A new class of Lyapunov functions for the constrained stabilization of linear systems

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Abstract

The constrained stabilization of linear uncertain systems is investigated via the set-theoretic framework of control Lyapunov R-functions. A novel composition rule allows the design of a composite control Lyapunov function with external level set that exactly shapes the maximal controlled invariant set and inner sublevel sets arbitrarily close to any choice of smooth ones, generalizing both polyhedral and truncated ellipsoidal control Lyapunov functions. The feasibility test of the proposed smooth control Lyapunov functions can be casted into matrix inequalities conditions. The constrained linear quadratic control is addressed as an application.

Key words: Generated Lyapunov functions; Lyapunov methods; Feedback stabilization; Constraints; Uncertain linear systems.

1 Introduction

The state-feedback stabilization of constrained uncertain linear systems, covering saturations of the control inputs, state constraints and model uncertainties, is equivalent to the design of a robust Control Lyapunov Function (CLF). Since the particular choice of the candidate CLF also provides an estimation of the controlled invariant set [Blanchini, 1999], the exact solution consists in providing the largest controlled invariant region of the state space, according to both state and control constraints [Balestrino et al., 2011a]. In general, non-trivial classes of candidate CLFs are required to shape the maximal controlled invariant set. For instance, Polyhedral CLFs (PCLFs) are a universal class of functions for the stabilizability of uncertain linear systems [Blanchini, 1995]; or equivalently Linear Differential Inclusions (LDIs). PCLFs can be smoothed with standard norms [Blanchini and Miani, 1999] in order to obtain an everywhere differentiable smoothed PCLF that can be used together with nonlinear gradient-based continuous controllers [Petersen and Barmish, 1987]. Recently, the class of Truncated Ellipsoids (TEs) [O’Dell and Misawa, 2002, Thibodeau et al., 2009] has been proposed as candidate LFs and CLFs for constrained uncertain linear systems to provide a good approximation of the maximal controlled invariant region with a reduced number of parameters [O’Dell and Misawa, 2002]. In [Thibodeau et al., 2009] a linear state-feedback control is designed by solving a Bilinear Matrix Inequality (BMI), maximizing the volume of the estimated controlled invariant set. The main contribution of this paper is the definition of a novel composition rule for merging two different CLFs, allowing the design of a non-homothetic smooth CLF with the following properties: a) the external level set exactly shapes the maximal controlled invariant set; b) the inner sublevel sets can be made arbitrarily close to any given choice of smooth ones. This properties allow to define a stabilizing nonlinear gradient-based control law that is continuous everywhere inside the maximal controlled invariant set. The results of [Balestrino et al., 2010, 2011a,b], where a basic composition rule is introduced, are extended to the class of constrained uncertain linear systems by deriving the more general class of so-called Control Lyapunov R-Functions (CLRFs). Moreover, a Linear Matrix Inequality (LMI) feasibility test for the candidate CLRF is here proposed. As in [Chesi and Hung, 2008, Hu and Blanchini, 2010], the synthesis condition is obtained via BMIs. CLRFs can smooth both PCLFs and TEs in a non-homothetic way and they can be made everywhere differentiable.

The novel smoothing technique follows from the framework of R-functions, referred in the next section. In Sections 3 and 4 the main results are provided. Section 5

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heuristically addresses the constrained LQ control problem as an application. All the proofs are in Appendix.

1.1 Notation

\( I_n \in \mathbb{R}^{n \times n} \) denotes the identity matrix. \( \mathbb{I}_r \) denotes \( \{n \in \mathbb{Z}^+: n \leq r\} \). \( T_r \) denotes \( \{1, \ldots , n\}^+ \cap \mathbb{N}^c \). \( \text{co}() \) denotes the convex hull. The closed \( k \)-level set of a continuous function \( V: \mathcal{X} \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \), i.e. \( \{x \in \mathcal{X}: V(x) \leq k\} \), is denoted by \( \mathcal{L}[V,k] \). A convex and compact set \( \mathcal{S} \) s.t. \( x \in \mathcal{S} \Rightarrow -x \in \mathcal{S} \) is called 0-symmetric [Blanchini, 1995]. A set \( \mathcal{X} \) is a controlled invariant set if \( \forall x(0) \in \mathcal{X} \) there exists an admissible control to keep \( x(t) \in \mathcal{X} \) \( \forall t \geq 0 \) [Hu and Blanchini, 2010]. If \( \mathcal{L}[V,k] \) is the largest controlled invariant set associated to the CLF \( V: \mathcal{X} \to \mathbb{R}^+ \), then the level set \( \{x \in \mathcal{X}: V(x) = k\} \) is called external \( k \)-level set, while the level sets such that \( V(x) = \kappa < k \) are addressed as internal \( k \)-level sets.

2 A novel composition rule for \( \mathbb{R} \)-functions

The framework of \( \mathbb{R} \)-functions has been firstly proposed in the setting of state-feedback stabilization in [Balestrino et al., 2010, 2011b]. Here a novel composition rule is defined.

Definition 1. A function \( r: \mathbb{R}^n \to \mathbb{R} \) is an \( \mathbb{R} \)-function if there exists a Boolean function \( \mathcal{R}: \mathbb{B}^n \to \mathbb{B} \), where \( \mathbb{B} = \{0, 1\} \), such that

\[
    h \left( r (x_1, x_2, \ldots , x_n) \right) = \mathcal{R} \left( h (x_1), h (x_2), \ldots , h (x_n) \right),
\]

where \( h(\cdot) \) is the standard Heaviside step function.

Informally, a real function \( r \) is an \( \mathbb{R} \)-function if it can change its sign only when some of its arguments \( x_i \) change their sign [Balestrino et al., 2011b]. The novel composition rule associated to the Boolean AND is the following.

\[
    r_{\wedge \phi} := r_1 \wedge \phi r_2 := \rho(\phi) \left( \phi r_1 + r_2 - \sqrt{(\phi r_1)^2 + r_2^2} \right),
\]

where \( \phi \in \mathbb{R}^+ \) and \( \rho(\phi) := (\phi + 1 - \sqrt{\phi^2 + 1})^{-1} \) is the normalizing factor such that \( \forall \phi \in \mathbb{R}^+ \), we have \( r_{\wedge \phi} = k \Leftrightarrow \{r_1 = r_2 = k\} \).

Lemma 1.

\[
    r_{\wedge \phi} > 0 \Leftrightarrow \{r_1 > 0 \text{ and } r_2 > 0\} \forall \phi \in \mathbb{R}^+.
\]

In the following we consider only 0-symmetric sets as admissible state space [Blanchini and Miani, 1999, Hu and Blanchini, 2010]. The following technical properties will be further used in the paper.

Proposition 1. For any \( r_1, r_2: \mathbb{R}^n \to \mathbb{R} \), function \( r_{\wedge \phi} : \mathbb{R}^n \to \mathbb{R} \) satisfies

\[
    \min \{r_1(x), r_2(x)\} \leq r_{\wedge \phi}(x) \leq \max \{r_1(x), r_2(x)\} \forall \phi \in \mathbb{R}^+, \forall x \in \mathbb{R}^n. \quad (4)
\]

Proposition 2. In the set \( \{x \in \mathbb{R}^n: r_1(x), r_2(x) > 0\} \), the composed function \( r_{\wedge \phi} = r_1 \wedge \phi r_2 \) converges pointwise to \( r_2 \) as the parameter \( \phi \in \mathbb{R}^+ \) approaches infinity (zero):

\[
    r_{\wedge \phi}(x) \xrightarrow{\phi \to \infty} r_2(x) \forall x; \quad r_{\wedge \phi}(x) \xrightarrow{\phi \to 0^+} r_1(x) \forall x. \quad (5)
\]

A geometric interpretation is now provided with an example in \( \mathbb{R}^2 \). Consider a polyhedral function (of the second order [Hu and Blanchini, 2010]) \( V_1(x) = \max \{x^T F_1^T F_1 x, x^T F_2^T F_2 x\} \), being \( F_i \) the \( i^{\text{th}} \) row of matrix \( F \), and a quadratic function \( V_2(x) = x^T P x \). To compose the positive definite functions \( V_1 \) and \( V_2 \) in their 1-level sets, respectively \( \mathcal{L}[V_1, 1] \) and \( \mathcal{L}[V_2, 1] \), define the \( \mathbb{R} \)-functions \( R_1(x) = 1 - V_1(x) \) and \( R_2(x) = 1 - V_2(x) \). Without loss of generality, these functions have been normalized so that their maximum value is 1. Then compute the \( \mathbb{R} \)-intersection \( \langle \wedge \phi \rangle \) \( R_{\wedge \phi} = R_1 \wedge \phi R_2 \) according to (2), for an arbitrary value of \( \phi \in \mathbb{R}^+ \). The composed function \( R_{\wedge \phi} \) is the (smooth) intersection between the polyhedral function and the quadratic one in the sense that \( R_{\wedge \phi} \) is positive inside the geometric intersection region \( \mathcal{L}[V_1, 1] \cap \mathcal{L}[V_2, 1] \), it is zero on the boundary, negative outside, and its maximum value is 1 at the origin. The positive definite function associated to \( R_{\wedge \phi} \) is \( V_{\wedge \phi} = 1 - R_{\wedge \phi} \), whose sublevel sets are shown in Figure 1.

![Fig. 1. Sublevel sets of the composed function for different values of \( \psi \): \( \mathcal{L}[V_{\wedge \phi}, 1] = \mathcal{L}[V_1, 1] \cap \mathcal{L}[V_2, 1] \) \( \forall \phi \).](image)

The novelty of the proposed composition rule consists in the fact that, unlike all composition rules proposed in the...
literature, a parameter ($\phi$) can be used to trade-off the shape of the sublevel sets of the composite function between the ones of the two generating functions, still preserving the overall domain $\mathcal{L}[\mathcal{V}_1, 1] = \mathcal{L}[\mathcal{V}_1, 1] \cap \mathcal{L}[\mathcal{V}_2, 1]$. Notice that the trade-off parameter $\phi$ provides an additional degree of freedom that could be exploited to improve the closed-loop performances with respect to the use of homothetic functions recovered in the two limit cases presented in Proposition 2.

For ease of reading, in the rest of the paper, the notation $\wedge$ is used in place of $\wedge_{\phi}$.

3 Stability analysis of nonlinear systems via Lyapunov R-functions

In this section, we consider the stability analysis of nonlinear dynamical systems $\dot{x} = f(x), x \in \mathcal{X} \subseteq \mathbb{R}^n$.

Given two differentiable LFs $\mathcal{V}_i$, $i \in \{1, 2\}$, respectively in $\mathcal{L}[\mathcal{V}_1, 1]$, $\mathcal{L}[\mathcal{V}_2, 1]$, the candidate LF $\mathcal{V}_L$ in $\mathcal{L}[\mathcal{V}_\wedge, 1] = \mathcal{L}[\mathcal{V}_1, 1] \cap \mathcal{L}[\mathcal{V}_2, 1]$ is derived as follows.

$$\begin{align*}
R_i := 1 - V_i, \quad i = 1, 2; & \quad R_{\wedge} := R_1 \wedge R_2; \\
V_{\wedge} := 1 - R_{\wedge}.
\end{align*}$$

Now if $R_i(x) = k - V_i(x)$, $i = 1, 2$, $k \in \mathbb{R}^+$, then, according to Lemma 1: $R_i(x) \geq 0 \Leftrightarrow \{R_i(x) \geq 0 \cap R_2(x) \geq 0\}$, i.e. $V_\wedge(x) \leq k \Leftrightarrow \{V_1(x) \leq k \wedge V_2(x) \leq k\}$ and hence $\mathcal{L}[V_\wedge, k] = \mathcal{L}[V_1, k] \cap \mathcal{L}[V_2, k]$. We consider the interior of $\mathcal{L}[V_\wedge, k]$ to avoid the lack of differentiability of $V_\wedge$ in its external level set.

**Theorem 1.** Assume that $\mathcal{V}_1, \mathcal{V}_2$ are two LFs such that $\nabla \mathcal{V}_i f(x) \leq -\eta \mathcal{V}_i(x), \quad i = 1, 2$, respectively in $\mathcal{L}[\mathcal{V}_i, 1] \subseteq \mathcal{X}$. Then $\mathcal{V}_\wedge$ is an LF in the interior of $\mathcal{L}[\mathcal{V}_\wedge, 1], \forall \phi \in \mathbb{R}^+$.

4 Constrained stabilization of uncertain linear systems via control Lyapunov R-functions

4.1 Problem statement and discussion

Consider the constrained stabilization of an uncertain linear system

$$\begin{align*}
\dot{x} & \in \text{co}\{A_i x + B_i u \mid i \in \mathbb{I}_p\} \quad \text{sub. to:} \\
& x \in \mathcal{X} \subseteq \mathbb{R}^n, \quad u \in \mathcal{U} \subseteq \mathbb{R}^m,
\end{align*}$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m} \forall i \in \mathbb{I}_p$, via a continuous state-feedback control $u(x)$ such that $x(t)$ asymptotically converges to the origin, in accordance to the state and control input constraints. The constraints are assumed to be convex and 0-symmetric. In particular $\mathcal{U} := \{u \in \mathbb{R}^m : \|u\|_{\infty} \leq 1\}$.

A polyhedral approximation (with arbitrary precision) $\mathcal{X}$ of the maximal controllable set for system (7) can be explicitly computed via sequential linear programming [Blanchini, 1995], obtaining a full column-rank matrix $F \in \mathbb{R}^{n \times n}$ such that $\mathcal{X} := \{x \in \mathbb{R}^n : \|F x\|_{\infty} \leq 1\} \subseteq \mathcal{X}$.

Several classes of explicit functions have been proposed in the literature as candidate CLFs, such as PCLFs [Blanchini, 1991], PCLFs of the second order [Hu and Blanchini, 2010], smooth PCLFs [Blanchini and Miani, 1998], TE CLFs [O’Dell and Misawa, 2002, Thibodeau et al., 2009] and smoothed TE CLFs [Balestrino et al., 2011b]. A continuous control, that guarantees the maximal invariant set, can be also designed via convex hull of ellipsoids [Hu and Lin, 2001].

4.2 Control Lyapunov R-functions

Function $\mathcal{V}_\ell$, corresponding to the smoothed intersection of a PCLF $\mathcal{V}_i(x) := \max_{i \in \mathbb{I}_p} \{x^T F_i x\}$ and a QCLF (for the unconstrained system) $\mathcal{V}_2(x) := x^T P x$ is here used as candidate CLF for (7). The following theorem is conclusive for the constrained stabilizability.

**Theorem 2.** Assume that there exist $K \in \mathbb{R}^{n \times n}$, $P > 0$, $\eta \in \mathbb{R}^+$ and $\gamma_{ijk} \in \mathbb{R}$, $\forall i \in \mathbb{I}_p$, $\forall j, k \in \mathbb{I}_s$, s.t.

$$\begin{align*}
(A_i + B_i K)^T F_k^T F_k + F_k^T F_k (A_i + B_i K) & \leq -\eta P f_k^T F_k + \sum_{j=1}^{s} \gamma_{ijk} (F_j^T F_j - F_k^T F_k) \\
(A_i + B_i K)^T P + P (A_i + B_i K) & \leq -\eta P f_k^T F_k + \sum_{j=1}^{s} \gamma_{ijk} (F_j^T F_j - F_k^T F_k)
\end{align*}$$

(8)

$$\begin{align*}
-I_m \leq K v^{(l)} \leq I_m, \quad \forall l,
\end{align*}$$

(9)

where $v^{(l)}$ are the vertices of the polyhedron $\mathcal{L}[\mathcal{V}_1, 1]$, and $F_k$ is the $k^{th}$ row of $F$. Then $\mathcal{V}_\ell$, merging $\mathcal{V}_1$ and $\mathcal{V}_2$, is a CLF for (7) in the set $\mathcal{L}[\mathcal{V}_1, 1] \cap \mathcal{L}[\mathcal{V}_2, 1] \forall \phi \in \mathbb{R}^+$.

The maximal controlled set, namely $\mathcal{L}[\mathcal{V}_1, 1] = \mathcal{X}$, is recovered for $\mathcal{V}_\ell$ by a-priori scaling $\mathcal{V}_2(x) = x^T P x$ such that $\mathcal{L}[\mathcal{V}_2, 1] \supseteq \mathcal{L}[\mathcal{V}_1, 1]$, as $\mathcal{L}[\mathcal{V}_\wedge, 1] = \mathcal{L}[\mathcal{V}_1, 1] \cap \mathcal{L}[\mathcal{V}_2, 1]$.

According to Theorem 2, in the interior of $\mathcal{L}[\mathcal{V}_1, 1]$, any trade-off shape obtained with $\phi \in \mathbb{R}^+$ is suitable. However, no explicit rule for selecting $\phi$ (and hence a particular shape) is here presented. This opens up the possibility of defining some criteria for the choice of $\phi$. This subject is not addressed in this paper and it is hence left for future investigations.

In view of Proposition 1, we have $\min\{\mathcal{V}_1, \mathcal{V}_2\} \leq \mathcal{V}_\wedge \leq \max\{\mathcal{V}_1, \mathcal{V}_2\}$, therefore $\mathcal{V}_\wedge$, merging $\mathcal{V}_1$ and $\mathcal{V}_2$, grows quadratically as well: $\exists c_1, c_2 \in \mathbb{R}^+$ s.t. $c_1 x^T x \leq \mathcal{V}_\wedge(x) \leq c_2 x^T x$. If the decreasing rate of $\mathcal{V}_\wedge$ is $\eta$, i.e. $\exists c \in \mathbb{R}^+$: $\mathcal{V}_\wedge(x(t)) \leq c \cdot e^{-\eta t} \mathcal{V}_\wedge(x(0))$, then the convergence rate (in terms of the 2-norm) is $\eta/2$: $c_1 \|x(t)\|^2_2 \leq$
\[ V(x(t)) \leq c \cdot e^{-\eta t} V(x(0)) \leq c \cdot e^{-\eta t} \|x(0)\|^2, \] that implies \[ \|x(t)\|^2 \leq \sqrt{\frac{c \cdot e^{-\eta t}}{c^2 c^{-1} \cdot e^{\eta t/2}}} \|x(0)\|^2 \forall t \geq 0. \]

Remark 1. The first inequality in (8) is a BMI in the variables \( K, P, \eta, \gamma_{ikj} \) (if \( P \) is fixed, then (8) becomes an LMI). Also in [Thibodeau et al., 2009] a BMI problem has to be solved for the synthesis of an unsmooth TE CLF together with a linear state-feedback controller. The advantage of the proposed approach with respect to [Thibodeau et al., 2009] is that if the BMI is feasible, then a smooth CLF is obtained. This implies that explicit nonlinear gradient-based controllers can be used [Petersen and Barmish, 1987, Blanchini and Miani, 1999], improving control performances.

Remark 2. The assumption of Theorem 2 on the existence of a linear control is adopted in the earlier works on stabilization of constrained linear systems by means of PCLFs [Vassilakos and Bitsoris, 1989, Blanchini, 1991], where the Linear Constrained Regulator Problem (LCRP) is addressed. More recently, the same assumption is required for the feasibility of the BMI problems proposed in [O’Dell and Misawa, 2002] for semi-ellipsoidal sets, in [Thibodeau et al., 2009] for TEs, in [Andrieu and Prieur, 2010] for the problem of unifying two different CLFs.

The choice of \( P = \frac{1}{2} \sum_{i=1}^{s} P_{ii} F_i^T F_i \) (the one associated to the shape of the PCLF) preserves the maximal controlled set \( L[V_1,1] \) because it guarantees \( L[V_2,1] \supseteq L[V_1,1] \). This particular choice corresponds to inequality (8) with \( \gamma_{ikj} = \gamma_{ik3} \), yielding a sufficient LMI. Without loss of generality, Theorem 2 is also valid for the CLRF merging the smoothed PCLF \( V_3(x) = \|F x\|^2_{2p} \), for \( p \) sufficiently large [Blanchini and Miani, 1999], and \( V_2(x) = x^T P x \). Unlike the standard 2p-norm of [Blanchini and Miani, 1999], we can provide a trade-off composition of (smoothed) polyhedral and quadratic functions.

Remark 3. Given a smooth CLRF, a known continuous stabilizer is the minimum effort control [Petersen and Barmish, 1987]:

\[
  u(x) = \arg \min_{v \in U} \|v\| \quad \text{sub. to} \quad \max_{t \in [0,T]} \nabla V_\lambda(x) (A_t x + B v) \leq -\eta V_\lambda(x). \tag{10}
\]

Note that the minimum effort control may be not continuous if applied to a polynomial function since differentiability fails [Blanchini and Miani, 1999]. More details about the explicit formulation of (10) and about the general case of uncertain matrix \( B(t) \) are addressed in [Blanchini and Miani, 1999].

Note that since \( \nabla V_1(x) = -\nabla R_1(x) \) and \( \nabla V_2(x) = -\nabla R_2(x) \), the gradient \( \nabla V_\lambda(x) \) is a nonlinear, positive, combination of \( \nabla V_1(x) \) and \( \nabla V_2(x) \), see (16).

5 Application to constrained linear quadratic optimal control

Designing the shape of the candidate CLRF, via the novel composition rule, suggests the application to the constrained LQ optimal control problem. In fact, while the external set can be designed in accordance to the shape of the maximal controllable set, the inner sublevel sets can be (independently) made arbitrarily close to the locally-optimal quadratic ones.

Consider the constrained (nominal) system \( \dot{x} = Ax + Bu, x, u \in U, \) with standard quadratic performance index \( J(x, u) = \int_0^\infty (\|x(t)\|^2 + \|u(t)\|^2)dt, \) where \( Q, R \succ 0 \). Let \( P^* \succ 0 \) be the solution of the Algebraic Riccati Equation (ARE).

For the unconstrained LQ optimal control problem it is possible to scale matrix \( P^* \) without loss of generality, because if \( \delta P^* \) is the solution of the ARE, then for any \( \delta \in \mathbb{R}^+ \), \( \delta P^* \) is the solution associated to \( Q \rightarrow \delta Q, R \rightarrow \delta R \), minimizing \( \int_0^\infty (\|x(t)\|^2 + \|u(t)\|^2)dt \). Therefore we assume that \( X \subseteq [x^T P^* x, 1] \).

A good control solution can be obtained by fixing \( P = P^* \) in the inequality (8) of Theorem 2. The proposed CLRF has maximal controlled invariant set and inner sublevel sets close to the quadratic optimal ones, as shown in the example of Section 5.1.

Considering the Hamilton–Jacobi–Bellman equation, a nonlinear control is here proposed:

\[
  u(x) = \arg \min_{v \in U} \{ \nabla V_\lambda(x) (Ax + B v) + x^T Q x + v^T R v \}
  \quad \text{sub. to} \quad \nabla V_\lambda(x) (Ax + B v) \leq -\eta V_\lambda(x). \tag{11}
\]

It can be proved that control (11) strictly follows from the minimal selection control [Freeman and Kokotović, 1996, Section 2.4] and therefore it is continuous [Freeman and Kokotović, 1996, Section 4.2].

Control (11) requires the on-line solution of one Quadratic Program (QP) in \( \mathbb{R}^m \). This kind of approach is “memoryless” and therefore differs from explicit MPC [Bemporad et al., 2002] where the state space is off-line partitioned in a certain number of polyhedral regions, whose number grows exponentially with the prediction horizon, leading to huge requirements of memory to be checked in the on-line search of the “current region”. While [Kojima and Morari, 2004] provides a sequence of sub-optimal QP solutions converging to the optimal one, no theoretical bounds of sub-optimality are discussed here.

5.1 Numerical simulations

Consider a constrained linear system together with the quadratic performance index \( J, \) where \( A = 1/4 [I_2, I_2], B = I_2 \) and \( F \) are from [Blanchini and...
Table 1

<table>
<thead>
<tr>
<th></th>
<th>sPCLF</th>
<th>$\phi = 1$</th>
<th>$\phi = 100$</th>
<th>OPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>IADU</td>
<td>1</td>
<td>1.100</td>
<td>0.696</td>
<td><strong>0.404</strong></td>
</tr>
<tr>
<td>J</td>
<td>1</td>
<td>0.940</td>
<td><strong>0.607</strong></td>
<td><strong>0.599</strong></td>
</tr>
</tbody>
</table>

Average control performances (over 10 simulations starting from random initial states), normalized w.r.t. the use of the smoothed PCLF [Blanchini and Miani, 1999].

Miani, 1998], $Q = 0.15I_2$, $R = 0.3I_2$.

The maximal controlled invariant set $\mathcal{L}[[Fx]^2, 1] \subseteq \mathcal{L}[x^TPx, 1]$ is actually recovered by the CLRF $V_\phi$ (6), see Figure 2, since the LMI problem (8) ($P$ is fixed as the solution of the ARE) is feasible.

Control (11) is associated to different differentiable CLFs: the smoothed PCLF [Blanchini and Miani, 1999], the smoothed TE, i.e. the standard CLRF ($\phi = 1$) of [Balestrino et al., 2011b], besides the novel CLRF with $\phi = 100$. Note that a non-differentiable CLF, for instance a PCLF or a TE, causes considerable control chattering [Blanchini and Miani, 1999] and hence much worse performances. The constrained optimal control is also used for comparisons. It can be computed via explicit MPC [Kojima and Morari, 2004] or via a (long-horizon) Receding Horizon Control (RHC) with no terminal constraint, provided that the final predicted state is inside the controlled domain of attraction of the unconstrained LQR [Chmielewski and Manousiouthakis, 1996].

Table 1 shows the values of the Integral of the Absolute value of the Time Derivative of the control $u$ (IADU) and of the performance cost $J$. The novel CLRF yields less stress on the control actuators and “good” closed-loop performances.

From our numerical experience, as one would expect when gradient-based controllers are used, the closed-loop behavior is “close” to the one induced by the CLF $V_1$ ($V_2$) if $\phi \ll 1$ ($\phi \gg 1$).

Fig. 2. Level sets of the novel CLRF merging a smoothed PCLF and a QCLF.

6 Conclusion and future work

The constrained stabilization of linear uncertain systems is addressed via the class of control Lyapunov R-functions, that are differentiable and non-homothetic.

The novel composition rule looks useful for constrained linear quadratic control problems because the proposed Lyapunov function can be designed with both a large domain of attraction and inner sublevel sets close to the optimal ones. Investigations for theoretical bounds of sub-optimality and for the best tuning of the free trade-off parameter, are not addressed and hence left for future work. Future investigations will also be focused on the assumption of the existence of a common controller.

Appendix

For ease of notation, let $r_1$ and $r_2$ denote, respectively $r_1(x)$ and $r_2(x)$.

Proof of Lemma 1. If $\{r_1 > 0 \land r_2 > 0\}$, then $r_\wedge > 0$ because $r_\wedge + r_2 > \sqrt{r_1^2 + r_2^2} \forall \phi \in \mathbb{R}^+$. Conversely, assume $r_\wedge > 0$. Trivially, $r_1$ and $r_2$ can not be both negative. Squaring both sides of the previous inequality, we obtain $r_1 r_2 > 0$, that leads to a contradiction if one between $r_1$ and $r_2$ is negative.

Proof of Proposition 1. For any $\phi \in \mathbb{R}^+$ we have

$$
\frac{\partial r_\wedge}{\partial r_1} = \phi \left(1 + \frac{-\phi r_1}{\sqrt{\phi^2 r_1^2 + r_2^2}}\right) \geq 0
$$

$$
\frac{\partial r_\wedge}{\partial r_2} = \phi \left(1 + \frac{-r_2}{\sqrt{\phi^2 r_1^2 + r_2^2}}\right) \geq 0.
$$

Therefore, let $\underline{r} := \min\{r_1, r_2\}$ and $\bar{r} := \max\{r_1, r_2\}$:

$$
r_\wedge = \rho(\phi) \left(\phi r_1 + r_2 - \sqrt{\phi^2 r_1^2 + r_2^2}\right)
$$

$$
\geq \rho(\phi) \left(\bar{r} - \sqrt{\phi^2 r_2^2 + \bar{r}^2}\right) = \bar{r}.
$$

Analogously $r_\wedge \leq \underline{r}$.

Proof of Proposition 2. The assumption that $r_1, r_2 > 0$ allows the division by $r_1$ and/or $r_2$.

$$
\lim_{\phi \to +\infty} r_\wedge = \lim_{\phi \to +\infty} \phi \left(\phi r_1 + r_2 - \sqrt{(\phi r_1)^2 + r_2^2}\right)
$$

$$
= \lim_{\phi \to +\infty} \frac{(\phi r_1 + r_2 - \sqrt{(\phi r_1)^2 + r_2^2})}{\phi + 1 - \sqrt{\phi^2 + 1}}
$$

$$
= \lim_{\phi \to +\infty} \frac{2\phi r_1 r_2}{\phi + 1 - \sqrt{\phi^2 + 1}} = r_2.
$$

Analogously, $\lim_{\phi \to +\infty} r_\wedge = r_1$.

Proof of Theorem 1. Define functions $R_i(x) = 1 - V_i(x)$, $i = 1, 2$ and $R_\wedge$, according to (2):

$$
R_\wedge = \frac{\phi R_1 + R_2 - \sqrt{(\phi R_1)^2 + R_2^2}}{\phi + 1 - \sqrt{\phi^2 + 1}}.
$$
The candidate LF $V_\lambda$ is positive definite in the set $\mathcal{L}[V_\lambda, 1] = \mathcal{L}[V_1, 1] \cap \mathcal{L}[V_2, 1]$ because $R_\lambda(x) = 1 \Leftrightarrow R_1(x) = R_2(x) = 1 \Leftrightarrow x = 0$. Moreover, as $V_1, V_2$ are differentiable, $V_\lambda$ is everywhere differentiable as well in the interior of the set $\mathcal{L}[V_\lambda, 1]$. Consider the time derivative

$$
\dot{R}_\lambda = \rho \left( \frac{\phi R_1}{\sqrt{\phi R_1^2 + R_2^2}} + R_2 \left( \frac{1}{\sqrt{\phi R_1^2 + R_2^2}} - \frac{R_2}{\sqrt{\phi R_1^2 + R_2^2}} \right) \right).
$$

The assumption on the decreasing rate is equivalent to $R_1(x) \geq \eta (1 - R_i(x))$, $i = 1, 2$, one has

$$
\dot{R}_\lambda \geq \eta \rho [\phi c_1 + c_2 - (\phi c_1 R_1 + c_2 R_2)].
$$

As $\rho (\phi c_1 R_1 + c_2 R_2) = R_\lambda$, $\dot{R}_\lambda \geq \eta [\rho (\phi c_1 + c_2) - R_\lambda]$, therefore $\rho (\phi c_1 + c_2) \geq 1$ has to be proved.

$$
\rho (\phi c_1 + c_2) \geq 1 \Leftrightarrow \frac{1}{\phi + 1 - \sqrt{\phi^2 + 1}} \left[ \phi + \phi \frac{\phi R_1}{\sqrt{(\phi R_1)^2 + R_2^2}} \right] + 1 + \frac{R_2}{\sqrt{(\phi R_1)^2 + R_2^2}} \geq 1 \Leftrightarrow \phi^2 + 1 \geq \frac{(\phi R_1)^2 + R_2^2}{\sqrt{(\phi R_1)^2 + R_2^2}}.
$$

Then square both sides of the latter inequality in (17):

$$
(\phi^2 + 1)(\phi^2 R_1^2 + R_2^2) \geq (\phi^2 R_1 + R_2)^2 \Leftrightarrow (R_1 - R_2)^2 \geq 0.
$$

\textbf{Proof of Theorem 2.} Inequality (8) is a sufficient condition for function $V_i(x) = \max_{i \in I_\lambda} \{ x^T F_i^T F_i x \}$ to be a PLF with decreasing rate $\eta$ for the state constrained closed-loop LDI $\dot{x} = (A_i + B_i K)x$, $i \in I_\lambda$. Moreover, function $V_2(x) = x^T P x$ is a QLF, with rate $\eta$, for the unconstrained closed-loop LDI. According to Theorem 1, the R-composition of $V_1$ and $V_2$ yields an LF $V_\lambda$ with decreasing rate $\eta$, therefore $V_\lambda$ is a CLF for the constrained system (7) in the set $\mathcal{L}[V_1, 1] \cap \mathcal{L}[V_2, 1]$, $\forall \phi \in \mathbb{R}^+$. 

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