

## ON THE WEAK UNIFORM ROTUNDITY OF BANACH SPACES

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We prove that if  $X_i, i = 1, 2, \dots$ , are Banach spaces that are weak\* uniformly rotund, then their  $l_p$  product space ( $p > 1$ ) is weak\* uniformly rotund, and for any weak or weak\* uniformly rotund Banach space, its quotient space is also weak or weak\* uniformly rotund, respectively.

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**1. Definitions and preliminaries.** In this note,  $X$  and  $Y$  denote Banach spaces and  $X^*$  and  $Y^*$  denote the conjugate spaces of  $X$  and  $Y$ , respectively. Let  $A \subset X$  be a closed subset and  $X/A$  denote the quotient space. We use  $S(X)$  for the unit sphere in  $X$  and  $P_{l_p}(X_i)$  for the  $l_p$  product space. We refer to [1, 3] for the following definitions and notations. For more recent treatment, one may see, for example, [2].

**DEFINITION 1.1.** A Banach space  $X$  is  $UR^{A'}$ , where  $A'$  is a nonempty subset of  $X^*$ , if and only if for any pair of sequences  $\{x_n\}$  and  $\{y_n\}$  in  $S(X)$ , if  $\|x_n + y_n\| \rightarrow 2$ , then  $f(x_n - y_n) \rightarrow 0$  for all  $f$  in  $A'$ .

**DEFINITION 1.2.** A Banach space  $X$  is WUR (weakly uniformly rotund) if and only if  $X$  is  $UR^{X^*}$ .

**DEFINITION 1.3.** The conjugate space  $X^*$  is  $W^*UR$  (weak\* uniformly rotund) if and only if  $X$  is  $UR^{Q(X)}$ , where  $Q: X \rightarrow X^{**}$  is the canonical embedding.

**2. Some results on the weak\* and weak uniform rotundity.** From the definition, we clearly have the following corollary.

**LEMMA 2.1.** *The Banach space  $X$  is  $W^*UR$  if and only if for any pair of sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$ , if  $\|x_n\| - \|y_n\| \rightarrow 0$ ,  $\{\|y_n\|\}$  is bounded, and  $\|x_n\| + \|y_n\| - \|x_n + y_n\| \rightarrow 0$ , then  $x_n - y_n \xrightarrow{w^*} \theta$ .*

**THEOREM 2.2.** *Suppose that  $X_i, i = 1, 2, \dots$ , are  $W^*UR$ , then for  $p > 1$ ,  $P_{l_p}(X_i)$  is  $W^*UR$ .*

**PROOF.** Let  $X_i = Y_i^*$ , then  $P_{l_p}(X_i) = [P_{l_q}(Y_i)]^*$  (where  $1/p + 1/q = 1$ ) (see [2]). Let  $\{x_n\} = \{(x_1^n, x_2^n, x_3^n, \dots, x_m^n, \dots)\} \in P_{l_p}(X_i)$ ,  $\{y_n\} = \{(y_1^n, y_2^n, y_3^n, \dots, y_m^n, \dots)\} \in P_{l_p}(X_i)$ ,  $\|x_n + y_n\| \rightarrow 2$ . Using the properties of  $l_p$  norm and Minkowski

inequality, one can see, for each  $i$ , that there exists a subsequence of  $\{n\}$ ,  $\{n_k^i\}$ , such that  $\lim_{k \rightarrow \infty} \|x_i^{n_k^i}\| = \lim_{k \rightarrow \infty} \|y_i^{n_k^i}\|$  and  $\lim_{k \rightarrow \infty} \|x_i^{n_k^i} + y_i^{n_k^i}\| = \lim_{k \rightarrow \infty} [\|x_i^{n_k^i}\| + \|y_i^{n_k^i}\|]$ . We now choose a subsequence with the diagonal method, without loss of generality, still use  $\{n\}$  as the index such that for each  $i$ , we have  $\lim_{n \rightarrow \infty} \|x_i^n\| - \lim_{n \rightarrow \infty} \|y_i^n\| = 0$  and  $\lim_{k \rightarrow \infty} [\|x_i^n\| + \|y_i^n\| - \|x_i^n + y_i^n\|] = 0$ . Since  $X_i$  is  $W^*UR$  for each  $i$ , by the lemma, we have

$$x_i^n - y_i^n \xrightarrow{w^*} \theta. \tag{2.1}$$

Suppose that  $P_{l_p}(X_i)$  is not  $W^*UR$ , then there exist sequences  $\{x_n\} \in S(P_{l_p}(X_i))$ ,  $\{y_n\} \in S(P_{l_p}(X_i))$ ,  $\|x_n + y_n\| \rightarrow 2$ , but  $x_n - y_n$  does not converge ( $w^*$ ) to  $\theta$ . So, there must be an  $a = (a_1, a_2, \dots, a_i, \dots)$  in  $P_{l_q}(Y_i)$ , with  $a_i \in Y_i$ , such that  $|(x^n - y^n)(a)|$  does not converge to 0. Therefore, there exist  $\epsilon > 0$  and a subsequence of  $\{n\}$  (for simplicity, we still use  $\{n\}$ ) such that  $|(x^n - y^n)(a)| > \epsilon$ , which implies that one can find an integer  $m$ , sufficiently large, so that

$$\sum_{i=1}^m |(x_i^n - y_i^n)(a_i)| > \frac{\epsilon}{2}. \tag{2.2}$$

Let  $(n_k)$  be the subsequence of  $\{n\}$  such that (2.1) holds. By (2.2), we have

$$\sum_{i=1}^m |(x_i^{n_k} - y_i^{n_k})(a_i)| > \frac{\epsilon}{2}. \tag{2.3}$$

Let  $k \rightarrow \infty$  in (2.3), we have a contradiction  $0 > \epsilon/2$ .

The proof is complete. □

**THEOREM 2.3.** *Suppose that  $X = Y^*$  and  $A$  is any  $w^*$  closed subspace of  $X$ . If  $X$  is  $W^*UR$ , then  $X/A$  is  $W^*UR$ .*

**PROOF.** Let  $D = \{y \in Y \mid x(y) = 0 \text{ for any } x \in A\}$ , then

$$A = \{x \in X \mid x(y) = 0 \text{ for any } y \in D\}, \tag{2.4}$$

see [4]. So, We have  $D^* \simeq X/A$ .

Suppose that  $X/A$  is not  $W^*UR$ , then there exist  $\{\tilde{x}_n\}$  and  $\{\tilde{y}_n\}$  in  $X/A$  such that  $\|\tilde{x}_n\| = \|\tilde{y}_n\| = 1$ ,  $\|\tilde{x}_n + \tilde{y}_n\| \rightarrow 2$ , but  $\tilde{x}_n - \tilde{y}_n$  does not converge ( $w^*$ ) to  $\theta$ . Here,  $\tilde{x} = \pi(x)$ , where  $\pi : X \rightarrow X/A$ .

Now, for each  $n$ , take  $x_n \in \tilde{x}_n$  and  $y_n \in \tilde{y}_n$ ,  $1 \leq \|x_n\| \leq 1 + 1/n$ ,  $1 \leq \|y_n\| \leq 1 + 1/n$ , then  $\lim_{n \rightarrow \infty} \|x_n + y_n\| = 2$ . Since  $X$  is  $W^*UR$ , we have  $x_n - y_n \xrightarrow{w^*} \theta$ ,  $\pi$  is  $w^* \text{-} w^*$  continuous. So, we must have  $\tilde{x}_n - \tilde{y}_n \xrightarrow{w^*} \theta$ . That contradicts the above, and the proof is complete. □

**THEOREM 2.4.** *Suppose that  $A$  is a closed subspace of  $X$  and  $X$  is  $WUR$  (Definition 1.2), then  $X/A$  is  $WUR$ .*

**PROOF.** The proof is similar to the proof of [Theorem 2.3](#). □

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