Minimal blocking sets of size $q^2 + 2$ of $Q(4, q)$, $q$ an odd prime, do not exist

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Abstract

It is known that every blocking set of $Q(4, q)$, $q > 2$ even, with less than $q^2 + 1 + \sqrt{q}$ points contains an ovoid, and hence $Q(4, q)$ has no minimal blocking set $\mathcal{B}$ with $q^2 + 1 < |\mathcal{B}| < q^2 + 1 + \sqrt{q}$. In contrast to this, it is even not known whether or not $Q(4, q)$, $q$ odd, has minimal blocking sets of size $q^2 + 2$. In this paper, the non-existence of a minimal blocking set of size $q^2 + 2$ of $Q(4, q)$, $q$ an odd prime, is shown. Strong geometrical information is obtained using an algebraic description of $W(3, q)$. Geometrical and combinatorial arguments complete the proof.

Key words: ovoid, blocking set, polar space, parabolic quadric
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1 Introduction

Consider the non-singular parabolic quadric $Q(4, q)$ in the 4-dimensional projective space $\text{PG}(4, q)$. It is known that $(q^2 + 1)(q + 1)$ points and the same number of lines of $\text{PG}(4, q)$ are contained in $Q(4, q)$, and that no higher dimensional subspaces of $\text{PG}(4, q)$ are completely contained in it. This quadric is also an example of a finite classical generalized quadrangle if we consider it as a pure point-line geometry.
An ovoid of $Q(4, q)$ is a set $\mathcal{O}$ of points of $Q(4, q)$, such that every line of $Q(4, q)$ meets $\mathcal{O}$ in exactly one point, necessarily, $|\mathcal{O}| = q^2 + 1$. A blocking set is a set $\mathcal{B}$ of points of $Q(4, q)$, such that every line of $Q(4, q)$ meets $\mathcal{B}$ in at least one point, necessarily $|\mathcal{B}| \geq q^2 + 1$ with equality if and only if $\mathcal{B}$ is an ovoid. A blocking set $\mathcal{B}$ is called minimal if $\mathcal{B} \setminus \{p\}$ is not a blocking set for any point $p \in \mathcal{B}$. A multiple line of $\mathcal{B}$ is a line of $Q(4, q)$ meeting $\mathcal{B}$ in at least two points.

It is a well known fact that the dual of the generalized quadrangle $Q(4, q)$, i.e., the point-line geometry obtained by interchanging the role of the points and the lines, is isomorphic to the generalized quadrangle $W(3, q)$, which is the generalized quadrangle with as pointset the points of PG$(3, q)$, and as lineset the totally isotropic lines with respect to a symplectic polarity of PG$(3, q)$. For more details we refer to [6].

An ovoid of $Q(4, q)$ translates under the duality to a spread of $W(3, q)$, this is a set of lines of $W(3, q)$ partitioning the pointset. A blocking set of $Q(4, q)$ translates to a cover of $W(3, q)$, this is a set of lines $\mathcal{C}$, such that every point of $W(3, q)$ lies on at least one line of $\mathcal{C}$. A multiple point of $\mathcal{C}$ is a point of $W(3, q)$ lying on at least two lines of $\mathcal{C}$.

It is known that $Q(4, q)$ has always an ovoid. Every elliptic quadric $Q^-(3, q)$ contained in $Q(4, q)$ is an example. Considering minimal blocking sets of $Q(4, q)$, $q$ even, the following result is known.

Result 1 (Eisfeld et al. [5]) Let $\mathcal{B}$ be a blocking set of the quadric $Q(4, q)$, $q$ even. If $q \geq 32$ and $|\mathcal{B}| \leq q^2 + 1 + \sqrt{q}$, then $\mathcal{B}$ contains an ovoid of $Q(4, q)$. If $q = 4, 8, 16$ and $|\mathcal{B}| \leq q^2 + 1 + \frac{q+4}{6}$, then $\mathcal{B}$ contains an ovoid of $Q(4, q)$.

No analogue theorem is known for $q$ odd. It is even not known whether or not $Q(4, q)$, $q$ odd, has a minimal blocking set of cardinality $q^2 + 2$, that is a blocking set of size $q^2 + 2$ that does not contain an ovoid. It is probably a bit unexpected but this problem seems to be quite hard. Using a combination of geometrical and algebraic methods we are able to solve the problem when $q$ is an odd prime. Our result is as follows.

Theorem 2 If $q$ is an odd prime, then $Q(4, q)$ does not have a minimal blocking set of size $q^2 + 2$.

We remark that this was proved earlier for $q = 5$ and $q = 7$, see [3].

Recently, it was proved that all ovoids of $Q(4, q)$, $q$ odd prime, are elliptic quadrics $Q^-(3, q)$, [2]. In the proof, one considers an ovoid $\mathcal{O}$ of $Q(4, q)$, $q = p^h$, $p$ an odd prime. Then it is proved that all hyperplanes of PG$(4, q)$ intersect $\mathcal{O}$ in $1 \mod p$ points. When $q = p$, combinatorial arguments prove that $\mathcal{O}$ is necessarily an elliptic quadric. The $1 \mod p$ result is proved using polynomial
techniques. In an earlier paper, [1], the 1 mod $p$ result is obtained using an algebraic description of the generalized quadrangle $W(3, q)$ in the field $\text{GF}(q^4)$. Using this description of $W(3, q)$ and the structure of the multiple points, we obtain comparable results for minimal blocking sets of $Q(4, q)$, $q$ odd, of size $q^2 + 2$. When $q$ is a prime, these results, together with geometrical and combinatorial arguments, exclude the existence of such a blocking set.

In Section 2, we will adapt the algebraic approach from [1] to obtain a $t \mod p$ result. In Section 3 we will derive some combinatorial properties of a blocking set of size $q^2 + 2$ of $Q(4, q)$. In the last section, we will exclude the existence of a minimal blocking set of size $q^2 + 2$ of $Q(4, q)$, $q$ an odd prime, using again geometrical and combinatorial arguments.

2 The intersection numbers

We make use of the fact that $Q(4, q)$ and $W(3, q)$ are dually isomorphic. Under this duality blocking sets of $Q(4, q)$ translates to covers of $W(3, q)$. In this section we prove a theorem on covers of $W(3, q)$ that have the property that the multiple points (the points covered more than once) form a sum of symplectic lines, see the next section for more details. By assigning the weight $w(L) = -1$ to these lines one is in the situation of the following theorem.

**Theorem 3** Consider $W(3, q)$ in $\text{PG}(3, q)$, $q = p^h$, $p$ a prime. Suppose that $w$ is a function from the lineset $\mathcal{L}$ of $W(3, q)$ to $\text{GF}(p)$ such that for every point $v$ we have

$$\sum_{L \in \mathcal{L} : L \ni v} w(L) = 1$$

Let $\mathcal{F}$ be a regular spread of $\text{PG}(3, q)$ consisting of lines of $W(3, q)$. Then

$$\sum_{L \in \mathcal{F}} w(L) = 1.$$
We emphasize that $X, Y$ are not points of $\text{PG}(3, q)$ here, but vectors of $V(4, q)$. From this it can be derived that two points that are represented by $u$ and $v$ are perpendicular if and only if

$$\gamma u^{q+1} - \gamma v^{q+1} + q^2 + q v - u v^{q^2 + q + 1} = 0 \quad (2)$$

Here $\gamma := \Gamma^{1-q}$.

The lines of $W(3, q)$, identified with their pointsets, are represented by two types of equations; the $q + 1$ solutions of such an equation are exactly the representatives of the points constituting the pointset of the line. Type (i) lines are represented by the equation

$$dU^{q+1} + U - \gamma d = 0,$$

where $d \in D := \{x \in \text{GF}(q^4) \mid x^{q^3 + q - \gamma - 1} x^{q^2 + 1} + 1 = 0\}$. Type (ii) lines are represented by the equation

$$U^{q+1} + e = 0,$$

where $e \in E := \{x \in \text{GF}(q^4) \mid x^{q^2 + 1} = 1\}$. It is also proved in [1] that the $q^2 + 1$ lines of type (ii) constitute a regular spread of $W(3, q)$.

**Remark.** Since the coefficient of $U^q$ in the equation of a symplectic line is zero, we find $\sum_{v \in L} v = 0$ for any symplectic line $L$. This implies that $\sum_{v \in \pi} v = 0$ for every plane $\pi$, since the $\pi$ is the union of the $q + 1$ symplectic lines on the point $u := \pi^\perp$.

**Proof of Theorem 3.** The lines $L \in \mathcal{L}$ are represented by elements $d \in D$ or $e \in E$ and we denote by $w_d$ or $w_e$ the weight $w(L)$ of $L$. Without loss of generality we may assume that the regular spread $\mathcal{F}$ consists of the symplectic lines represented by the elements of $E$. Consider a point of $u \in W(3, q)$. All symplectic lines on $u$ lie in $u^\perp$ and all symplectic lines in $u^\perp$ pass through $u$. Consider a symplectic line not passing through $u$. Then $L$ meets $u^\perp$ in a point $v$. If $L$ has type (i), represented by $d \in D$, then, using equation (2), one can prove

$$v^q = -u(du^{q+1} + u - \gamma d^q)^{q-1}. \quad (3)$$

If $L$ has type (ii), represented with parameter $e \in \text{GF}(q^4)$, then, using equation (2) it is proved that

$$v^q = \gamma^{-1} u e (u^{q+1} + e)^{q-1}. \quad (4)$$

This was also used in [1]. It follows that

$$- \sum_{L \in \mathcal{L}} w_L \sum_{v \in L \cap u^\perp} v^q = \sum_{d \in D} w_d (du^{q+1} + u - \gamma d^q)^{q-1} - \sum_{e \in E} w_e \gamma^{-1} u e (u^{q+1} + e)^{q-1}.$$
In fact, every symplectic line appears on the left and the right hand side. We show that the contribution on both sides is equal for every line $L$ of $F$. If $L$ is not contained in $u^\perp$, so that $L$ meets $u^\perp$ in a unique point $v$, then this follows from (3) and (4). Now consider the case when $L$ is contained in $u^\perp$. Then the contribution on the left hand side is $w_L(\sum_{v \in L} v)^q$, which is zero by the remark of this section. The contribution on the right side is also zero, because then $L$ is a symplectic line on $u$, so $u$ satisfies the equation of $L$.

The left hand side of the equation is zero, since

$$\sum_{L \in \mathcal{L}} \sum_{v \in L \cap u^\perp} w_L v^q = \sum_{v \in u^\perp} v^q = \left( \sum_{v \in u^\perp} v \right)^q = 0^q = 0.$$ 

The first equality sign follows from the hypothesis in Theorem 3, and the third equality sign follows from the remark in this section. Consider the polynomial

$$f(U) := \sum_{d \in \mathcal{D}} w_d U(dU^{q+1} + U - \gamma d^q)^{q-1} - \sum_{e \in \mathcal{E}} w_e \gamma^{-1} U e(U^{q+1} + e)^{q-1}.$$ 

This polynomial has degree at most $q^2$ and by the previous arguments, $f(u) = 0$ for all points $u$. As there are $q^3 + q^2 + q + 1$ points, it follows that $f(U)$ is identically zero. Looking at the coefficient of $U^q$ in $f(U)$, we conclude that

$$\sum_{d \in \mathcal{D}} w_d = 0.$$ 

Hypothesis (1) shows that

$$(q + 1) \sum_{L \in \mathcal{L}} w(L) = \sum_{u} \sum_{L \in \mathcal{L} : L \ni u} w(L) = \sum_{u} 1 = q^3 + q^2 + q + 1.$$ 

As this is a calculation in GF($p$), then $\sum_{L \in \mathcal{L}} w(L) = 1$. Hence the sum of $\sum_{d \in \mathcal{D}} w_d$ and $\sum_{e \in \mathcal{E}} w_e$ is also one, so $\sum_{e \in \mathcal{E}} w_e = 1$. Because the $e \in \mathcal{E}$ represent the lines of the spread $\mathcal{F}$, this proves the theorem. 

3 The structure of the multiple points

Suppose that $\mathcal{B}$ is a blocking set of size $q^2 + 1 + r$ of $Q(4, q)$. A line of $Q(4, q)$ is called a multiple line or an excess line of $\mathcal{B}$ when it contains at least two points of $\mathcal{B}$. The excess $e_L$ of a line $L$ of $Q(4, q)$ is by definition $|L \cap \mathcal{B}| - 1$. Counting pairs $(u, L)$ with lines $L \in \mathcal{B}$ and points $u \in Q(4, q)$ with $u \in L$, one finds that the sum of the excesses over all lines of $Q(4, q)$ is $r(q + 1)$. Consider a line $L$ of $Q(4, q)$. Every point of $L \cap \mathcal{B}$ lies on $q + 1$ lines of $Q(4, q)$ while the points of $\mathcal{B}$ that are not on $L$ lie on a unique line of $Q(4, q)$ that meets $L$. 

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Hence, if $\mathcal{M}$ is the set consisting of the $q^2 + q + 1$ lines of $Q(4, q)$ that meet $L$, then
\[
\sum_{M \in \mathcal{M}} (1 + e_M) = (1 + e_L) \cdot (q + 1) + (|\mathcal{B}| - 1 - e_L) \cdot 1.
\]
As $|\mathcal{M}| = q^2 + q + 1$, this gives
\[
\sum_{M \in \mathcal{M}} e_M = |\mathcal{B}| - q^2 - 1 + e_L q.
\]

Now suppose that $|\mathcal{B}| = q^2 + 2$. Then the right hand side is equal to $e_L q + 1$. As the sum of the excesses over all lines of $Q(4, q)$ is $r(q + 1) = q + 1$, it follows that $e_L \leq 1$. Furthermore, if $L$ is a multiple line, then $e_L = 1$ and $L$ meets every other multiple line. It follows that there exist $q + 1$ multiple lines and that they mutually meet. Hence, for $|\mathcal{B}| = q^2 + 2$, there exists a point $m \in Q(4, q)$ with the property that the $q + 1$ lines of $Q(4, q)$ on $m$ are the multiple lines; they meet $B$ in two points while every other line of $Q(4, q)$ meets $B$ in a unique point. We remark that this property can also be derived from a general theorem in [5].

Consider a solid (3-space) of the ambient projective space $\text{PG}(4, q)$ of $Q(4, q)$. Then $S \cap Q(4, q)$ is either a quadric $Q^+(3, q)$, a quadric $Q^-(3, q)$ or a cone with a point vertex over a $Q(2, q)$; we say that $S$ has hyperbolic, elliptic or parabolic type in the respective cases.

If $S$ is a hyperbolic solid, then the existence of $m$ shows that $|S \cap \mathcal{B}| = q + 2$ if $m \in S$, and $|S \cap \mathcal{B}| = q + 1$, if $m \not\in S$. If $S$ is a parabolic solid, then similarly $m \in S$ implies that $|S \cap \mathcal{B}| \in \{2, q + 2\}$ if $m \in S$, and $|S \cap \mathcal{B}| \in \{1, q + 1\}$ if $m \not\in S$. Hence, for hyperbolic and parabolic solids $S$ we have that
\[
|S \cap \mathcal{B}| \equiv \begin{cases} 2 \mod q, & \text{if } m \in S, \\ 1 \mod q, & \text{if } m \not\in S. \end{cases} \quad (5)
\]

In the forth section we shall see that if (5) also holds for the elliptic solids, then $\mathcal{B}$ is the union of a point and an elliptic quadric. For odd primes we can show that this always holds:

**Lemma 4** Suppose that $\mathcal{B}$ is a blocking set of size $q^2 + 2$ of $Q(4, q)$, $q$ an odd prime. Then (5) holds for the elliptic solids.

**Proof.** We recall that exactly $q + 2$ lines are blocked twice by $\mathcal{B}$ and that all these lines pass through a point $m$ of $Q(4, q)$. We now use that $Q(4, q)$ and $W(3, q)$ are dual. Under the duality the blocking set $\mathcal{B}$ corresponds to a cover $\mathcal{C}$ of $W(3, q)$, and the point $m$ translates to a symplectic line $M$. Every point
of $W(3, q)$ is covered by exactly one line of $C$, except for the points of $M$ which are covered by two lines of $C$. Define a function $w$ from the set consisting of the lines of $W(3, q)$ to the field $GF(q)$ as follows. If $M \not\in C$ put $w(L) = 1$ for $L \in C$, $w(M) := -1$, and $w(L) = 0$ for the remaining symplectic lines. If $M \in C$ put $w(L) = 1$ for $L \in C \setminus \{M\}$ and $w(L) = 0$ for the remaining symplectic lines. Then hypothesis (1) of Theorem 3 is satisfied.

As the regular spreads of $PG(3, q)$ consisting of symplectic lines correspond under the duality to solids of $PG(4, q)$ meeting $Q(4, q)$ in an elliptic quadric $Q^-(3, q)$, the assertion follows from Theorem 3.

Remarks. (1) If $|B| = q^2 + 2$, then we have proved above the existence of a point $m$ lying on all multiple lines. We mention that $m \in B$ if and only if $B \setminus \{m\}$ is an ovoid of $Q(4, q)$.

(2) Consider a blocking set $B$ of $Q(4, q)$ with $|B| = q^2 + 1 + r$. We also consider the corresponding cover $C$ of $W(3, q) \subseteq PG(3, q)$. If $r$ is not too large, then it was shown in [5], then there exists $r$ lines $M_1, \ldots, M_r$ of $PG(3, q)$ (repeated lines are allowed) with the following property. The number of lines of $C$ on a point $v$ of $W(3, q)$ is one plus the number of lines $M_i$ on $v$. Here the lines $M_i$ can be symplectic but need not to be symplectic. However, if they are all symplectic, then the technique of Section 2 can be applied. In that case, going back to $B$ in $Q(4, q)$ one obtains points $m_1, \ldots, m_r$ of $Q(4, q)$ corresponding to $M_1, \ldots, M_r$, and the intersection of a solid with $B$ is modulo $p$ congruent to one plus the number of points $M_i$ in such a solid.

If a line $M_i$ is not symplectic, then the translation to $Q(4, q)$ gives a regulus consisting of multiple lines, and then the opposite regulus will also have only multiple lines.

(3) S. De Winter [4] has constructed a minimal blocking set of $Q(4, q)$ of size $q^2 + 3$ for $q = 5$. If one analyzes the structure of the multiple lines in his example, one sees that the multiple lines are the $2(q + 1)$ lines of a hyperbolic quadric $Q^+(3, q)$.

4 The final step

In this section, $B$ denotes a blocking set of $Q(4, q)$ of size $q^2 + 2$. It has been shown in Section 3 that there exists a point $m$ of $Q(4, q)$ with the property that the $q + 1$ lines of $Q(4, q)$ on $m$ meet $B$ in two points while every other line of $Q(4, q)$ meets $B$ in a unique point. If $q$ is a prime, we have also seen that for every solid $S$ we have
\[ |S \cap B| \equiv \begin{cases} 2 \mod q, & \text{if } m \in S, \\ 1 \mod q, & \text{if } m \notin S. \end{cases} \]  

(6)

We shall not assume in this section that \( q \) is a prime, but we shall assume that (6) holds for all solids. We also assume that \( q \) is odd. Our goal is to show that \( B \) is not minimal. This will also prove Theorem 2.

We proceed in an indirect way and assume for the rest of the section that \( B \) is minimal. This implies that \( m \notin B \), since the special properties of \( m \) imply otherwise that \( B \setminus \{m\} \) is also a blocking set. We shall derive a contradiction in a series of lemmas.

**Lemma 5** If \( t_i \) is the number of solids meeting \( B \) in precisely \( i \) points, then 

\[ t_{q+2} \geq \frac{1}{2} q(q^2 + q) \quad \text{and} \quad t_2 \leq \frac{1}{2} (q^2 + q + 2)(q - 1). \]

**Proof.** Only solids through \( m \) can meet \( B \) in \( q + 2 \) points. The solid \( m^⊥ \) meets \( B \) in \( 2q + 2 \) points. Every other solid \( S \) on \( m \) meets \( m^⊥ \) in a plane. Note that \( m^⊥ \cap Q(4, q) \) is a cone with vertex \( m \) over a \( Q(2, q) \).

There are \( (q^2 + q)/2 \) planes \( \pi \) of \( m^⊥ \) on \( m \) that meet \( Q(4, q) \) in the union of two lines; these meet \( B \) in four points. From (6) it follows that every solid \( S \) with \( S \neq m^⊥ \) on such a plane \( \pi \) contains at least \( q - 2 \) more points of \( B \). As \( |B| = q^2 + 2 = (2q + 2) + q(q - 2) \), it follows that every solid \( S \neq m^⊥ \) on \( \pi \) meets \( B \) in precisely \( q + 2 \) points. Hence \( t_{q+2} \geq \frac{1}{2} q(q^2 + q) \).

Apart from the planes just considered, there are \( (q^2 + q + 2)/2 \) other planes \( \pi \) of \( m^⊥ \) on \( m \). These meet \( Q(4, q) \) either in one line or just in the point \( m \), so they meet \( B \) either in no or two points. As \( |B| = q^2 + 2 \) is an odd number, not all solids different from \( m^⊥ \) on such a plane can meet \( B \) in exactly two points. This implies that \( t_2 \leq \frac{1}{2} (q^2 + q + 2)(q - 1) \). \( \square \)

For the rest of the section we use \( i_S := |S \cap B| \) with \( S \) any solid of \( \text{PG}(4, q) \), and \( \theta_n := \frac{q^{n+1} - 1}{q - 1} = q^n + q^{n-1} + \ldots + q + 1 \) for any integer \( n \geq 0 \).

**Lemma 6** Put \( c := \frac{1}{2} (q^2 + 1) \). Then

\[ \sum_S (i_S - 1)(i_S - q - 1)(i_S - c) \geq \frac{1}{2} \left( q^5 + 4q^4 + q^3 + 7q^2 + 3 \right). \]

Here the sum runs over all solids of \( \text{PG}(4, q) \).

**Proof.** Put \( b := |B| = q^2 + 2 \). Standard counting arguments show

\[ \sum_S i_S = b\theta_3 \quad \text{and} \quad \sum_S i_S(i_S - 1) = b(b - 1)\theta_2. \]
We know that every line of the quadric meets $B$ in one or two points. Hence, any three points of $B$ span a plane. Therefore

$$\sum_S i_S(i_S - 1)(i_S - 2) = b(b - 1)(b - 2)\theta_1.$$ 

It follows that

$$\sum_S (i_S - 1)(i_S - q - 1)(i_S - c) =$$

$$\sum_S \left( i_S(i_S - 1)(i_S - 2) - i_S(i_S - 1)(c + q - 1) + i_S c(q + 1) - c(q + 1)^2 \right) =$$

$$b(b - 1)(b - 2)\theta_1 - (c + q - 1)b(b - 1)\theta_2 + c(q + 1)b\theta_3 - c(q + 1)\theta_4 =$$

$$\frac{1}{2} \left( q^5 + 4q^4 + q^3 + 7q^2 + 3 \right).$$

This is the claim. \(\square\)

**Lemma 7** There exists a solid \(S \neq m^\perp\) meeting \(B\) in more than \((q^2 + 1)/2\) points.

**Proof.** Again put \(c = (q^2 + 1)/2\). Recall that \(q\) is odd, which implies that \(q \geq 3\) and \(c - q - 2 \geq 0\). We already know that every solid meets \(B\) in one or two modulo \(q\) points. Hence, a solid meets \(B\) in 1, 2, \(q + 1\) or at least \(q + 2\) points. Recall that \(m^\perp\) is a solid that meets \(B\) in \(2q + 2\) points. If \(\mathcal{L}'\) is the set consisting of all solids \(S\) with \(S \neq m^\perp\) and \(|S \cap B| > q + 2\), then the preceding lemma implies that

$$\sum_{S \in \mathcal{L}'} (i_S - 1)(i_S - q - 1)(i_S - c) \geq$$

$$\frac{1}{2} \left( q^5 + 4q^4 + q^3 + 7q^2 + 3 \right) + t_{q+2}(q + 1)(c - q - 2)$$

$$- t_2(q - 1)(c - 2) - (2q + 1)(q + 1)(2q + 2 - c)$$

Using the bounds for \(t_{q+2}\) and \(t_2\), a calculation shows that the right hand side is positive. Thus, some solid of \(\mathcal{L}'\) meets \(B\) in more than \(\frac{1}{2}(q^2 + 1)\) points. \(\square\)

**Lemma 8** The final contradiction.

**Proof.** Let \(S\) be a solid meeting \(B\) in more than \(\frac{1}{2}(q^2 + 1)\) points. As lines of the quadric meet \(B\) in at most two points, and lines of the quadric with two points pass through \(m\), it follows that all parabolic solids different from \(m^\perp\) and all hyperbolic solids meet \(B\) in at most \(q + 2\) points. Hence \(S\) is an elliptic solid, that is \(S \cap Q(4, q)\) is a \(Q^-(3, q)\). Denote by \(\alpha\) the number of points of \(S \cap Q(4, q)\) that are not in \(B\). Then \(B' := B \setminus (B \cap S)\) contains \(\alpha + 1\) points.
Assume that the $Q^-(3, q)$ contains a conic $C$ such that no point of this conic belongs to $B$. We count pairs $(u, v) \in C \times B'$ for which $uv$ is a line of the quadric. A point $u \in C$ lies on $q + 1$ lines of the quadric, which meet $B$ and thus $B'$. Hence each $u \in C$ occurs in $q + 1$ such pairs. Thus, the number of such pairs is at least $(q + 1)^2$. A point $v \in B'$ can be perpendicular to zero, one, two or $q + 1$ points of $C$. However, as the quadric $Q(4, q)$ has only two points that are perpendicular to all points of the conic $C$, there are at most two points $v$ in $B'$ that occur in $q + 1$ pairs $(u, v)$. Hence, the number of pairs is at most
\[ 2(q + 1) + (|B'| - 2)2 = 2\alpha + 2q. \]
It follows that $2\alpha + 2q \geq (q + 1)^2$, that is $\alpha \geq \frac{1}{2}(q^2 + 1)$. Then $|S \cap B| = q^2 + 1 - \alpha \leq \frac{1}{2}(q^2 + 1)$, and this is a contradiction. Hence, every conic of the elliptic quadric $S \cap Q(4, q)$ meets $B$.

Count pairs $(u, v)$ with perpendicular points $u$ and $v$ where $u \in S \cap Q(4, q)$, $u \notin B$ and $v \in B'$. For $v \in B'$, the subspace $v^\perp \cap S$ is a plane that meets the quadric in a conic, and we have just seen that at most $q$ points of such a conic do not lie in $B$. Hence, each point $v \in B'$ occurs in at most $q$ such pairs. Each point $u \in S \cap Q(4, q)$ with $u \notin B$, lies on $q + 1$ lines of the quadric, which meet $B$ and hence which meet $B'$. Thus, every such point $u$ occurs in at least $q + 1$ such pairs. It follows that $\alpha(q + 1) \leq |B'|q$. As $|B'| = \alpha + 1$, this gives $\alpha \leq q$.

Hence $|S \cap B| \geq q^2 + 1 - q$ and at most $q + 1$ points of $B$ do not lie in $S$. As the global assumption in this section is that $B$ is minimal, it is not possible that all points of $S \cap Q(4, q) = Q^-(3, q)$ lie in $B$. Let $u$ be a point of $S \cap Q(4, q)$ does not lie in $B$. We have just seen that the $q + 1$ lines of the quadric on $u$ meet $B'$. Hence $|B'| \geq q + 1$. As $|S \cap B| + |B'| = |B| = q^2 + 2$, it follows that $|S \cap B| = q^2 + 1 - q$ and $|B'| = q + 1$. The argument also shows that each of the $q$ points $u$ of $S \cap Q(4, q)$ that is not in $B$ is perpendicular to each point of $B'$. But $q$ points of $S \cap Q(4, q)$ span at least a conic-plane and thus have at most two common perpendicular points in $Q(4, q)$. This is a contradiction. □

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References


