Blow-up in the Parabolic Problems under Nonlinear Boundary Conditions

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Abstract—In this paper, I consider nonlinear parabolic problems under nonlinear boundary conditions. I establish respectively the conditions on nonlinearities to guarantee that \( u(x,t) \) exists globally or blows up at some finite time. If blow-up occurs, an upper bound for the blow-up time is derived, under somewhat more restrictive conditions, lower bounds for the blow-up time are also derived.

Index Terms—Nonlinear Boundary Conditions; Blow-Up; Heat Equation

I. INTRODUCTION

In this paper, we investigate the blow-up phenomenon of the classical solution of the following initial-boundary value problem

\[
\begin{aligned}
  &u_t = \sum_{i,j=1}^{N} a^{ij}(x) u_{x_{j}x_{i}} + f(u), \quad (x,t) \in \Omega \times (0,t^*) \quad (1) \\
  &\sum_{i,j=1}^{N} a^{ij}(x) u_{x_{j}x_{i}} = g(u), \quad (x,t) \in \partial \Omega \times (0,t^*), \\
  &u(x,0) = u_0(x) \geq 0, \quad x \in \Omega,
\end{aligned}
\]

where \( n \) is the unit outward normal on the boundary \( \partial \Omega \), \( \Omega \) is a bounded star-shaped region in \( \mathbb{R}^N, N \geq 2 \) , \( t^* \) is the blow-up time if blow-up occurs, or else \( t^* = \infty \), and \( (a^{ij}(x))_{N \times N} \) is a differentiable positive definite matrix. It is well known that the data \( f \) and \( g \) may greatly affect the behavior of \( u(x,t) \) with the development of time. From the physical standpoint, \( f \) is the heat source function, \( g \) is the heat-conduction function transmitting into interior of \( \Omega \) from the boundary of \( \Omega \).

The study of the blow-up phenomena in parabolic problems has received a great deal of attention in the last decades (we refer the reader especially to the books of Straughan [1] and Quitmair-Souplet [2], the survey papers of Levine [3] and Galaktionov [4] and the references therein). Therefore, nowadays a variety of methods are known and used in the study of various questions regarding the blow-up phenomena in parabolic problems. But, most of the methods used to show that solutions blow-up provide only an upper bound for the blow-up time, while in applications, due to the explosive nature of the solutions, it is more important to determine the lower bounds on the blow-up time. To our knowledge, there seems to have been relatively little work devoted to obtaining lower bounds on blow-up time if blow-up occurs.

In [5], Payne and Schaefer used a differential inequality technique to obtain a lower bound on blow-up time for solutions of the semilinear heat equation

\[
u_t = \Delta u + f(u)
\]

under homogeneous Dirichlet boundary conditions, where suitable constraints were imposed on \( f \) which allowed, for instance, \( f(u) = u^p, p > 1 \), and \( f(u) = 2 \cosh(\gamma u - 1), \gamma > 0 \). A second method based on a comparison principle was also presented there. They also consider the initial-boundary value problem for the semilinear heat equation (4) under a Robin boundary condition [6].

Payne and Schaefer [7] considered

\[
u_t = \Delta u.
\]

Under suitable conditions on the nonlinearities, they determined a lower bound of the blow-up time when blow-up occurs.

In addition, a sufficient condition which implies that blow-up does occur was determined.

In [8] Payne, Philippin and Vernier Piro considered

\[
u_t = \Delta u - f(u)
\]

and established conditions on nonlinearities to guarantee that \( u(x,t) \) exists for all time \( t > 0 \) or blows up at some finite time \( t^* \). Moreover, an upper bound for \( t^* \) was derived. Under somewhat more restrictive conditions, a lower bound for \( t^* \) was derived.

Recently, Philippin and Vernier [9] investigated

\[
u_t = \nabla(\nabla u |^{p-2} \nabla u), (x,t) \in \Omega \times (0,t^*)
\]

and showed that blow-up occurs at some finite time under certain conditions on the nonlinearities and the data, upper and lower bounds for the blow-up time were obtained when blow-up occurs.

Further extensions were accomplished ([10-20] and the references therein).
Motivated by the above work, we intend to study the global existence and the blow-up phenomena for problem (1)-(3). The main contribution of this paper are: (a) the problems considered in this paper are nonlinear equations with inhomogeneous Neumann boundary dissipation, these problems possess representative; (b) we give the reason and the process of the definition of auxiliary functional; (c) since the models are general, the estimates are concise and precise.

The present work is organized as follows. In Section 2, I show the conditions on the nonlinearities which ensure that the solution blows up at some finite time and obtain an upper bound of the blow-up time. Section 3 is devoted to showing a lower bound of blow-up time under some assumptions. In Section 4, I establish the conditions on the nonlinearities which guarantee that $u(x,t)$ exists globally. In the final section, I briefly show how to obtain an analogous result for a domain $\Omega \in \mathbb{R}^2$ and how to extend this work to systems of equations.

II. A criterion for blow-up

In this section, we establish conditions on the data of problem (1)-(3) under which $u(x,t)$ will blow up at finite time $t^*$ and derive under these conditions an upper bound $T$ for $t^*$.

**Theorem 2.1.** Let $u(x,t)$ be the solution of (1)-(3) and assume the following conditions on the data

$$
\xi f(\xi) \geq 2(1+\alpha)F(\xi), \quad \xi \geq 0, \quad (8)
$$

$$
\xi g(\xi) \geq 2(1+\beta)G(\xi), \quad \xi \geq 0, \quad (9)
$$

with

$$
F(\xi) := \int_0^\xi f(\eta) d\eta, \quad g(\xi) := \int_0^\xi g(\eta) d\eta.
$$

In (8), (9), $\alpha$ and $\beta$ are constants satisfying the condition $0 \leq \beta < \alpha$.

Moreover we assume $\Theta(0) \geq 0$ with

$$
\Theta(t) = 2\int_{\Omega} G(u) dS - \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\nu (x) u_{x_i} u_{x_j} dx + 2\int_{\Omega} F(u) dx.
$$

Then $u(x,t)$ blows up at time $t^* < T$ with

$$
T = \frac{\Phi(0)}{2\beta(1+\beta)\Theta(0)} (\beta > 0),
$$

where $\Phi(t) = \int_{\Omega} u^2 dx$. Where $\beta > 0$, we have $t^* = \infty$.

**Proof of Theorem 2.1.** Using Green formula and the hypotheses stated in Theorem 2.1, we get

$$
\Phi'(t) = 2\int_{\Omega} u_t dx = 2\int_{\Omega} \left[\int_0^t \sum_{i,j=1}^N a_{ij}^\nu (x) u_{x_i} u_{x_j} dx + f(u)\right] dx
$$

$$
= 2\int_{\Omega} u g(u) dS + \int_{\Omega} f(u) dx - 2\int_{\Omega} \sum_{i,j=1}^N a_{ij}^\nu (x) u_{x_i} u_{x_j} dx
$$

$$
\geq 2(1+\beta)\int_{\Omega} G(u) dS - \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\nu (x) u_{x_i} u_{x_j} dx
$$

$$
+ 2(1+\alpha)\int_{\Omega} F(u) dx \geq 2(1+\beta)\Theta(t),
$$

and

$$
\Theta'(t) = 2\int_{\Omega} u g(u) dS + 2\int_{\Omega} f(u) dx
$$

$$
- \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\nu (x) u_{x_i} u_{x_j} \right) dx = 2\int_{\Omega} a_{ij}^\nu (x) u_{x_i} u_{x_j} dx
$$

$$
+ 2\int_{\Omega} u g(u) dS + 2\int_{\Omega} f(u) dx = 2\int_{\Omega} u g(u) dS
$$

$$
+ 2\int_{\Omega} u f(u) dx + 2\int_{\Omega} \sum_{i,j=1}^N a_{ij}^\nu (x) u_{x_i} u_{x_j} dx
$$

$$
- 2\int_{\Omega} \sum_{i,j=1}^N a_{ij}^\nu (x) u_{x_i} u_{x_j} dx = 2\int_{\Omega} u^2 dx
$$

which with $\Theta(0) > 0$ imply $\Theta(t) > 0$ for all $t \in (0,t^*)$.

Using (10), (11) and Hölder inequality, we obtain

$$
\Theta(t) \Phi'(t) \leq \frac{1}{2(1+\beta)} (\Phi'(t))^2 = \frac{2}{1+\beta} \left( \int_{\Omega} u^2 dx \right)^2
$$

$$
\leq \frac{2}{1+\beta} \int_{\Omega} u^2 dx \int_{\Omega} \frac{1}{1+\beta} \Theta(t) \Phi(t).
$$

Multiplying the above inequality by $\Phi^{1-\beta}$, we deduce

$$
(\Theta^{1-\beta})' \geq 0
$$

integrating (12) over $[0,t]$ and noting $\Phi(0) > 0$ (by $\Theta(0) > 0$), we get

$$
\Theta(t) \Phi^{1-\beta}(t) \geq \Theta(0) \Phi^{1-\beta}(0) = M > 0.
$$

that is

$$
\Theta(t) \geq M \Phi^{1-\beta}(t).
$$

By (8) and (13), we obtain

$$
\Phi'(t) \geq 2(1+\beta)\Theta(t) \geq 2M(1+\beta)\Phi^{1-\beta}(t).
$$

If $\beta > 0$, (14) can be written as

$$
(\Phi^{1-\beta})' = -\beta \Phi^{1-\beta}(t) \Phi'(t) \leq -2M \beta (1+\beta).
$$

Noting (8), $\Theta(t) > 0$ and $\Theta(0) > 0$, we deduce

$$
\Phi(t) > 0.
$$

From (15) and (16), it follows that

$$
0 < \Phi(t) \leq \Phi^{1-\beta}(t) - 2M \beta (1+\beta) t,
$$

that is

$$
\Phi(t) \geq \frac{1}{2\beta(1+\beta)} \Phi^{1-\beta}(0) - 2M \beta (1+\beta) t,
$$

which implies $\Phi(t) \to +\infty$ as

$$
t \to T = \frac{(\Phi(t))^{\beta}}{2\beta(1+\beta)M} = \frac{\Phi(0)}{2\beta(1+\beta)\Theta(0)}
$$

(by the definition of $M$). Therefore for $\beta > 0$,

$$
t^* \leq T = \frac{\Phi(0)}{2\beta(1+\beta)\Theta(0)}.
$$

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If $\alpha = \beta = 0$, we have

$$\Phi(t) \geq \Phi(0)e^{\beta t},$$

valid for $t > 0$, implying that $t' = \infty$. The proof of Theorem 2.1 is completed.

III. LOWER BOUND FOR BLOW-UP TIME

In this section, under the assumption that $\Omega \subset \mathbb{R}^2$ is a convex bounded star-sharped domain in two orthogonal directions, we establish a lower bound for the blow-up time $t'$. Now we state the result as follows.

Theorem 3.1. Assumed that $\Omega \subset \mathbb{R}^2$ is a bounded star-sharped convex domain in two orthogonal directions. Let $u(x, t)$ be the nonnegative solution of problem (1) and $u(x, t)$ blows up at $t'$, moreover the nonnegative $f$ and $g$ satisfy the conditions

$$0 \leq f(s) \leq k_1 s^{\sigma-1}, s > 0,$$
$$0 \leq g(s) \leq k_2 s^{\frac{\sigma}{2}}, s > 0,$$

for $k_1 > 0, k_2 > 0, \sigma \geq 1$. Define

$$\phi(t) = \int_{\Omega} u^{2\sigma} dx.$$  (22)

Then $\phi(t)$ satisfies inequality $\phi'(t) \leq \varphi(\phi)$ for some computable function $\varphi(\phi)$. It follows that $t'$ is bounded below by

$$t' \geq \int_{\eta(t)}^{\infty} \frac{d\eta}{\varphi(\eta)}.$$  (23)

In order to prove this theorem, we first give the following lemma.

Lemma 3.1. Let $\Omega$ be a bounded star-sharped region in $\mathbb{R}^2, N \geq 2$. Then for any nonnegative $C^1$ function $u$ and $r > 0$, we have

$$\int_{\Omega} u^2 ds \leq \frac{N}{\rho_0} \int_{\Omega} u^2 dx + rd \int_{\Omega} u^{2-r} |\nabla u| dx,$$

where $\rho_0 := \min_{\partial \Omega}(x \cdot n), \quad d := \max_{\partial \Omega} |x|$.

Proof. Since $\Omega$ is a bounded star-sharped region, we know $\rho_0 > 0$. Integrating the identity

$$\text{div}(u^r \mathbf{x}) = Nu^r + ru^{r-1}(x \cdot \nabla u)$$

over $\Omega$, and using divergence theorem, we get

$$\int_{\Omega} u^r \cdot (x \cdot n) dx = \int_{\Omega} Nu^r dx + \int_{\Omega} ru^{r-1}(x \cdot \nabla u) dx,$$

By the definitions of $\rho_0$ and $d$, it follows that

$$\rho_0 \int_{\Omega} u^r dx \leq \int_{\partial \Omega} u^r (x \cdot n) dx \leq N \int_{\Omega} u^r dx + r \int_{\Omega} u^{r-1} |x| |\nabla u| dx \leq N \int_{\Omega} u^r dx + rd \int_{\Omega} u^{2-r} |\nabla u| dx,$$

which implies the desire conclusion.

Proof of Theorem 3.1. Since $(a^{(i)}(x))_{N \times N}$ is positive definite matrix, then there exists a constant $\theta > 0$ such that

$$\sum_{i, j = 1}^{N} a^{(i)}(x) \eta_i \eta_j \geq \theta |\eta|^2$$

for a.e. $x \in \Omega$ and all $\eta \in \mathbb{R}^N$.

Differentiating (22) we obtain

$$\phi'(t) = 2\sigma \int_{\Omega} u^{2\sigma-1} u dx = 2\sigma \int_{\Omega} u^{2\sigma-1} \left[ \sum_{i, j = 1}^{N} a^{(i)}(x) u_{x_i}, u_{x_j} \right]$$
$$+ f(u) \int_{\Omega} dx = -2\sigma(2\sigma - 1) \int_{\Omega} u^{2\sigma-2} \times \sum_{i, j = 1}^{N} a^{(i)}(x) u_{x_i}, u_{x_j}$$
$$+ 2\sigma \int_{\Omega} u^{2\sigma-1} f(u) dx + 2\sigma \int_{\Omega} u^{2\sigma-1} g(u) dx$$
$$\leq -2\sigma \sigma(2\sigma - 1) \int_{\Omega} u^{2\sigma-2} |\nabla u|^2 dx$$
$$+ 2\sigma k_1 \int_{\Omega} u^{2\sigma} dx + 2\sigma k_2 \int_{\Omega} u^{\sigma} dx$$

where we have used successively the differential equation (1), the divergence theorem, the boundary condition (2) and the assumption (21).

Next, Application of Lemma 2.1 leads to the inequality

$$\int_{\Omega} u^{2\sigma} dx \leq \frac{3}{\rho_0} \int_{\Omega} u^{2\sigma} dx + 2\sigma \rho_0 \int_{\Omega} u^{2\sigma-1} |\nabla u|^2 dx.$$

Furthermore, since $|\nabla u|^2 = \sigma^{2\sigma-2} |\nabla u|^2$, we replace (23) by

$$\phi'(t) \leq -\frac{2(2\sigma - 1)}{\sigma} \int_{\Omega} |\nabla u|^2 dx + 2\sigma k_2 \int_{\Omega} u^{2\sigma} dx$$
$$+ \frac{5\sigma d}{2\rho_0} \int_{\Omega} u^{2\sigma-1} |\nabla u| dx + 2\sigma k_2 \int_{\Omega} u^{\sigma} dx.$$  (24)

We now use the Schwarz inequality on the two integrals in the bracket in (24) and then the arithmetic-geometric mean inequality to arrive at

$$\phi'(t) \leq -\frac{2(2\sigma - 1)}{\sigma} \int_{\Omega} |\nabla u|^2 dx$$
$$+ \frac{3\sigma k_1}{\rho_0} \int_{\Omega} u^{2\sigma} dx + \frac{5\sigma^2 d}{2\rho_0} \int_{\Omega} u^{2\sigma-1} |\nabla u| dx + 2\sigma k_2 \int_{\Omega} u^{\sigma} dx$$

for some positive $\lambda$ which is to be determined. To bound the latter integral, we use an integral inequality which was derived in [18, (15)-(23)] and is restricted to $N = 3$ dimensions, namely,

$$\int_{\Omega} u^{2\sigma} dx \leq \frac{3}{2\rho_0} \int_{\Omega} u^{2\sigma} dx$$
$$+ \left( \frac{d}{2\rho_0} + 1 \right) \left( \int_{\Omega} u^{2\sigma} dx \right)^{\frac{1}{2}} \times \left( \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$  (26)

Using the inequalities
\[ (a+b) \frac{\partial u}{\partial t} \leq \frac{1}{2} \left( a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial y^2} \right) \]  
\[ a' b' \leq ra + qb, \quad r + q = 1, \]  
for positive \(a\) and positive \(b\), it follows that

\[ \int_0^\infty u^2 \, dx \leq 2^{\frac{1}{3}} \left( \frac{3 \sigma}{2} \frac{\rho}{\rho_0} + \frac{3 \sigma^2 k d}{2 \rho_0} \right)^{\frac{3}{2}} \left( \frac{5 \sigma k d}{2 \rho_0} + \frac{5 \sigma^2 k d}{2 \rho_0} \right) \]  
\[ \leq 3^{\frac{3}{2}} 2^{\frac{1}{2}} \left( \frac{d}{\rho_0} + 1 \right)^{\frac{3}{2}} \left( \frac{3}{4} \right)^{\frac{3}{2}} \chi \]  
for some positive \(\chi\) that is to be determined. Collecting terms in (25) and (29) and choosing \(\chi\) and \(\lambda\) so that

\[ -2\theta(2\sigma - 1) + \frac{5k d \lambda}{2\rho_0} \]  
\[ + 3^{\frac{3}{2}} 2^{\frac{1}{2}} \left( \frac{d}{\rho_0} + 1 \right)^{\frac{3}{2}} \left( \frac{3\sigma k d}{2\rho_0} + \frac{5\sigma^2 k d}{2\rho_0} \right) \]  
we have

\[ \phi'(t) \leq C_1 \phi + C_2 \phi^2 + C_3 \phi ^3, \]  
where

\[ C_1 = \frac{3\sigma k d}{\rho_0} \]  
\[ C_2 = 3^{\frac{3}{2}} 2^{\frac{1}{2}} \left( \frac{3\sigma k d}{\rho_0} + \frac{5\sigma^2 k d}{2\rho_0} \right) \]  
\[ C_3 = 2^{\frac{1}{2}} \left( \frac{3\sigma k d}{\rho_0} + \frac{5\sigma^2 k d}{2\rho_0} \right) \left( \frac{d}{\rho_0} + 1 \right)^{\frac{3}{2}} \left( \frac{3}{4} \right)^{\frac{3}{2}} \chi. \]  
Under the assumption that \(\phi\) blows up at finite time \(t'\), we obtain a lower bound for \(t'\) by integration of (30), i.e.,

\[ t' \geq \int_0^\infty \frac{d\eta}{\phi^2 (\eta) + \phi^3 (\eta)} \]  
The proof of Theorem 3.1 is completed.

IV. CRITERION FOR GLOBAL EXISTENCE

In this section, we establish the conditions on the nonlinearity of the following initial-boundary value problem to guarantee that \(u(x,t)\) exists globally.

\[ u_i = (g(u)u)_i - f(u), \quad (x,t) \in \Omega \times (0, t') \]  
\[ \frac{\partial u}{\partial n} = h(u), \quad (x,t) \in \partial \Omega \times (0, t') \]  
\[ u(x,0) = u_0(x), \quad x \in \Omega, \]  
where \(\frac{\partial u}{\partial n}\) is the outward normal derivative of \(u\) on the boundary \(\partial \Omega\) assumed sufficiently smooth, \(\Omega\) is a bounded star-shaped region in \(\mathbb{R}^N\), \(N \geq 2\), \(g\) is a positive non-increasing function, with \(g(u) \geq g_m\) for all \(u \geq 0\), \(h\) is non-negative, with \(f(0) = 0\), \(t'\) is the blow-up time if blow-up occurs, or else \(t' = \infty\). The notations \(u_{\eta} = \frac{\partial u}{\partial \eta}, u_{\eta i} = \frac{\partial^2 u}{\partial x_i \partial \eta}\) will be used throughout this work, and summation from 1 to \(N\) is understood for repeated indices.

**Theorem 4.1.** Assume that the nonnegative functions \(f\) and \(g\) satisfy

\[ 0 \leq h(s) \leq k_1 \left( \int_0^t \frac{1}{g(\eta)} \, d\eta \right)^p, \quad s > 0, \]  
\[ f(s) \geq k_2 g(s) \left( \int_0^t \frac{1}{g(\eta)} \, d\eta \right)^p, \quad s > 0 \]  
for \(k_1 > 0, k_2 > 0, p > 1, q > 1, \) and \(2q < p + 1\)

Then the (nonnegative) solution \(u(x,t)\) of problem (31)-(33) does not blow up, so that \(u(x,t)\) exists for all \(t > 0\).

Proof of Theorem 4.1. Set

\[ \phi(t) = \int_0^\infty v^2 (u(x,t)) \, dx, \]  
\[ v(s) := \int_0^\infty \frac{1}{g(\eta)} \, d\eta. \]  

Differentiating (34), we obtain

\[ \phi'(t) = 2 \int_0^\infty \frac{\partial v}{\partial \eta} \left[ \left( g(u)u \right)_i - f(u) \right] \, dx \]  

\[ = 2 \int_0^\infty \frac{\partial u}{\partial \eta} \, ds - 2 \int_0^\infty \left( \left| \nabla u \right|^2 - g(v) \left| \nabla u \right|^2 - f(u)v \right) \, dx \]  

\[ \leq 2 k_1 \int_0^\infty v^{q+1} \, ds - 2 g_m \int_0^\infty \left| \nabla v \right|^2 \, dx - 2 k_2 \int_0^\infty v^{q+1} \, dx, \]  
where we have used successively the differential equation (31), the divergence theorem, the fact that \(g' \leq 0\) and the hypotheses stated in Theorem 31.

Application of Lemma 3.1 leads to the inequality

\[ \int_0^\infty v^{q+1} \, ds \leq \frac{N}{\rho_0} + \int_0^\infty v^2 \, dx \]  

Inserting (36) into (35), we get

\[ \phi'(t) \leq \frac{2k_1 N}{\rho_0} \int_0^\infty v^{q+1} \, dx + \frac{2(q+1)k_2 d}{\rho_0} \int_0^\infty v^2 \, dx \]  

\[ - 2 g_m \int_0^\infty \left| \nabla v \right|^2 \, dx - 2 k_2 \int_0^\infty v^{q+1} \, dx. \]  
Choosing \(k = \frac{k_1 d(q+1)}{2\rho_0 g_m}\), we have
\[
\int_\Omega v^q |\nabla v| \, dx \leq \frac{K}{2} v^q dx + \frac{1}{2K} \int_\Omega |\nabla v|^2 \, dx
\]
(38)

\[
= k_d (q+1) \int_\Omega v^q dx + \frac{\rho \sigma_{nn}}{k_d (q+1)} \int_\Omega |\nabla v|^2 \, dx.
\]

Inserting (38) into (37), we have
\[
\phi'(t) \leq \frac{2k_d \mathcal{N}}{\rho_d^2} \int_\Omega v^{p+1} dx + 2g_{\text{en}} \kappa^2 \int_\Omega v^q - 2k_2 \int_\Omega v^{p+1} dx.
\]
(39)

In view of \(2q < p+1\), we see \(\tau = \frac{p+1 - 2q}{p-q} < 1\). Using Holder inequality and Young inequality, we obtain
\[
\int_\Omega v^q \leq \left( \int_\Omega v^{p+1} dx \right)^{q/(q+1)} \left( \int_\Omega v^{p+1} dx \right)^{1/q} \leq \left( \int_\Omega v^{p+1} dx \right)^{q/(q+1)} \left( \int_\Omega v^{p+1} dx \right)^{q/(p+1)} \leq \left( \int_\Omega v^{p+1} dx \right)^{q/(q+1)} \left( \int_\Omega v^{p+1} dx \right)^{q/(p+1)} \leq (1-\tau) \int_\Omega v^{\alpha} + \tau \frac{r-1}{r} \int_\Omega v^{\alpha} dx.
\]

Combining (39) with (38), we have
\[
\phi'(t) \leq L_1 \int_\Omega v^{p+1} dx - L_2 \int_\Omega v^{p+1} dx,
\]
with
\[
L_1 = \frac{2Nk_d \mathcal{N}}{\rho_d^2} + 2g_{\text{en}} \kappa^2 \tau \frac{r-1}{r} > 0,
\]
\[L_2 = 2k_2 - 2g_{\text{en}} \kappa^2 (1-\tau) \]
for \(\tau > 0\) small enough.

\[
\int_\Omega v^{p+1} dx \leq \left( \int_\Omega v^{p+1} dx \right)^{q/(q+1)} \left( \int_\Omega v^{p+1} dx \right)^{1/q} \leq \left( \int_\Omega v^{p+1} dx \right)^{q/(q+1)} \left( \int_\Omega v^{p+1} dx \right)^{q/(p+1)} \leq \left( \int_\Omega v^{p+1} dx \right)^{q/(q+1)} \left( \int_\Omega v^{p+1} dx \right)^{q/(p+1)} \leq (1-\tau) \int_\Omega v^{\alpha} + \tau \frac{r-1}{r} \int_\Omega v^{\alpha} dx.
\]

V. CONCLUDING REMARKS

Concluding remarks

A result analogous to theorem 3.1 can be obtained for a bounded domain \(\Omega \subset \mathbb{R}^2\) by first using Schwarz's inequality to write
\[
\int_\Omega u^2 dx \leq \left( \int_\Omega u^2 dx \right)^{1/2} \left( \int_\Omega u^2 dx \right)^{1/2},
\]
then deriving an integral inequality similar to (14) for \(\int_\Omega u^2 dx\) by the method used in going from (15) to (23) in [18]. In this manner, we obtain a first-order differential inequality of the form
\[
\phi(t) \leq K_1 \phi + K_2 \phi^2 + K_3 \phi^3
\]
for computable constants \(K_1, K_2, K_3\), from which a lower bound on the blow-up time follows.

The results in this paper can also be extended to the following initial-boundary value problem for the nonlinear system of \(n\) parabolic equations which are coupled through the nonlinear terms and the boundary data
\[
(u_{ij}, j = 1,2,...,n) \leq \sum_{i,j=1}^{n} a_{ij}^{(x)}(u_{ij})_{ij} + f_{ij}(u) \quad \text{for } i = 1,2,...,n
\]
(48)
\[
\sum_{i,j=1}^{n} a_{ij}^{(x)}(u_{ij})_{ij} + g_{n} = g_{n}(u)
\]
(49)
\[
u_n(x,0) = u_n(x) \geq 0
\]
(50)
where \(\alpha = 1,2,...,n, \nu_n = (u_1,...,u_n)\) and \((a^{(x)}_{ij}(x))_{N\times N}\) is differentiable positive definite matrices. We shall use comma notation to denote partial differentiation with respect to \(x\), and the summation convention on repeated indices, where the Greek indices sum over \(1,2,...,n\). We impose the following constraint on the nonlinear terms and the boundary data in (31)
\[
u_n^{(y)} \leq B_1(u_{ij}^{(y)}) \quad \text{for } i = 1,2,...,n
\]
\[
u_n^{(u)} \leq B_2(u_{ij}^{(u)})
\]
for positive constants \(B_1, B_2\) and \(\sigma > 1\), and assume the existence of solutions \(u_n, \alpha = 1,2,...,n\, of\) the system (31), one or more of which becomes unbounded in finite time \(\tau^*\).

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REFERENCES


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