49] or [23]) and Gronwall lemma yield:

**Lemma 2.3.** *Under Assumption 2.1, there exist $T > 0$ and a unique solution $\phi_{\rm eik} \in C^\infty([0, T] \times \mathbb{R}^n)$ to:*

$$\partial_t \phi_{\rm eik} + \frac{1}{2}|\nabla \phi_{\rm eik}|^2 + V = 0 \quad ; \quad \phi_{{\rm eik}|t=0} = \phi_0 \, . \tag{2.1}$$

*This solution is subquadratic: $\partial^\alpha_x \phi_{\rm eik} \in L^\infty([0, T] \times \mathbb{R}^n)$ as soon as $|\alpha| \geqslant 2$.*

*Remark* 2.4. In [14], examples are given, that show that if either the potential $V$ or the initial phase $\phi_0$ has a super-quadratic growth at infinity, the above result fails. Sub-quadratic potentials play a special role in the mathematical analysis of Schrödinger equations: the results of [28, 29] imply local in time Strichartz estimates for the semi-group associated to $-\Delta + V$. On the other hand, in space dimension $n = 1$, $-\partial^2_x - x^4$ is not essentially self-adjoint on $C^\infty_0(\mathbb{R})$ (see [26, Chap. 13, Sect. 6, Cor. 22]). If $V$ tends to $+\infty$ at infinity, with super-quadratic growth, the available results are very different from those of the sub-quadratic case, see e.g. [55, 56].

To prove this lemma, we introduce the Hamiltonian flow:

$$\begin{cases} \partial_t x(t, y) = \xi\,(t, y) \quad ; \quad x(0, y) = y, \\ \partial_t \xi(t, y) = -\nabla V\,(t, x(t, y)) \quad ; \quad \xi(0, y) = \nabla \phi_0(y). \end{cases} \tag{2.2}$$

The time $T$ is such that the map $y \mapsto x(t, y)$ is a diffeomorphism of $\mathbb{R}^n$ for $t \in [0, T]$. Therefore, the Jacobi determinant

$$J_t(y) = \det \nabla_y x(t, y),$$

is bounded from above, and from below away from zero, for $t \in [0, T]$. The justification of the leading order asymptotics sketched in §1.1 is:

**Proposition 2.5.** *Let $\kappa \geqslant 1$ and $f \in C^\infty(\mathbb{R}_+; \mathbb{R})$. Assume that there exists a smooth function $a_0$ independent of $\varepsilon$ such that*

$$a^\varepsilon_0 \to a_0 \text{ in } H^s(\mathbb{R}^n), \quad \forall s \geqslant 0.$$

*Then under Assumption 2.1, for all* $\varepsilon \in ]0,1]$, (1.1)–(1.2) *has a unique solution* $u^\varepsilon \in C^\infty([0,T]\times\mathbb{R}^n)\cap C([0,T];H^s)$ *for all* $s>n/2$, *where* $T$ *is given by Lemma 2.3. Moreover, there exist* $a, G \in C^\infty([0,T]\times\mathbb{R}^n)$, *independent of* $\varepsilon \in ]0,1]$, *where* $a \in C([0,T];L^2\cap L^\infty)$, *and* $G$ *is real-valued with* $G \in C([0,T];L^\infty)$, *such that:*

$$\left\| u^\varepsilon - a e^{i\varepsilon^{\kappa-1}G} e^{i\phi_{\rm eik}/\varepsilon}\right\|_{L^\infty([0,T];L^2\cap L^\infty)} \to 0 \quad \text{as } \varepsilon \to 0.$$

*The profile* $a$ *solves the initial value problem:*

$$\partial_t a + \nabla \phi_{\rm eik}\cdot \nabla a + \frac{1}{2} a \Delta \phi_{\rm eik} = 0 \quad ; \quad a_{|t=0} = a_0, \tag{2.3}$$

*and* $G$ *depends nonlinearly on* $a$*:*

$$\begin{aligned} a(t,x) &= \frac{1}{\sqrt{J_t(y(t,x))}} a_0\left(y(t,x)\right),\\ G(t,x) &= -\int_0^t f\left( J_s(y(t,x))^{-1} \left|a_0(y(t,x))\right|^2\right) ds. \end{aligned}$$

*In particular, if* $\kappa > 1$, *then*

$$\left\| u^\varepsilon - a e^{i\phi_{\rm eik}/\varepsilon}\right\|_{L^\infty([0,T];L^2\cap L^\infty)} \to 0 \quad \text{as } \varepsilon \to 0,$$

*and no nonlinear effect is present in the leading order behavior of* $u^\varepsilon$. *If* $\kappa = 1$, *nonlinear effects are present at leading order, measured by* $G$.

We see that the critical nonlinear effect (case $\kappa = 1$) is a self-modulation of the amplitude. In the context of laser physics, this phenomenon is known as *phase self-modulation* (see e.g. [57, 4, 24]).

*Sketch of the proof.* The proof given in [14] consists in changing the unknown function, by setting

$$a^\varepsilon = u^\varepsilon e^{-i\phi_{\rm eik}/\varepsilon},$$

where $\phi_{\rm eik}$ is given by Lemma 2.3. Then (1.1)–(1.2) is equivalent to:

$$\partial_t a^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi_{\rm eik} = i\frac{\varepsilon}{2}\Delta a^\varepsilon - i\varepsilon^{\kappa-1} f\left(|a^\varepsilon|^2\right)a^\varepsilon \quad ; \quad a^\varepsilon_{|t=0} = a^\varepsilon_0.$$

Energy estimates show that the above equation has a unique, smooth solution $a^\varepsilon \in C([0,T];H^s)$ for all $s > n/2$, uniformly bounded for $\varepsilon \in ]0,1]$. This step uses the facts that $\phi_{\rm eik}$ is sub-quadratic and $i\Delta$ is skew-symmetric. We can then neglect the terms $\varepsilon\Delta a^\varepsilon$ and $a_0^\varepsilon - a_0$, so that $\|a^\varepsilon - \widetilde{a}^\varepsilon\|_{L^\infty([0,T];H^s)} = o(1)$, where:

$$\partial_t \widetilde{a}^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla\widetilde{a}^\varepsilon + \frac{1}{2}\widetilde{a}^\varepsilon\Delta\phi_{\rm eik} = -i\varepsilon^{\kappa-1} f\left(|\widetilde{a}^\varepsilon|^2\right)\widetilde{a}^\varepsilon \quad ; \quad \widetilde{a}^\varepsilon_{|t=0} = a_0. \tag{2.4}$$

Recall that $J_t(y)$ is the Jacobi determinant. Denote

$$A^\varepsilon(t,y) := \widetilde{a}^\varepsilon\left(t, x(t,y)\right)\sqrt{J_t(y)}.$$

We see that so long as $y \mapsto x(t,y)$ defines a global diffeomorphism (which is guaranteed for $t \in [0,T]$ by construction), (2.4) is equivalent to:

$$\partial_t A^\varepsilon = -i\varepsilon^{\kappa-1} f\left(J_t(y)^{-1}\left|A^\varepsilon\right|^2\right)A^\varepsilon \quad ; \quad A^\varepsilon(0,y) = a_0(y).$$

This ordinary differential equation along the rays of geometrical optics can be solved explicitly, after we have remarked the identity $\partial_t |A^\varepsilon|^2 = 0$:

$$A^\varepsilon(t,y) = a_0(y) \exp\left( -i\varepsilon^{\kappa-1} \int_0^t f\left( J_s(y)^{-1} |a_0(y)|^2 \right) ds \right).$$

Back to the initial solution $u^\varepsilon$, this yields the proposition. □

*Remark* 2.6. A similar result is proved in [15] for the equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = V(x)u^\varepsilon + V_\Gamma\left(\frac{x}{\varepsilon}\right)u^\varepsilon + \lambda\varepsilon|u^\varepsilon|^{2\sigma}u^\varepsilon,$$

where $V_\Gamma$ is lattice-periodic. The presence of this rapidly oscillatory potential changes dramatically the geometry of the propagation. Using the corresponding Bloch theory, a similar phase self-modulation phenomenon is proved, under the assumption that the initial data are well-prepared. Removing this assumption, or considering highly nonlinear régimes (as in §3) are interesting open questions, and have physical motivations in the context of Bose–Einstein condensation.

## 3. Highly nonlinear WKB analysis: $\kappa = 0$

We saw in §1.1 that constructing a formal asymptotic expansion for (1.1)–(1.2) is a delicate issue when $\kappa = 0$. We also point out that another problem arises, even if one has managed to construct an approximate solution $v^\varepsilon$ that solves

$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta v^\varepsilon = Vv^\varepsilon + f\left(|v^\varepsilon|^2\right)v^\varepsilon + \varepsilon^N r_N^\varepsilon \quad ; \quad v^\varepsilon_{|t=0} = u^\varepsilon_{|t=0}, \tag{3.1}$$

where $N$ is large, and $r_N^\varepsilon$ is bounded in $L^2$ for instance. Setting $w^\varepsilon = u^\varepsilon - v^\varepsilon$, and supposing that $u^\varepsilon$ and $v^\varepsilon$ remain bounded in $L^\infty(\mathbb{R}^n)$ on a time interval $[0,t]$, the usual $L^2$ estimate for Schrödinger equations yields:

$$\varepsilon\|w^\varepsilon(t)\|_{L^2} \leqslant C\int_0^t \|w^\varepsilon(\tau)\|_{L^2}d\tau + 2\varepsilon^N\int_0^t \|r_N^\varepsilon(\tau)\|_{L^2}d\tau.$$

We infer, using Gronwall lemma:

$$\|w^\varepsilon(t)\|_{L^2} \leqslant C\varepsilon^{N-1}e^{Ct/\varepsilon}.$$

The exponential factor shows that this method may yield interesting results only up to time of the order $c\varepsilon|\log\varepsilon|^\theta$ for some $c, \theta > 0$. Note that in some functional analysis contexts, this may be satisfactory (see e.g. [21], or the appendices in [8, 13]). However, it seems reasonable to wish to have a description of the solution of (1.1)–(1.2) at least on a time interval independent of $\varepsilon$. We list below several approaches that yield such information.

*Remark* 3.1. In a slightly different context, a fairly explicit example in [12] shows that one may find a function satisfying (3.1) for $N$ arbitrarily large, such that

$$\liminf_{\varepsilon\to 0} \|u^\varepsilon(t^\varepsilon) - v^\varepsilon(t^\varepsilon)\|_{L^2} > 0, \quad \text{for } t^\varepsilon = \varepsilon^\beta \text{ and } 0 < \beta < 1.$$

Therefore, the stability issue in this highly nonlinear régime is really delicate.

3.1. **Modulated energy functional.** A general technique was introduced by Y. Brenier in [5]. It yields the convergence of some physically important quantities (such as the Wigner measure, see e.g. [31, 44]), but not of the wave function $u^\varepsilon$ itself. In the case of the nonlinear Schrödinger equation, it has been used by P. Zhang [58] (see also [59] for the case of the Schrödinger–Poisson equation). More recently, F. Lin and P. Zhang have adapted this approach in the case of the Gross-Pitaevskii equation, in the exterior of an obstacle [43]. We shall present the technique of Brenier in the case of (1.1)–(1.2), using the simplified approach of [43]. In all this paragraph, we will assume $V \equiv 0$: no external potential is present.

The first step consists in guessing a suitable approximate solution. Even though the system (1.6) is not closed, the analysis of §2 shows that so long as $\Phi$ is smooth and $\nabla\Phi$ is a global diffeomorphism, the second equation of (1.6) is of the form:

$$\dot{\mathtt{a}}_0 = i\Xi \mathtt{a}_0,$$

where $\dot{\mathtt{a}}_0$ stands for the differentiation along the rays associated to $\nabla\Phi$, and $\Xi$ is real-valued. In particular, the modulus of $\mathtt{a}_0$ is constant along these rays. Setting $(\rho, v) = (|\mathtt{a}_0|^2, \nabla\Phi)$ as a new unknown function, (1.6) yields:

$$\begin{cases} \partial_t v + v\cdot\nabla v + f'(\rho)\nabla\rho = 0 \quad ; \quad v_{|t=0} = \nabla\phi_0, \\ \partial_t \rho + v\cdot\nabla\rho + \rho \operatorname{div} v = 0 \quad ; \quad \rho_{|t=0} = |a_0|^2. \end{cases} \tag{3.2}$$

If $f' > 0$, we get a compressible Euler equation, which is hyperbolic symmetric in the sense of Friedrichs. We shall assume now that $f' \equiv 1$, that is, we consider a cubic, defocusing nonlinearity in (1.1). Note that older formal approaches suggest the introduction of (3.2) as a limiting equation. In [40, Chap. III], we find:

$$\begin{cases} \partial_t \Phi^\varepsilon + \frac{1}{2}|\nabla\Phi^\varepsilon|^2 + |\mathtt{a}^\varepsilon|^2 = \varepsilon^2 \frac{\Delta \mathtt{a}^\varepsilon}{2\mathtt{a}^\varepsilon} \quad ; \quad \Phi^\varepsilon_{|t=0} = \phi_0 \,, \\ \partial_t \mathtt{a}^\varepsilon + \nabla\Phi^\varepsilon\cdot\nabla \mathtt{a}^\varepsilon + \frac{1}{2}\mathtt{a}^\varepsilon\Delta\Phi^\varepsilon = 0 \quad ; \quad \mathtt{a}^\varepsilon_{|t=0} = a_0^\varepsilon \,. \end{cases} \tag{3.3}$$

Of course, this choice is not adapted when the amplitude $\mathtt{a}^\varepsilon$ vanishes, so it must be left out for a rigorous mathematical analysis, when $a_0^\varepsilon \in L^2(\mathbb{R}^n)$. Passing formally to the limit $\varepsilon \to 0$, the right hand side of the equation for $\Phi^\varepsilon$ vanishes, and using the hydrodynamical variables as above, we retrieve (3.2).

The modulated energy functional associated to (1.1)–(1.2) when $V \equiv 0$ and $f(y) = y$ is:

$$H^\varepsilon(t) = \frac{1}{2}\int_{\mathbb{R}^n} |(\varepsilon\nabla - iv)u^\varepsilon(t,x)|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^n} (\rho^\varepsilon(t,x) - \rho(t,x))^2\, dx,$$

where we have set $\rho^\varepsilon = |u^\varepsilon|^2$. We find that the time derivative of this modulated energy functional is:

$$\begin{aligned} \frac{d}{dt}H^\varepsilon(t) = &\frac{\varepsilon^2}{4}\int \nabla(\operatorname{div} v)\cdot\nabla\rho^\varepsilon - \sum_{j,k}\int \partial_j v_k \operatorname{Re}\Big( (\varepsilon\partial_j - iv_j)u^\varepsilon \overline{(\varepsilon\partial_k - iv_k)u^\varepsilon}\Big) \\ &+ \frac{3}{2}\int (\rho^\varepsilon - \rho)^2 \operatorname{div} v. \end{aligned}$$

The last two terms are estimated by $\|\nabla v(t)\|_{L^\infty} H^\varepsilon(t)$. For the first term, write

$$\begin{aligned}\varepsilon^2 \int \operatorname{div}(\nabla v)\cdot \nabla |u^\varepsilon|^2 &= \varepsilon \int \operatorname{div}(\nabla v)\cdot \left(\overline{u}^\varepsilon \varepsilon \nabla u^\varepsilon + u^\varepsilon \varepsilon \nabla \overline{u}^\varepsilon\right)\\ &= \varepsilon \int \operatorname{div}(\nabla v)\cdot \left(\overline{u}^\varepsilon (\varepsilon\nabla - iv)u^\varepsilon + u^\varepsilon \overline{(\varepsilon\nabla - iv)u^\varepsilon}\right).\end{aligned}$$

Since $\|u^\varepsilon(t)\|_{L^2} = \|a_0\|_{L^2}$ and $v \in L^\infty([0,T];W^{2,\infty})$, Young's inequality yields:

$$\frac{d}{dt} H^\varepsilon(t) \leqslant C\left(H^\varepsilon(t) + \varepsilon^2\right),$$

so long as $v$ remains smooth, that is, before shocks appear in (3.2). We conclude thanks to Gronwall lemma:

**Theorem 3.2.** *Let $n \geqslant 1$, and assume that $\kappa = 0$, $V \equiv 0$ and $f(y) = y$. Assume that there exists a smooth function $a_0$ independent of $\varepsilon$ such that*

$$a_0^\varepsilon \to a_0 \text{ in } H^s(\mathbb{R}^n), \quad \forall s \geqslant 0.$$

*Assume also that $\phi_0 \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is such that $\nabla\phi_0 \in H^s(\mathbb{R}^n)$ for all $s \geqslant 0$. Then there exists $T > 0$ independent of $\varepsilon > 0$ such that (1.1)–(1.2) has a unique solution $u^\varepsilon \in C^\infty([0,T]\times\mathbb{R}^n) \cap C([0,T];H^s)$ for all $s > n/2$. In addition, as $\varepsilon \to 0$,*

$$\|(\varepsilon\nabla - iv)u^\varepsilon\|^2_{L^\infty([0,T];L^2)} + \left\||u^\varepsilon|^2 - \rho\right\|^2_{L^\infty([0,T];L^2)} = \mathcal{O}\left(\varepsilon^2 + \left\||a_0^\varepsilon|^2 - |a_0|^2\right\|^2_{L^2}\right).$$

In the above theorem, we have not tried to compute the lowest possible value for the Sobolev regularity $s$ given by the proof, nor shall we try in the other sections.

*Remark* 3.3. In the more general case where the nonlinearity is $f(y) = y^\sigma$, with $\sigma \in \mathbb{N}$, a generalization of the above modulated energy functional was introduced in [3]. In particular, the analogue of Theorem 3.2 is proved. This includes for instance the case of the quintic, defocusing nonlinearity.

One might be afraid that the above result is somehow contradictory with Remark 3.1, or with the results of [13]. A typical example in [13], under the assumptions of Theorem 3.2, consists in choosing $a_0^\varepsilon = a_0$ independent of $\varepsilon$, and considering $v^\varepsilon$ solving (1.1)–(1.2) with $\widetilde{a}_0^\varepsilon = (1+\varepsilon^{1-\alpha})a_0$ $(0<\alpha<1)$. Then for $t^\varepsilon$ of order $\varepsilon^\alpha$,

$$\liminf_{\varepsilon\to 0} \|u^\varepsilon(t^\varepsilon) - v^\varepsilon(t^\varepsilon)\|_{L^2} > 0.$$

Yet, there is no contradiction with Theorem 3.2: the instability mechanism in [13] is the appearance of an extra oscillatory factor in $v^\varepsilon$. This oscillation shows up essentially through a multiplicative factor of the form $e^{ig(t,x)/\varepsilon^\alpha}$. It does not affects the modulus of the wave function, and vanishes in the limit $\varepsilon \to 0$ of $\varepsilon\nabla v^\varepsilon$.

We can therefore conclude that the modulated energy functional shares several features with the Wigner measure. It is a rather general tool: in [43], the authors consider a nonlinear Schrödinger equation with a boundary condition, aspect which apparently cannot be recovered with the approach of E. Grenier recalled in the next paragraph. On the other hand, by definition, it ignores the oscillatory phenomena that occur at a scale of order $\varepsilon^\alpha$ for $0<\alpha<1$ (for instance). The next section shows how to get a more precise description, under similar assumptions.

3.2. **Point-wise asymptotics without potential.** In this paragraph, we keep assuming $V \equiv 0$. Note that in (3.3), the supposedly small term on the right hand side is of order $\varepsilon^2$, while $\varepsilon$ should be enough to neglect a term in the limit $\varepsilon \to 0$. We have seen in §1.1 that the equation for the phase is obtained after simplification by the leading order amplitude. This explains the singular factor on the right hand side of (3.3). The main technical ingredient in [32] consists in shifting the source term in (3.3) to the next order, that is, the equation for the amplitude: we now seek $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$, where the amplitude $a^\varepsilon$ is *complex-valued* (even if $a_0^\varepsilon$ is real-valued), $\Phi^\varepsilon$ is real-valued, and:

$$\begin{cases} \partial_t \Phi^\varepsilon + \frac{1}{2}\left|\nabla\Phi^\varepsilon\right|^2 + f\left(|a^\varepsilon|^2\right) = 0 \quad ; \quad \Phi^\varepsilon_{|t=0} = \phi_0 \,, \\ \partial_t a^\varepsilon + \nabla\Phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \Phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon_{|t=0} = a_0^\varepsilon \,. \end{cases} \tag{3.4}$$

Another originality of this approach lies in the fact that the phase $\Phi^\varepsilon$ depends on $\varepsilon$, through the coupling of the two equations. The idea of E. Grenier consists in somehow performing the usual WKB analysis "the other way round": first, solve (3.4), then show that $\Phi^\varepsilon$ and $a^\varepsilon$ have asymptotic expansions as $\varepsilon \to 0$. In particular, this resolves the stability issue pointed out at the beginning of §3.

To solve (3.4), consider the new unknown

$$\mathbf{u}^\varepsilon = \begin{pmatrix} \operatorname{Re} a^\varepsilon \\ \operatorname{Im} a^\varepsilon \\ \nabla\Phi^\varepsilon \end{pmatrix} \in \mathbb{R}^{n+2}.$$

The system (3.4) is equivalent to a quasi-linear equation of the form:

$$\partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon = \frac{\varepsilon}{2} L\mathbf{u}^\varepsilon\,, \quad \text{with} \quad L = \begin{pmatrix} 0 & -\Delta & 0 & \dots & 0 \\ \Delta & 0 & 0 & \dots & 0 \\ 0 & 0 & & 0_{n\times n} & \end{pmatrix}, \tag{3.5}$$

$$\text{and} \quad A(\mathbf{u},\xi) = \sum_{j=1}^n A_j(\mathbf{u})\xi_j = \begin{pmatrix} v\cdot\xi & 0 & \frac{1}{2}\operatorname{Re} a\, {}^t\xi \\ 0 & v\cdot\xi & \frac{1}{2}\operatorname{Im} a\, {}^t\xi \\ 2f'\operatorname{Re} a\,\xi & 2f'\operatorname{Im} a\,\xi & v\cdot\xi I_n \end{pmatrix},$$

where $f'$ stands for $f'(|a|^2)$. The system (3.5) is hyperbolic symmetric when $f' > 0$, and we can consider the following symmetrizer:

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{4f'(|a|^2)} I_n \end{pmatrix},$$

which is symmetric and positive for $f' > 0$.

*Remark* 3.4. The argument of $f'$ is morally bounded (this will result from the analysis), but may have zeroes: the assumption $f' \geqslant 0$ cannot be considered by this approach. For instance, justifying a WKB analysis for the quintic, defocusing NLS remains an open problem.

An advantage for this choice of $S$ is that $SL$ remains a skew-symmetric operator: the possible loss of derivative caused by the second order operator $L$ does not affect the usual energy estimates in $H^s(\mathbb{R}^n)$. One can then prove existence and uniqueness for (3.5) in Sobolev spaces of sufficiently large order. Since the last $n$ components define initially, and remain, an irrotational function, this implies that we can solve (3.4). The natural limit is given by:

$$\left\{\begin{aligned}&\partial_t\Phi+\frac{1}{2}\left|\nabla\Phi\right|^2+f\left(|a|^2\right)=0\quad;\quad\Phi_{|t=0}=\phi_0\,,\\&\partial_t a+\nabla\Phi\cdot\nabla a+\frac{1}{2}a\Delta\Phi=0\quad;\quad a_{|t=0}=a_0\,.\end{aligned}\right.\tag{3.6}$$

Local existence in Sobolev spaces for (3.6) follows from the same arguments, and one has:

**Theorem 3.5.** *Let $n\geqslant 1$, and assume that $\kappa=0$, $V\equiv 0$ and $f\in C^\infty(\mathbb{R}_+;\mathbb{R})$ with $f'>0$. Assume that there exists a smooth function $a_0$ independent of $\varepsilon$ such that*

$$a_0^\varepsilon\to a_0 \text{ in } H^s(\mathbb{R}^n),\quad \forall s\geqslant 0.$$

*Assume also that $\phi_0\in C^\infty(\mathbb{R}^n;\mathbb{R})$ is such that $\nabla\phi_0\in H^s(\mathbb{R}^n)$ for all $s\geqslant 0$. Then there exists $T>0$ independent of $\varepsilon>0$ such that (1.1)–(1.2) has a unique solution $u^\varepsilon=a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$ in $C^\infty([0,T]\times\mathbb{R}^n)\cap C([0,T];H^s)$ for all $s>n/2+2$. Moreover, $a^\varepsilon$ and $\Phi^\varepsilon$ are bounded in $L^\infty([0,T];H^s)$, uniformly in $\varepsilon\in]0,1]$ and, for all $s>n/2+1$, there exists $C_s$ such that*

$$\|\nabla\Phi^\varepsilon-\nabla\Phi\|_{L^\infty([0,T];H^s)}+\|a^\varepsilon-a\|_{L^\infty([0,T];H^s)}\leqslant C_s\left(\varepsilon+\|a_0^\varepsilon-a_0\|_{H^s}\right).$$

*Therefore, $\|\Phi^\varepsilon(t)-\Phi(t)\|_{H^s}\leqslant \widetilde{C}_s t\left(\varepsilon+\|a_0^\varepsilon-a_0\|_{H^s}\right)$, $\forall t\in[0,T]$.*

Theorem 3.5 does not suffice to describe the asymptotic behavior of $u^\varepsilon$ on the time interval $[0,T]$ though:

$$u^\varepsilon-ae^{i\Phi/\varepsilon}=a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}-ae^{i\Phi/\varepsilon}=\left(a^\varepsilon-a\right)e^{i\Phi^\varepsilon/\varepsilon}+a\left(e^{i\Phi^\varepsilon/\varepsilon}-e^{i\Phi/\varepsilon}\right).$$

Therefore, we have

$$\left|u^\varepsilon-ae^{i\Phi/\varepsilon}\right|\leqslant|a^\varepsilon-a|+2|a|\left|\sin\left(\frac{\Phi^\varepsilon-\Phi}{2\varepsilon}\right)\right|$$

Taking the $L^2$ norm, we infer:

$$\begin{aligned}\left\|u^\varepsilon(t)-a(t)e^{i\Phi(t)/\varepsilon}\right\|_{L^2}&\leqslant\|a^\varepsilon(t)-a(t)\|_{L^2}+2\|a(t)\|_{L^2}\left\|\sin\left(\frac{\Phi^\varepsilon(t)-\Phi(t)}{2\varepsilon}\right)\right\|_{L^\infty}\\&\lesssim\left(\varepsilon+\|a_0^\varepsilon-a_0\|_{H^s}\right)\left(1+\frac{t}{\varepsilon}\right),\end{aligned}$$

for $s>n/2+1$. Even if $a_0^\varepsilon-a_0=\mathcal{O}(\varepsilon^N)$ for $N$ large, the above estimate shows that $ae^{i\Phi/\varepsilon}$ is a good approximation of $u^\varepsilon$ as $t\to 0$, but not necessarily at time $t=T$ for instance. To have a better error estimate, it is necessary to compute the next term in the asymptotic expansion of $(\phi^\varepsilon,a^\varepsilon)$ in powers of $\varepsilon$. Assume furthermore that there exists $a_1\in\cap_{s\geqslant 0}H^s$ such that

$$a_0^\varepsilon=a_0+\varepsilon a_1+o(\varepsilon)\quad\text{in } H^s,\ \forall s\geqslant 0.\tag{3.7}$$

For times of order $\mathcal{O}(1)$, the initial corrector $a_1$ must be taken into account:

**Proposition 3.6.** *Define* $(a^{(1)},\Phi^{(1)})$ *by*

$$\left\{\begin{aligned}&\partial_t\Phi^{(1)}+\nabla\Phi\cdot\nabla\Phi^{(1)}+2f'\left(|a|^2\right)\mathrm{Re}\left(\overline{a}a^{(1)}\right)=0,\\&\partial_t a^{(1)}+\nabla\Phi\cdot\nabla a^{(1)}+\nabla\Phi^{(1)}\cdot\nabla a+\frac{1}{2}a^{(1)}\Delta\Phi+\frac{1}{2}a\Delta\Phi^{(1)}=\frac{i}{2}\Delta a,\\&\Phi^{(1)}_{|t=0}=0\quad;\quad a^{(1)}_{|t=0}=a_1.\end{aligned}\right.$$

*Then* $a^{(1)},\Phi^{(1)}\in L^\infty([0,T];H^s)$ *for every* $s\geqslant 0$*, and*

$$\|a^\varepsilon-a-\varepsilon a^{(1)}\|_{L^\infty([0,T];H^s)}+\|\Phi^\varepsilon-\Phi-\varepsilon\Phi^{(1)}\|_{L^\infty([0,T];H^s)}\leqslant C_s\varepsilon^2,\quad\forall s\geqslant 0\,.$$

Despite the notations, it seems unadapted to consider $\Phi^{(1)}$ as being part of the phase. Indeed, we infer from Proposition 3.6 that

$$\left\|u^\varepsilon-ae^{i\Phi^{(1)}}e^{i\Phi/\varepsilon}\right\|_{L^\infty([0,T];L^2\cap L^\infty)}=\mathcal{O}(\varepsilon).$$

Relating this information to the WKB methods presented in §1.1, we would have:

$$\mathsf{a}_0=ae^{i\Phi^{(1)}}.$$

Since $\Phi^{(1)}$ depends on $a_1$ while $a$ does not, we retrieve the fact that in super-critical régimes, the leading order amplitude in WKB methods depends on the initial first corrector $a_1$.

*Remark* 3.7. The term $e^{i\Phi^{(1)}}$ does not appear in the Wigner measure of $ae^{i\Phi^{(1)}}e^{i\Phi/\varepsilon}$. Thus, from the point of view of Wigner measures, the asymptotic behavior of the exact solution is described by the Euler-type system (3.2).

*Remark* 3.8. If we assume that $a_0$ is real-valued, then so is $a$. If moreover $a_1$ is purely imaginary (for instance, if $a_1=0$), then we see that $a^{(1)}$ is purely imaginary, hence, $\Phi^{(1)}\equiv 0$.

*Remark* 3.9. The proof of Theorem 1.1 follows. Consider initial data of the form

$$u_0(x)=\lambda^{-\frac{n}{2}+s}a_0\left(\frac{x}{\lambda}\right),\quad\lambda\to 0.$$

Set $\varepsilon=\lambda^{s_c-s}$: $\varepsilon$ and $\lambda$ go simultaneously to zero, by assumption. Define

$$\psi^\varepsilon(t,x)=\lambda^{\frac{n}{2}-s}u\left(\lambda^{\frac{n}{2}+1-s}t,\lambda x\right)\,.$$

It solves:

$$i\varepsilon\partial_t\psi^\varepsilon+\frac{\varepsilon^2}{2}\Delta\psi^\varepsilon=|\psi^\varepsilon|^2\psi^\varepsilon\quad;\quad\psi^\varepsilon_{|t=0}=a_0(x)\,.\tag{3.8}$$

The idea of the proof is that for times of order $\mathcal{O}(1)$, $\psi^\varepsilon$ has become $\varepsilon$-oscillatory. This is rather clear from (3.6): even though $\Phi_{|t=0}=0$, we have $\partial_t\Phi_{|t=0}\neq 0$, and rapid oscillations at scale $\varepsilon$ appear instantly. Back to $u$, this yields the theorem (up to replacing $a_0$ by $|\log\lambda|^{-1}a_0$).

To conclude this paragraph, we point out an open problem concerning the time $T_c$ when shocks appear for (3.2). First, the break-up for (3.2) does not allow us to deduce anything concerning the behavior of the solution of (3.4). More generally, the notion of caustic in this case is not so clear. Geometrically, as $t\to T_c$, the rays for (3.6) tend to form an envelope. In the linear case $f\equiv 0$, this geometrical phenomenon goes along with an analytical one:

$$\liminf_{\varepsilon\to 0}\|u^\varepsilon(t)\|_{L^\infty}\to+\infty\quad\text{as } t\to T_c.$$

For instance, $\|u^\varepsilon(t)\|_{L^\infty} \approx (\varepsilon + |T_c - t|)^{-n/2}$ for all $t$ in the case of a focal point (all the rays meet at one point as $t \to T_c$).

It is not clear at all that a similar phenomenon occurs for (1.1) when $\kappa = 0$. Suppose for instance that the nonlinearity is cubic, defocusing, $f(y) = y$, and that the initial profile $a_0^\varepsilon$ does not depend on $\varepsilon$, $a_0^\varepsilon = a_0$. The standard conservations of mass and energy for nonlinear Schrödinger equations yield:

$$\|u^\varepsilon(t)\|_{L^2} = \|a_0\|_{L^2} = \mathcal{O}(1),$$
$$\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}^2 + \|u^\varepsilon(t)\|_{L^4}^4 = \|\varepsilon\nabla a_0 + ia_0\nabla\phi_0\|_{L^2}^2 + \|a_0\|_{L^4}^4 = \mathcal{O}(1).$$

In space dimension $n \leqslant 3$, the solution $u^\varepsilon$ remains in $H^1(\mathbb{R}^n)$ for all time, therefore we know that the $L^2$ and $L^4$ norms of $u^\varepsilon(t,\cdot)$ remain bounded by a constant independent of $\varepsilon$. This *suggests* that the $L^\infty$ norm of $u^\varepsilon(t,\cdot)$ may remain bounded, if we can somehow inverse the Hölder inequality

$$\|u^\varepsilon(t)\|_{L^4}^4 \leqslant \|u^\varepsilon(t)\|_{L^2}^2 \|u^\varepsilon(t)\|_{L^\infty}^2.$$

One could then distinguish two notions of caustic: a geometrical one (present in all the cases), and an analytical one (possibly absent in the highly nonlinear case).

3.3. **Point-wise asymptotics with an external potential.** Physical motivations may lead to the study of (1.1)–(1.2) when the external potential $V$ is not zero. Mathematically, a special role is played by sub-quadratic potentials, as we have noticed in §2; see Remark 2.4. We therefore suppose that Assumption 2.1 is satisfied.

The analysis presented in §3.1 suggests that in this case, we have to consider solutions to a compressible Euler equation with (possibly) unbounded external force and initial velocity:

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla V + f'(\rho)\nabla\rho = 0 \quad ; \quad v_{|t=0} = \nabla\phi_0, \\ \partial_t\rho + v\cdot\nabla\rho + \rho\,\mathrm{div}\, v = 0 \quad ; \quad \rho_{|t=0} = |a_0|^2. \end{cases} \tag{3.9}$$

The existence of such solutions is not standard. The naive approach presented in [14] consists in resuming the idea of E. Grenier, writing the unknown phase $\Phi^\varepsilon$ as

$$\Phi^\varepsilon = \phi_{\mathrm{eik}} + \phi^\varepsilon,$$

and considering (3.4) where $\nabla\phi^\varepsilon$ has replaced $\nabla\Phi^\varepsilon$ as an unknown function. This procedure is similar to linearizing (3.4) in $\Phi^\varepsilon$, around $\phi_{\mathrm{eik}}$. Of course, extra terms appear at this stage. Note that the space where we seek $\Phi^\varepsilon$ is of mixed type: $\Phi^\varepsilon$ is the sum of a smooth, sub-quadratic (and possibly unbounded) function, and the phase $\phi^\varepsilon(t,\cdot)$ which is sought in Sobolev spaces $H^s(\mathbb{R}^n)$. Nevertheless, $\phi^\varepsilon$ must not be considered as small, as shown by the analysis of §3.2.

The good news is that the extra terms that have appeared can be treated as *semi-linear* perturbations in the energy estimates. This is due to the fact that the phase $\phi_{\mathrm{eik}}$ is sub-quadratic in space. Therefore, the analysis of §3.2 is easily adapted: provided that we assume $f' > 0$, an analogue of Theorem 3.5 is available. Note that unless $f' = $ Const. (in which case the symmetrizer $S$ is constant), we need the extra decay assumption on the initial profile:

$$xa_0 \in \cap_{s\geqslant 0} H^s, \quad \text{and } xa_0^\varepsilon \to xa_0 \text{ in } H^s(\mathbb{R}^n), \quad \forall s \geqslant 0.$$

In particular, a local solution to (3.9) is constructed. We refer to [14] for precise statements in this case.

*Remark* 3.10. For Schrödinger–Poisson equations in space dimension $n \geqslant 3$, the idea of E. Grenier was adapted in [2], under more general geometrical assumptions. For instance, solutions that do not necessarily have a zero limit at spatial infinity are considered. Under the assumptions of [59], a point-wise asymptotics of the wave function is given, which is more precise that the results in [59].

3.4. **The case of focusing nonlinearities.** Note that in §3.2, the study of (3.4) involves a quasi-linear system whose principal part writes:

$$\square_{f'} = \partial_t^2 - \operatorname{div}\left(f'(|u^\varepsilon|^2)\nabla\cdot\right).$$

This has the same form as the principal part for (3.6), which is the limiting system expected in general, whichever formal approach is followed. When $f' > 0$, we face a quasi-linear wave equation. We have pointed out some open problems under the weaker assumption $f' \geqslant 0$ (a case where loss of hyperbolicity may occur). When $f' < 0$, the above operator becomes elliptic: it does not seem adapted to work in Sobolev spaces any more. On the other hand, data and solutions with analytic regularity seem appropriate.

In [30], P. Gérard works with the analytic regularity, when the space variable $x$ belongs to the torus $\mathbb{T}^n$, without external potential ($V \equiv 0$). Note that the only assumption needed on the nonlinearity $f$ is analyticity near the range of $|a_0|^2$. This includes the focusing case $f' < 0$, as well as the defocusing quintic case $f(y) = y^2$ for instance.

The initial phase $\phi_0$ is supposed real analytic, and the initial amplitude is analytic in the sense of J. Sjöstrand [50]: there exist $\ell > 0$, $A > 0$, $B > 0$ such that, for all $j \geqslant 0$, $a_j$ is holomorphic in $\{|\operatorname{Im} x| < \ell\}$, and

$$|a_j(x)| \leqslant AB^j j!$$

Denoting $\overline{a}(t,x)$ the complex conjugate of $a(\overline{t},\overline{x})$, P. Gérard considers the system:

$$\begin{cases} \partial_t v^\varepsilon = -v^\varepsilon \cdot \nabla v^\varepsilon - \nabla f\left(a_0\overline{a}_0\right), \\ \partial_t a^\varepsilon = -v^\varepsilon \cdot \nabla a^\varepsilon - \dfrac{1}{2} a^\varepsilon \operatorname{div} v^\varepsilon + i\dfrac{\varepsilon}{2}\Delta a^\varepsilon - \dfrac{i a^\varepsilon}{\varepsilon}\left(f\left(a^\varepsilon \overline{a}^\varepsilon\right) - f\left(a_0\overline{a}_0\right)\right). \end{cases}$$

A solution of the form

$$u^\varepsilon = a^\varepsilon e^{i\phi/\varepsilon}, \quad a^\varepsilon(t,x) = \sum_{j\geqslant 0} \varepsilon^j a^{(j)}(t,x),$$

where the sum is defined in the sense of J. Sjöstrand, is thus obtained. Setting

$$v^\varepsilon = e^{i\phi/\varepsilon} \sum_{j\leqslant 1/(C_0\varepsilon)} \varepsilon^j a^{(j)}$$

for $C_0$ sufficiently large, the approximate solution $v^\varepsilon$ satisfies:

$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta v^\varepsilon = f\left(|v^\varepsilon|^2\right) v^\varepsilon + \mathcal{O}\left(e^{-\delta/\varepsilon}\right),$$

for some $\delta > 0$. Essentially, this source term is sufficiently small to overcome the difficulty pointed out at the beginning of §3: for small time independent of $\varepsilon$, the exponential growth provided by Gronwall lemma is more than compensated by the term $e^{-\delta/\varepsilon}$. We refer to [30] for precise statements and elements of proof.

3.5. **The integrable case.** In the one-dimensional case, $n = 1$, for a cubic nonlinearity ($f(y) = \pm y$), the Schrödinger equation is completely integrable. This property remains with a time-independent external potential which is a polynomial of degree at most two [1, p. 375].

In the absence of potential, several papers have studied the semi-classical limit for (1.1)–(1.2) for the cubic NLS in space dimension one. See for instance [33] in the defocusing case, and [35, 53] in the focusing case. A very interesting aspect of this approach is that it yields a description of the solution $u^\varepsilon$ even after shocks have appeared for the limiting Euler equation (3.2). This description involves theta functions, and the so-called Whitham equations (see [52]). In particular, this approach seems to confirm the formal discussion of the end of §3.2: in the defocusing case, the $L^\infty$ norm of the solution $u^\varepsilon$ remains bounded as $\varepsilon \to 0$, for all time.

Unfortunately, it seems that all the results in the integrable case have been written in a way that makes any comparison with the other results mentioned above very difficult. The last step of inverse scattering is not always performed, which should yield a point-wise asymptotics of the wave function $u^\varepsilon$. Moreover, the spaces in which it would be available are not completely clear. The space $L^\infty_{\rm loc}(\mathbb{R}_x)$ seems the most natural candidate. A bridge between the approaches of §3.1 and §3.2 on the one hand, and the approaches in the integrable case on the other hand, would certainly be welcome in the community of semi-classical analysis for nonlinear Schrödinger equations.

## 4. Propagation of concentrated initial data

4.1. **Defocusing nonlinearity.** We now consider (1.7) with $\lambda > 0$. By scaling, we may assume $\lambda = 1$. The general heuristic argument is the following. For $t$ close to zero, the solution $\mathtt{u}^\varepsilon$ remains concentrated near the point $x_0$, at a scale of order $\varepsilon$. Since the potential $V$ does not depend on $\varepsilon$, we have $V\mathtt{u}^\varepsilon \sim V(x_0)\mathtt{u}^\varepsilon$: the potential can be considered as constant at leading order. Introduce the function $\psi^\varepsilon$ given by the scaling

$$\mathtt{u}^\varepsilon(t,x) = \frac{1}{\varepsilon^{n/2}}\psi^\varepsilon\left(\frac{t}{\varepsilon},\frac{x-x_0}{\varepsilon}\right)e^{i(x_0\cdot\xi_0/\varepsilon - V(x_0)t/\varepsilon)}.$$

The Cauchy problem (1.7) is equivalent to:

$$i\partial_t\psi^\varepsilon + \frac{1}{2}\Delta\psi^\varepsilon = \left(V(x_0+\varepsilon x) - V(x_0)\right)\psi^\varepsilon + |\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \quad ; \quad \psi^\varepsilon(0,x) = R(x)e^{ix\cdot\xi_0}.$$

The above argument suggests that we have $\psi^\varepsilon \sim \psi$, where $\psi$ is independent of $\varepsilon$ and solves:

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = |\psi|^{2\sigma}\psi \quad ; \quad \psi(0,x) = R(x)e^{ix\cdot\xi_0}. \tag{4.1}$$

Under suitable assumptions on $\sigma$ and $R$, there is scattering for this equation (see e.g. [17, 18, 45]): there exist $\psi_\pm \in L^2(\mathbb{R}^n)$ such that

$$\left\|\psi(t) - e^{i\frac{t}{2}\Delta}\psi_\pm\right\|_{L^2} \to 0 \quad \text{as } t \to \pm\infty. \tag{4.2}$$

The standard asymptotics of the free Schrödinger group $e^{i\frac{t}{2}\Delta}$ then yields:

$$\psi(t,x) \underset{t\to\pm\infty}{\sim} \frac{e^{i|x|^2/(2t)}}{(it)^{n/2}}\widehat{\psi}_\pm\left(\frac{x}{t}\right),$$

where the Fourier transform is given by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

Back to $\mathtt{u}^\varepsilon$, this yields, for $t \gg \varepsilon$ and so long as we consider the external potential as constant:

$$\mathtt{u}^\varepsilon(t,x) \sim \frac{1}{(it)^{n/2}} \widehat{\psi}_+ \left(\frac{x-x_0}{t}\right) e^{i\frac{|x-x_0|^2}{2\varepsilon t}} e^{i(x_0\cdot\xi_0/\varepsilon - V(x_0)t/\varepsilon)}. \tag{4.3}$$

Indeed, we have the following rigorous result:

**Proposition 4.1** ([16], Proposition 6.3)**.** *Let $V$ satisfying Assumption 2.1. Let $R \in \Sigma := H^1 \cap \mathcal{F}(H^1)$, and*

$$\frac{2-n+\sqrt{n^2+12n+4}}{4n} \leqslant \sigma < \frac{2}{n-2}.$$

*Then for any $\Lambda > 0$, the following holds:*
1. *There exists $\varepsilon(\Lambda) > 0$ such that for $0 < \varepsilon \le \varepsilon(\Lambda)$, the initial value problem* (1.7) *has a unique solution $\mathtt{u}^\varepsilon \in C([-\Lambda\varepsilon, \Lambda\varepsilon]; \Sigma)$.*
2. *This solution satisfies the following asymptotics,*

$$\limsup_{\varepsilon\to 0} \sup_{|t|\leqslant \Lambda\varepsilon} \left\|\mathtt{u}^\varepsilon(t) - \mathtt{v}^\varepsilon(t)\right\|_{L^2} = 0 \ ,$$

*where $\mathtt{v}^\varepsilon$ is given by* $$\mathtt{v}^\varepsilon(t,x) = \frac{1}{\varepsilon^{n/2}} \psi\left(\frac{t}{\varepsilon}, \frac{x-x_0}{\varepsilon}\right) e^{i(x_0\cdot\xi_0/\varepsilon - V(x_0)t/\varepsilon)},$$

*and $\psi \in C(\mathbb{R}; \Sigma)$ is given by* (4.1)*.*

A transition is expected to occur in the above boundary layer, that is for $|t| = \Lambda\varepsilon$ and $\Lambda \gg 1$. The heuristic argument consists in saying that because of dispersion for $\psi$, the external potential $V$ can no longer be considered as constant. On the other hand, and for the same reason, the nonlinearity ceases to be relevant at leading order: for $\varepsilon \ll \pm t \leqslant T$, we expect $\mathtt{u}^\varepsilon \sim \mathtt{u}^\varepsilon_\pm$, where

$$i\varepsilon\partial_t \mathtt{u}^\varepsilon_\pm + \frac{\varepsilon^2}{2}\Delta \mathtt{u}^\varepsilon_\pm = V\mathtt{u}^\varepsilon_\pm \quad ; \quad \mathtt{u}^\varepsilon(0,x) = \frac{1}{\varepsilon^{n/2}} \psi_\pm\left(\frac{x}{\varepsilon}\right) e^{i(x_0\cdot\xi_0/\varepsilon - V(x_0)t/\varepsilon)}, \tag{4.4}$$

and $\psi_\pm$ are given by (4.2). The value of $T$ is not arbitrary: the asymptotic behavior of $\mathtt{u}^\varepsilon_\pm$ involves the classical trajectories associated to $V$. These trajectories may refocus at one point; this is the case when $V$ is an isotropic harmonic potential for instance.

Proving the above asymptotics for $\varepsilon \ll \pm t \leqslant T$ is actually an open problem for general potentials satisfying Assumption 2.1, even for time-independent potentials. It has been proved when $V = V(x)$ is exactly a polynomial of degree at most two, in [11] for the case of refocusing(s), and in [16] for the complementary case.

The restriction to this class of polynomial potentials is certainly purely technical, and we know explain it. The proof of the asymptotics for $\varepsilon \ll \pm t \leqslant T$ relies on the use of operators well suited to the propagation of classical trajectories associated to $V$. In the linear setting, good candidates to meet this requirement are given by the action of Heisenberg derivatives (see e.g. [48]):

$$U^\varepsilon(t)\varepsilon\nabla U^\varepsilon(-t) \text{ and } U^\varepsilon(t)\frac{x-x_0}{\varepsilon}U^\varepsilon(-t), \quad \text{where } U^\varepsilon(t) = e^{-i\frac{t}{\varepsilon}\left(-\frac{\varepsilon^2}{2}\Delta + V\right)}.$$

The main technical remark in [11, 16] is that when $V$ is a polynomial of degree at most two, then the above two Heisenberg derivatives are very interesting for nonlinear problems too. Indeed, we can find $p = p(t)$, and $\phi = \phi(t,x)$ real-valued, such that, for instance:

$$U^\varepsilon(t)\frac{x-x_0}{\varepsilon}U^\varepsilon(-t) = p(t)e^{i\phi(t,x)/\varepsilon}\nabla\left(e^{-i\phi(t,x)/\varepsilon}\cdot\right). \tag{4.5}$$

In [16], it is proved that an operator of the form of the right hand side of (4.5) commutes with $U^\varepsilon(t)$ if and only if $V$ is a polynomial of degree at most two, and $\phi$ solves the eikonal equation (2.1). The fact that an Heisenberg derivative commutes with the group $U^\varepsilon(t)$ is a straightforward consequence of its definition. The right hand side of (4.5) implies two important things:

- This Heisenberg derivative acts on gauge invariant nonlinearities $G(|u|^2)u$ like a derivative.
- Weighted Gagliardo–Nirenberg inequalities are available, of the form

$$\|\varphi\|_{L^r} \leqslant C_r|p(t)|^{-\delta(r)}\|\varphi\|_{L^2}^{1-\delta(r)}\left\|U^\varepsilon(t)\frac{x-x_0}{\varepsilon}U^\varepsilon(-t)\varphi\right\|_{L^2}^{\delta(r)}.$$

To illustrate the use of these properties, we recall [11, Corollary 1.3]:

**Proposition 4.2.** *Let $R\in\Sigma$. Assume that $\mathtt{u}^\varepsilon$ solves* (1.7) *with $x_0=\xi_0=0$ and*

$$V(x)=\frac{|x|^2}{2}\cdot$$

*Let $\psi_\pm = W_\pm^{-1}R$ be given by* (4.2) *(upon suitable assumptions on $\sigma$, see e.g. Prop. 4.1). Then for any $2<r<\frac{2n}{n-2}$, the following asymptotics holds in $L^2\cap L^r$:*

- *If $0<t<\pi$, then* $\mathtt{u}^\varepsilon(t,x)\underset{\varepsilon\to 0}{\sim}\left(\frac{1}{i\sin t}\right)^{n/2}\widehat{\psi_+}\left(\frac{x}{\sin t}\right)e^{i\frac{|x|^2}{2\varepsilon\tan t}}$.
- *If $-\pi<t<0$, then* $\mathtt{u}^\varepsilon(t,x)\underset{\varepsilon\to 0}{\sim}\left(\frac{1}{i\sin t}\right)^{n/2}\widehat{\psi_-}\left(\frac{x}{\sin t}\right)e^{i\frac{|x|^2}{2\varepsilon\tan t}}$.

*Remark* 4.3. The result of [9] shows that in the above case, Wigner measure is not a good tool to characterize the behavior of $\mathtt{u}^\varepsilon$. More precisely, we can find $R_1,R_2\in\Sigma$ such that the Wigner measures for the corresponding solutions $\mathtt{u}_1^\varepsilon$ and $\mathtt{u}_2^\varepsilon$ coincide at time $t=-\pi/2$, but are different at time $t=\pi/2$. The crossing of a focal point may lead to an ill-posed Cauchy problem as far as Wigner measures are concerned.

We see that the formal asymptotics (4.3) is valid only in the transition régime $t=\Lambda\varepsilon$, with $\Lambda\gg 1$. For larger times, the trigonometric functions in the above result account for the dynamical influence of the harmonic potential.

In the above case of an isotropic harmonic potential, the above result can be iterated in time. Recall that the (nonlinear) scattering operator $S$ associated to (4.1) maps $\psi_-$ to $\psi_+$, given by (4.2).

**Corollary 4.4.** *Under the assumptions of Proposition 4.2, consider $k\in\mathbb{N}$. For $k\pi<t<(k+1)\pi$, and $2<r<\frac{2n}{n-2}$, the following asymptotics holds in $L^2\cap L^r$:*

$$\mathtt{u}^\varepsilon(t,x)\underset{\varepsilon\to 0}{\sim}\frac{e^{-in\frac{\pi}{4}-ink\frac{\pi}{2}}}{|\sin t|^{n/2}}\widehat{S^k\psi_+}\left(\frac{x}{\sin t}\right)e^{i\frac{|x|^2}{2\varepsilon\tan t}},$$

*where $S^k$ denotes the $k^{\text{th}}$ iterate of the scattering operator $S$.*

The phase shift $e^{-ink\frac{\pi}{2}}$ corresponds to successive Maslov indices: this is a linear phenomenon [25]. On the other hand, we see that a nonlinear phenomenon occurs at leading order at time $t = k\pi$, which is measured by the scattering operator $S$.

4.2. **Focusing nonlinearity.** When $\lambda < 0$ in (1.1)–(1.3), we assume similarly that $\lambda = -1$. We let $R = Q$, the unique positive, radially symmetric ([39]), solution of:

$$-\frac{1}{2}\Delta Q + Q = |Q|^{2\sigma}Q.$$

Now, the focusing nonlinearity is an obstruction to dispersive phenomena. The solution $u^\varepsilon$ is expected to keep the ground state as a leading order profile. Nevertheless, the point where it is centered in the phase space, initially $(x_0, \xi_0)$, should evolve according to the Hamiltonian flow (2.2). In the absence of external potential, $V \equiv 0$, we have explicitly:

$$u^\varepsilon(t,x) = Q\left(\frac{x - x(t)}{\varepsilon}\right)e^{ix\cdot\xi(t)/\varepsilon + i\theta(t)/\varepsilon},$$

where $(x(t), \xi(t)) = (x_0 + t\xi_0, \xi_0)$ solves (2.2) with initial data $(x_0, \xi_0)$, and $\theta(t) = t - t|\xi_0|^2/2$. When $V$ is not trivial, seek $u^\varepsilon$ of the form of a rescaled WKB expansion:

$$u^\varepsilon(t,x) \sim \left(\sum_{j\geqslant 0}\varepsilon^j U_j\left(\frac{t}{\varepsilon}, \frac{x - x(t)}{\varepsilon}\right)\right)e^{i\phi(t,x)/\varepsilon}.$$

Note that this scaling meets the exact result of the case $V \equiv 0$. Plugging this expansion into (1.1)–(1.3) and canceling the $\mathcal{O}(\varepsilon^0)$ term, we get:

$$i\partial_t U_0 + \frac{1}{2}\Delta U_0 + U_0\left(-\partial_t\phi - \frac{1}{2}|\nabla\phi|^2 - V + |U_0|^{2\sigma}\right) - i\left(\dot{x}(t) - \nabla\phi\right)\cdot\nabla U_0 = 0.$$

Impose the leading order profile to be the standing wave given by

$$U_0(t,x) = e^{it}Q(x).$$

Then the above equation becomes:

$$U_0\left(-\partial_t\phi - \frac{1}{2}|\nabla\phi|^2 - V\right) - i\left(\dot{x}(t) - \nabla\phi\right)\cdot\nabla U_0 = 0.$$

Since $U_0 e^{-it}$ is real-valued, and since we seek a real-valued phase $\phi$, this yields:

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + V = 0 \quad ; \quad \phi(0,x) = x\cdot\xi_0.$$
$$\dot{x}(t) = \nabla\phi(t,x).$$

The first equation is the eikonal equation (2.1). We infer that we have exactly

$$\nabla\phi\left(t, x(t)\right) = \xi(t).$$

The form of $U_0$ and the exponential decay of $Q$ show that we can formally assume that $x = x(t) + \mathcal{O}(\varepsilon)$. In this case,

$$\nabla\phi(t,x) = \nabla\phi\left(t, x(t)\right) + \mathcal{O}(\varepsilon) = \xi(t) + \mathcal{O}(\varepsilon) = \dot{x}(t) + \mathcal{O}(\varepsilon).$$

Thus, we have canceled the $\mathcal{O}(\varepsilon^0)$ term, up to adding extra terms of order $\varepsilon$, that would be considered in the next step of the analysis, which we stop here. Back to $u^\varepsilon$, this formal computation yields

$$u^\varepsilon(t,x) \sim Q\left(\frac{x - x(t)}{\varepsilon}\right)e^{i\phi(t,x)} \sim Q\left(\frac{x - x(t)}{\varepsilon}\right)e^{ix\cdot\xi(t)/\varepsilon + i\theta(t)/\varepsilon},$$

where $\theta(t) = t\left(1 - |\xi_0|^2/2 - V(x_0)\right) + \int_0^t x(s)\cdot\nabla V(x(s))ds.$

To give the above formal analysis a rigorous justification, the following assumptions are made in [38]:

**Assumption 4.5.** *The nonlinearity is $L^2$-subcritical: $\sigma < 2/n$.*
*The potential $V = V(x)$ is real-valued, and can be written as $V = V_1 + V_2$, where*

- $V_1 \in W^{3,\infty}(\mathbb{R}^n)$.
- $\partial^\alpha V_2 \in W^{2,\infty}(\mathbb{R}^n)$ *for every multi-index $\alpha$ with $|\alpha| = 2$.*

For instance, $V$ can be an harmonic potential.

**Theorem 4.6** ([38]). *Let $x_0, \xi_0 \in \mathbb{R}^n$. Under Assumption 4.5, the solution $u^\varepsilon$ to (1.1)–(1.3) with $R = Q$ can be approximated as follows:*

$$u^\varepsilon(t,x) = Q\left(\frac{x - x(t)}{\varepsilon}\right)e^{ix\cdot\xi(t)/\varepsilon + i\theta^\varepsilon(t)/\varepsilon} + \mathcal{O}(\varepsilon) \quad \text{in } L^\infty_{\rm loc}(\mathbb{R}_t; X^\varepsilon),$$

*where $(x(t), \xi(t))$ is given by the Hamiltonian flow, the real-valued function $\theta^\varepsilon$ depends on $t$ only, and $X^\varepsilon$ is defined by the norm*

$$\|f\|^2_{X^\varepsilon} = \frac{1}{\varepsilon^n}\|f\|^2_{L^2} + \frac{1}{\varepsilon^{n-2}}\|\nabla f\|^2_{L^2}.$$

*Remark* 4.7. The assumption $\sigma < 2/n$ is crucial for the above result to hold. Indeed, if $\sigma = 2/n$ and $V$ is the isotropic harmonic potential

$$V(x) = \frac{|x|^2}{2},$$

then we have explicitly, when $x_0 = \xi_0 = 0$ (see [10, 38]):

$$u^\varepsilon(t,x) = \frac{1}{(\cos t)^{n/2}} Q\left(\frac{x}{\varepsilon\cos t}\right)e^{i\frac{\tan t}{\varepsilon} - i\frac{|x|^2}{2\varepsilon}\tan t}, \quad 0 \leqslant t < \frac{\pi}{2},$$

so the profile $Q$ is modulated as time evolves, in a fashion similar to §4.1.

The proof of the above result heavily relies on the orbital stability of the ground state, which holds when $\sigma < 2/n$. For $v \in H^1(\mathbb{R}^n)$, denote

$$\mathcal{E}(v) = \frac{1}{2}\|\nabla v\|^2_{L^2} - \frac{1}{\sigma+1}\|v\|^{2\sigma+2}_{L^{2\sigma+2}}.$$

The ground state $Q$ is the unique solution, up to translation and rotation, to the minimization problem:

$$\mathcal{E}(Q) = \inf\{\mathcal{E}(v)\ ;\ v \in H^1(\mathbb{R}^n) \text{ and } \|v\|_{L^2} = \|Q\|_{L^2}\}.$$

The orbital stability is given by the following result:

**Proposition 4.8** ([54]). *Let $\sigma < 2/n$. There exist $C, h > 0$ such that if $\phi \in H^1(\mathbb{R}^n)$ is such that $\|\phi\|_{L^2} = \|Q\|_{L^2}$ and $\mathcal{E}(\phi) - \mathcal{E}(Q) < h$, then:*

$$\inf_{y\in\mathbb{R}^n, \theta\in\mathbb{T}} \left\|\phi - e^{i\theta}Q(\cdot - y)\right\|^2_{H^1} \leqslant C\left(\mathcal{E}(\phi) - \mathcal{E}(Q)\right).$$

The strategy in [38] consists in applying the above result to the function

$$v^\varepsilon(t,x) = u^\varepsilon\left(t, \varepsilon x + x(t)\right)e^{-i(\varepsilon x + x(t))\cdot\xi(t)/\varepsilon}.$$

For $A>0$ sufficiently large, let $\chi$ be a smooth non-negative cut-off function, supported in $\{x \in \mathbb{R}^n; |x| \leqslant 2A\}$, and constant equal to 1 in $\{x \in \mathbb{R}^n; |x| \leqslant A\}$. Introduce the error estimate $\eta^\varepsilon(t)$ given by $\eta^\varepsilon = \eta_1^\varepsilon + \eta_2^\varepsilon + \eta_3^\varepsilon + \eta_4^\varepsilon$, where:

$$\begin{aligned}
\eta_1^\varepsilon(t) &= \int_{\mathbb{R}^n} x\chi(x) m^\varepsilon(t,x)dx - \|Q\|_{L^2}^2 x(t),\\
\eta_2^\varepsilon(t) &= \int_{\mathbb{R}^n} \nabla V_2(x) m^\varepsilon(t,x)dx - \|Q\|_{L^2}^2 \nabla V_2\left(x(t)\right),\\
\eta_3^\varepsilon(t) &= \int_{\mathbb{R}^n} \xi^\varepsilon(t,x)dx - \|Q\|_{L^2}^2 \xi(t),\\
\eta_4^\varepsilon(t) &= \int_{\mathbb{R}^n} \chi(x)V(x) m^\varepsilon(t,x)dx - \|Q\|_{L^2}^2 V\left(x(t)\right),\\
m^\varepsilon(t,x) &= \frac{1}{\varepsilon^n}|u^\varepsilon(t,x)|^2 \quad ; \quad \xi^\varepsilon(t,x) = \frac{1}{\varepsilon^{n-1}} \operatorname{Im}\left(\overline{u}^\varepsilon \nabla u^\varepsilon\right).
\end{aligned}$$

Noting that $\eta^\varepsilon(0) = \mathcal{O}(\varepsilon^2)$, the proof in [38] shows that $\eta^\varepsilon(t) = \mathcal{O}(\varepsilon^2)$ for $t \in [0, T_0]$ for some $T_0 > 0$ independent of $\varepsilon$. The proof eventually relies on Gronwall lemma and a continuity argument. In order to invoke these arguments, S. Keraani uses Proposition 4.8 and the scheme of the proof of J. Bronski and R. Jerrard [6], based on duality arguments and estimates on measures. Finally, the time $T_0$ given by the proof depends only on constants of the motion, so the argument can be repeated indefinitely, to get the $L^\infty_{\rm loc}$ estimate of Theorem 4.6.

In the particular case where the external potential $V$ is an harmonic potential (isotropic or anisotropic), the proof can be simplified. We invite the reader to pay attention to the short note [37], where this simplification is available.

The phase shift $\theta^\varepsilon$ in Theorem 4.6 is not known in general. It is easy to guess from the arguments given above that in the proof given by S. Keraani, it stems from the use of Proposition 4.8. On the other hand, as noted in [38], a time-dependent phase shift does not alter the Wigner measure of $u^\varepsilon$, which is an important physical quantity.

**Acknowledgments.** The author is grateful to Thomas Alazard for his careful reading of the manuscript, and for his comments.

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