ANALYSIS OF RANGE AND POSITION COMPARISON METHODS AS A MEANS TO PROVIDE GPS INTEGRITY IN THE USER RECEIVER*

Dr. Young C. Lee
The MITRE Corporation, McLean, Virginia

ABSTRACT

Integrity is the ability of a system to let its users know whether the system is operating out of its specified performance limits. This paper analyzes two receiver-based methods for assuring the integrity of Global Positioning System (GPS) signals-in-space for the purpose of determining whether the methods can do the job satisfactorily. These methods are called the range comparison method and the position comparison method. Equations relating unknown satellite range errors to the quantity measured in each method are analyzed. The equations reveal important characteristics of the methods, including their mathematical equivalence. The performance of the two methods is then derived as a function of the range error magnitude of the failed satellite. Finally, numerical results for the performance of the methods are shown, both alone and in combination with monitoring of the receiver clock bias estimate.

INTRODUCTION

Integrity is the ability of a system to let its users know whether the system is operating out of its specified performance limits. The integrity of GPS signals-in-space is essential if the system is to be used by civil aviation for nonprecision approach guidance. One alternative, probably the simplest, is to perform the integrity function in the user receiver. This would obviate the need for any communication links from ground monitoring station(s) that would be necessary in other methods. This paper analyzes two such self-contained integrity evaluation methods, which are called the range comparison (RCM) and position comparison (PCM) methods. It also evaluates the detection performance of these methods, both alone and in combination with monitoring of the receiver clock bias estimate.

For both of the above methods, it is assumed that five satellites are in view of the user. This assumption is influenced by a coverage study [1] showing that five or more satellites are in view to the user in the conterminous U.S. 98 percent of the time. The current study is motivated by a surmise that while four satellites are needed for a navigation solution, the additional satellite might give an indication on the state (normal and abnormal) of the GPS signals-in-space and even perhaps indicate which satellite signal(s) may not be useable.

In the RCM method, illustrated in Figure 1, the user receiver first estimates user position and clock bias based on four satellites at a time. Each of the five 4-satellite navigation solutions is then used to predict the pseudorange to the fifth satellite not included in that particular solution. The differences between predicted pseudoranges and the corresponding measured pseudoranges are used as the basis for detecting an abnormal state. In the PCM method, illustrated in Figure 2, the user receiver estimates position and clock bias using all five satellites at once and then using sets of four satellites one set at a time. The differences between the five possible 4-satellite solutions and the 5-satellite solution are used as the basis for detecting an abnormal state.

* This paper is based upon navigation system studies performed by The MITRE Corporation for the Systems Engineering Service, Federal Aviation Administration under Contract No. DTFA01-84-C-00001. The data presented herein do not necessarily reflect the official views or policy of the FAA.

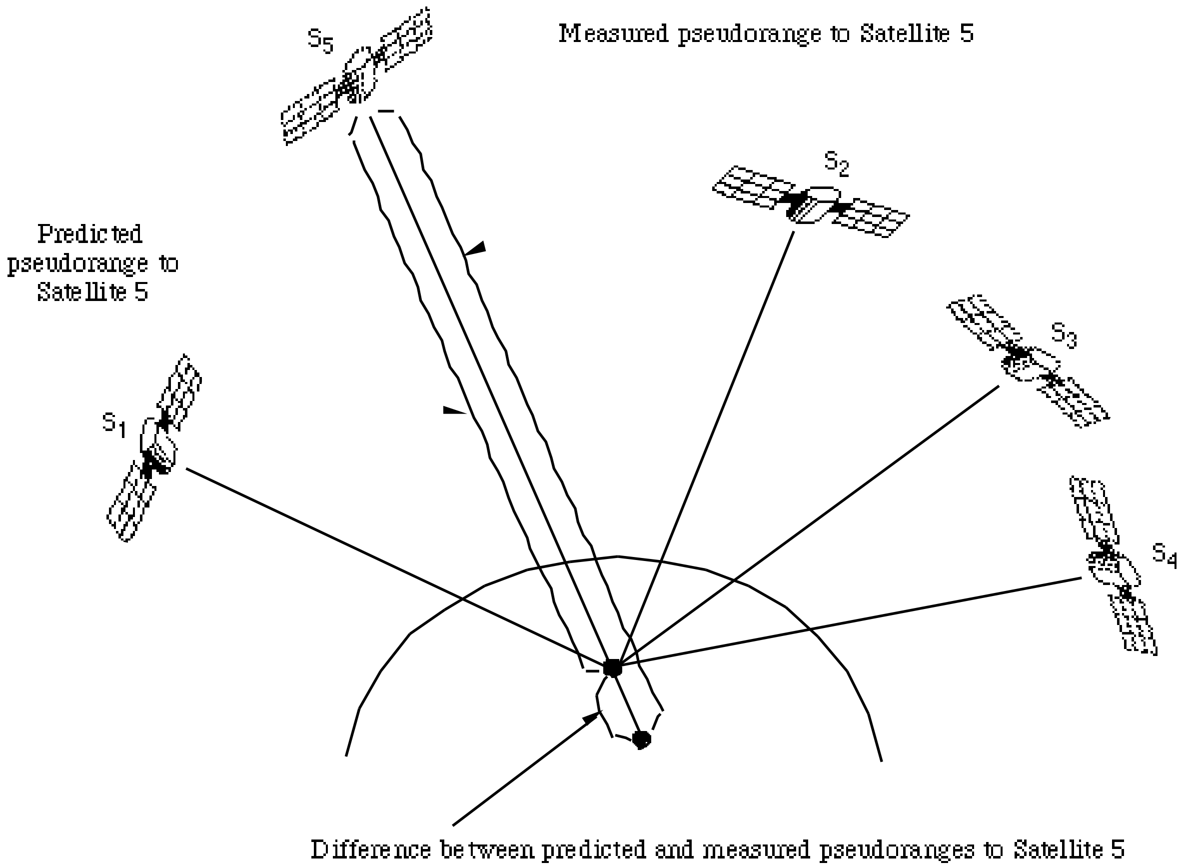


Figure 1. The Range Comparison Method (RCM)

It should be emphasized that the two user receiver methods examined here do not make assumptions on how the GPS signal errors may vary in time. Rather, the methods make use of snapshot pseudorange data at each integrity decision time. Also, the methods depend solely on the satellite pseudorange measurement data and do not use any other information. However, the methods can be used in combination with other methods to improve the integrity monitoring performance. Such methods include receiver clock bias estimate monitoring or use of barometric altimetry data. The performance improvement with various combined methods was previously assessed for the GPS P code at the Aerospace Corporation. The following analysis considers the C/A code and the effects of selective availability (SA).

Equations for the quantities observed in the RCM and PCM methods were originally derived in [2] and [3], respectively. The next section examines the characteristics of the two methods revealed by those equations. The subsequent section first defines normal and abnormal states and the probabilities of detection and false alarm. On the basis of the definitions, the section then derives an appropriate decision criterion and analytic expressions for the probabilities of detection, missed detection, and false alarm, and gives numerical results. The last section summarizes the results of the paper. The appendix at the end contains what was originally published in [2], namely, the derivations of the equations relating observed quantities to the satellite range errors.

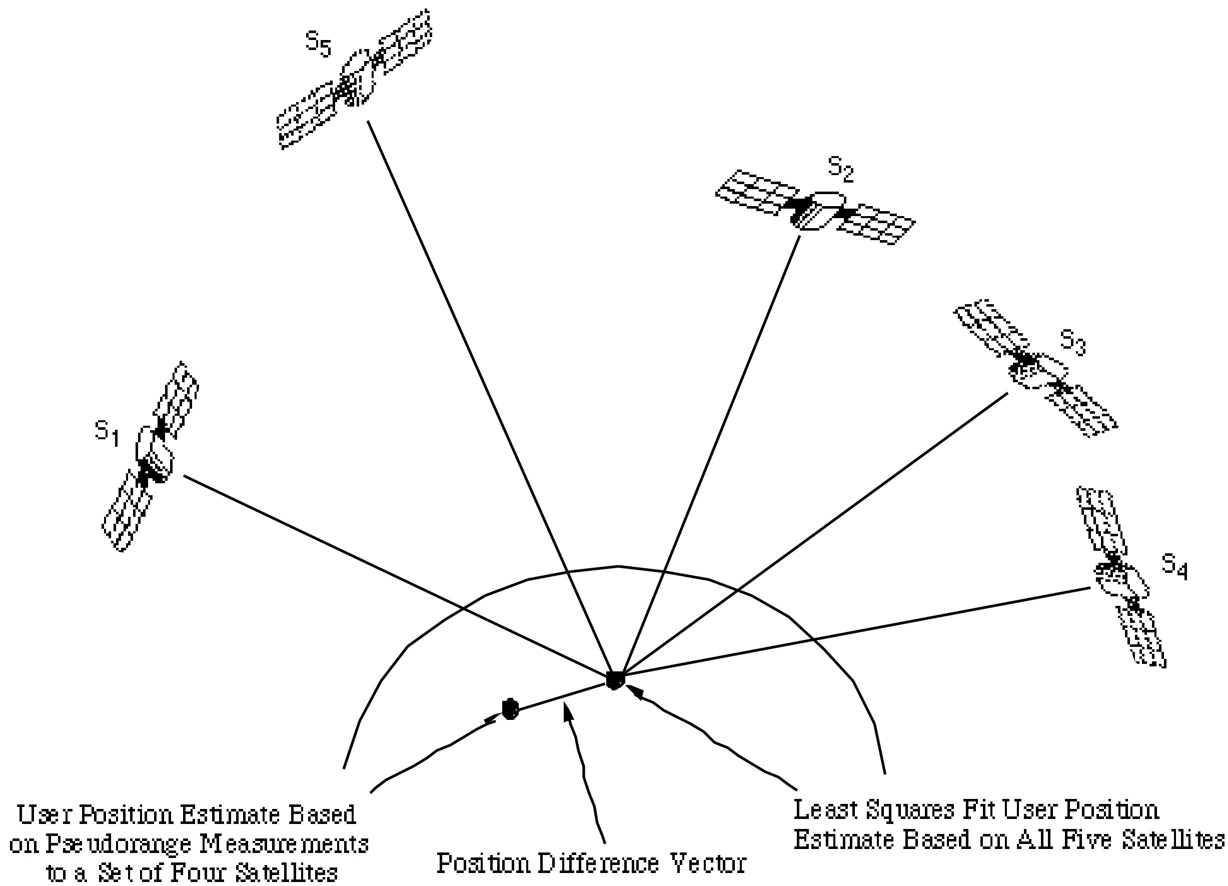


Figure 2. The Position Comparison Method (PCM)

MATHEMATICAL EQUATIONS RELATING OBSERVATIONS TO SATELLITE RANGE ERRORS

This section examines the results of the analysis in the appendix, which mathematically relates the observed quantities in the two methods to errors in the satellite range measurements. First, a typical solution method for GPS navigation is briefly reviewed to facilitate the discussion.

GPS Navigation Solution

A user position vector (x, y, z) and user clock bias B_u are estimated, based on the pseudorange measurements from n (≥ 4) satellites, by solving the following set of simultaneous equations:

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} & 1 \\ e_{21} & e_{22} & e_{23} & 1 \\ & \cdots & & 1 \\ e_{n1} & e_{n2} & e_{n3} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ B_u \end{bmatrix} = \begin{bmatrix} \bar{e}_1 \cdot \bar{R}_1 - \rho_1 + B_1 \\ \bar{e}_2 \cdot \bar{R}_2 - \rho_2 + B_2 \\ \cdots \\ \bar{e}_n \cdot \bar{R}_n - \rho_n + B_n \end{bmatrix} \quad (1)$$

where

$\bar{e}_i$: unit vector from user to the i^{th} satellite
$\bar{R}_i$: i^{th} satellite position vector from the center of the earth
ρ_i : pseudorange to the i^{th} satellite
B_i : range equivalent of the i^{th} satellite clock offset

Because of the great distances between user and satellites, it may reasonably be assumed that errors in the direction cosines e_{i1}, e_{i2}, and e_{i3} that constitute the matrix on the left-hand side of the above equation are negligible. Therefore, the solution errors can be expressed as a function of the errors introduced in the right-hand side. These errors can be divided into two groups. One group, denoted by N_i, is the aggregate user equivalent range error to satellite i due to the inherent GPS system inaccuracy or the failure. The other group, denoted by S_i, is due to intentional degradation of accuracy due to SA. The error in the i^{th} row to be denoted by E_i, and called i^{th} satellite range error, is then a sum of N_i and S_i:

$$E_i = N_i + S_i \tag{2}$$

Observations in each method can be mathematically equated with the satellite range error E_i's as demonstrated in the equations that follow.

Equations for the Range Comparison Method

As was described at the beginning, the RCM method compares the measured pseudorange to each satellite with the predicted pseudorange based on the pseudorange measurements to the other four satellites. The five differences between the measured and predicted pseudoranges constitute the observation space to detect an abnormal state. The equations derived in the appendix, relating the satellite range errors to the observed differences, reveal three important characteristics of this method:

1. Each difference can be expressed as a linear combination of the five satellite range errors. That is, each of the observations D_i (i = 1, 2,...5) can be expressed as

$$D_i = a_{i1} E_1 + a_{i2} E_2 + a_{i3} E_3 + a_{i4} E_4 + a_{i5} E_5 \tag{3}$$

   where

   E_j: satellite range error defined in (2)
   a_{ij}: coefficients that depend on the satellite geometry.

2. The above equations for all the D_i's (i = 1, 2,...5) are proportional to each other with proportionality factors that can be computed as a function of only the satellite geometry. This means that, out of the five observations that can be obtained in this method, any one of the observations contains all of the information.

3. Since only a single linear equation is provided for a hypothesis test, this method can only detect an abnormal state where the error in user position estimate exceeds a threshold; it is not possible to identify any bad satellite(s), using only five satellites.

Equations for the Position Comparison Method

The PCM method compares the user position estimate based on all five satellites with those based on sets of four satellites. The differences between the five possible 4-satellite solutions and the 5-satellite solution comprise the observation space to detect an abnormal state. Since the solution difference for each satellite involves four quantities (three for position in x, y, and z coordinates and one for user clock bias), the comparisons for five different satellites generate a total of 20 differences. The equations relating the satellite range errors to the observations for the PCM method can be derived in a similar manner as were derived for the RCM method in the Appendix. The derivations for the PCM method in [3] reveal the following characteristics of the PCM:

1. Like the case of the RCM, each difference can be expressed as a linear combination of the five satellite range errors. That is, each of the P_i's (i = 1,2, ...20)

$$P_i = b_{il} E_l + b_{i2} E_2 + b_{i3} E_3 + b_{i4} E_4 + b_{i5} E_5 \quad (4)$$

   E_j: satellite range error
   bij: coefficients that depend on the satellite geometry.

2. Like the case of the RCM, the above equations for all the P_i's (i = 1,2,...20) are proportional to each other with proportionality factors that depend only on the satellite geometry. Therefore, only a single equation is provided for a hypothesis test.

3. Therefore, it is only possible for this method to detect an abnormal state; it is not possible to identify any bad satellite(s), using only five satellites.

Mathematical Equivalence of the Range and Position Comparison Methods

It is noted that the characteristics of the two individual methods described above are similar to each other. Knowing, in particular, that only a single equation is available for a hypothesis test in each method, one may wonder what the relationship may be between the equations from the two methods. Reference [3] shows that the two equations are indeed identical with respect to hypothesis testing. Therefore, the RCM and PCM methods are mathematically equivalent.

Analysis of the two methods implies that any user receiver integrity evaluation methods that depend solely upon pseudorange measurements from five satellites, without making any other assumptions or using any other information, will all be mathematically equivalent. This includes, for example, a method in which user position is estimated based on all of the five satellites in view, and then pseudorange predictions based on that position estimate are compared with the actually measured pseudoranges to the corresponding satellites.

PERFORMANCE ESTIMATION

This section estimates how well the comparison methods can provide the GPS signal integrity. The integrity performance is characterized by the probabilities of detection (P_D), missed detection (P_M), and false alarm (P_F).

Definitions of Probabilities of Detection and False Alarm

As discussed in the preceding section, the comparison methods provide only a single equation for a binary hypothesis test. Therefore, the most appropriate hypothesis test for determining the system state would be to take any one of the differences and apply the following criterion:

$$\text{If } |\Delta q| < d, \text{ say } H_o \quad \text{(normal state)} \qquad (5)$$
$$\geq d, \text{ say } H_i \quad \text{(abnormal state)}$$

where

Δq: the difference observed from the comparison
d: a decision threshold determined as a function of the particular observation used for the test.

In a binary decision problem such as the above, two decision criteria are most widely used, the Bayes or Neyman-Pearson. Whereas the Bayes criterion requires an assignment of a cost for each possible course of action and a priori probabilities of normal and abnormal states, the Neyman-Pearson criterion requires neither of them. Since it is difficult to assign realistic costs or a priori probabilities in the application at hand, the Neyman-Pearson criterion is to be used. The criterion maximizes P_D (or minimizes P_M) under the constraint that P_F does not exceed a certain value chosen by the user.

Before P_D and P_F can be computed, it is important to define precisely what constitutes an error or an abnormal state to guard against. In this analysis, an abnormal state is defined to be a state in which any of the satellites used by the GPS user for navigation is malfunctioning. A satellite is defined to be malfunctioning if the magnitude of the satellite range error has some fixed value M large enough to be considered excessive relative to the two-sigma pseudorange error of about 60 meters under SA. P_D is the probability that an abnormal state is declared when the system is indeed in an abnormal state. P_F is the probability that an abnormal state is declared when the system is actually in a normal state.

Strictly speaking, a GPS user should not be concerned with how large satellite range errors may be unless the position accuracy requirement cannot be met for that particular phase of flight. The observations in the comparison methods, however, are directly related to satellite range errors, as was shown earlier, but not to the user position error. For this reason, an abnormal state is defined here in terms of satellite range error rather than user position error.

Other assumptions that form the basis of the analysis are as follows:

1. All satellites have equal probabilities of malfunctioning.

2. Only one satellite is malfunctioning at any one time.

3. Satellite range error being M or -M is equally likely.

4. Both S_i and N_i (equation (2)) are normally distributed with zero mean and standard deviations of σ_s and σ_n, respectively. Thus, $E_i = S_i + N_i$ is also normally distributed with zero mean and a standard deviation σ_e

   where

$$\sigma_e = \sqrt{\sigma_s^2 + \sigma_n^2} \qquad (6)$$

5. If no satellite is malfunctioning, the satellite range errors for all the satellites have mutually independent normal distributions with zero mean and identical variance σ_e^2. If any one satellite is malfunctioning, the satellite range errors for all the satellites, other than the malfunctioning one, have mutually independent normal distribution with zero mean and the same identical variance.

As shown in Equations (3) and (4), an observation Δq in the two comparison methods is related to satellite range errors in a form

$$\Delta q = a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4 + a_5 E_5 \qquad (7)$$

In the following, P_F and P_D are derived analytically using equation (7) and the assumptions stated above.

Probability of False Alarm Versus Decision Threshold

In the usual state, i.e., if no fixed bias error is associated with any E_i, Δq is also normally distributed with zero mean and standard deviation $\sigma_{\Delta q}$

where

$$\sigma_{\Delta q} = \sigma_e \sqrt{\sum_{j=1}^{5} a_j^2} \qquad (8)$$

Then, from Equation (5) and the definition of P_F

$$P_F = P\,[\,|\,\Delta q\,| > d \mid \Delta q \text{ is } N(0, \sigma_{\Delta q})] \qquad (9)$$

Therefore, for any selected value of P_F, d can be determined from a standard normal probability distribution function table.

Probability of Detection and Missed Detection

In an abnormal state, i.e., if E_i = M or - M, Δq is normally distributed with a mean of $a_i M$ or $-a_i M$ and standard derivation $\sigma_{\Delta qi}$

$$\sigma_{\Delta qi} = \sigma_e \sqrt{\sum_{j \neq i}^{5} a_j^2} \qquad (10)$$

Since E_i being +M or -M is equally likely, the probability of missed detection when the i^{th} satellite is malfunctioning, to be denoted by P_{mi}, is given by

$$\begin{aligned} P_{mi} = {} & \frac{1}{2}\, P\,[\,|\,\Delta q\,| < d \mid \Delta q \text{ is } N(a_i M, \sigma_{\Delta qi})] \\ & + \frac{1}{2}\, P\,[\,|\,\Delta q\,| < d \mid \Delta q \text{ is } N(-a_i M, \sigma_{\Delta qi})] \end{aligned} \qquad (11)$$

Once P_{mi}'s are obtained for all i = 1, 2...5, P_M and P_D can be obtained from

$$P_M = \frac{1}{5} (P_{m1} + P_{m2} + \cdots + P_{m5}) \quad (12)$$

$$P_D = 1 - P_M \quad (13)$$

Numerical Results

For the evaluation, GPS satellite geometries as viewed from the Chicago area over a 24-hour period are used. The analysis assumes the planned GPS constellation consisting of 18 satellites in six planes plus three active spares. Probabilities are computed based on the satellite geometry as viewed from Chicago every half hour and averaged over the entire 24-hour period. Out of 48 samples of GPS satellite geometries, one case in which only four satellites are in view of the user is excluded for the computation. In cases where there are more than five satellites in view, the five satellites that give the best Horizontal Dilution of Precision (HDOP) in combination are selected.

According to the procedure outlined in the preceding section, a P_F value should first be selected to determine the decision threshold. For the example, a P_F value of 0.004 was selected. Then from Equation (9) and from a normal probability distribution function table, the decision threshold d is determined as a function of a_j's, which vary depending on the particular observation picked for the hypothesis test. In the presence of SA, σ_e is obtained from Equation (6) whereas under the absence of SA, $\sigma_e = \sigma_n$. σ_n is chosen to be 9.7 m, which is the inherent C/A code accuracy when Kalman filtering is used to reduce the influence of random error. On the other hand, σ_s is chosen from d_{rms} = 50 m divided by a typical HDOP of l.5. The malfunction detection performance of the comparison methods is shown in Figure 3.

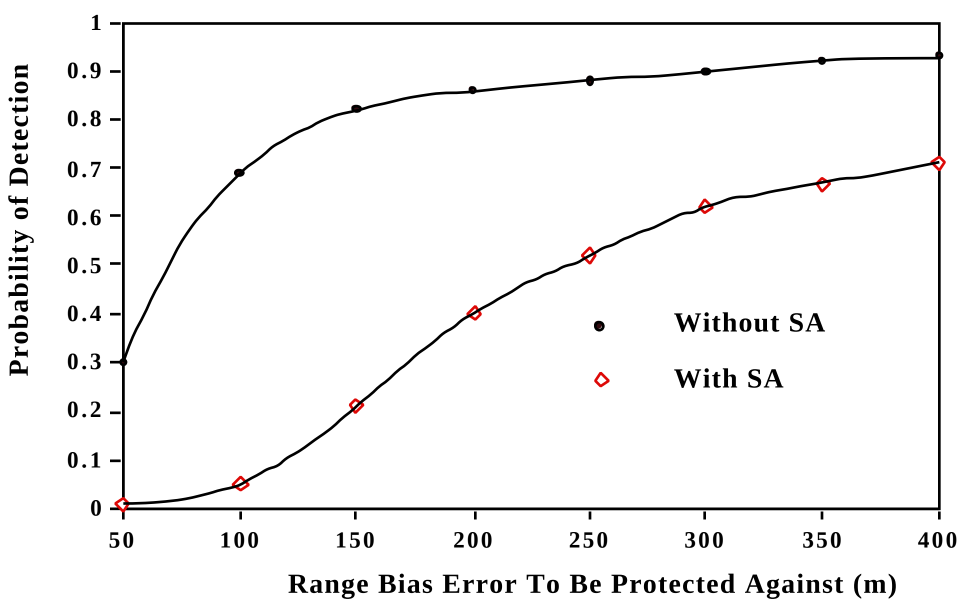


Figure 3. Detection Performance of the Comparison Methods at Chicago (Proability of false alarm = 0.004)

According to the figure, it is obvious that even without SA, the comparison methods alone would not be able to detect an abnormal state with a high enough detection probability. This is because there is a significant fraction of time that the geometry does not support consistency checking among redundant pseudoranges. On the other hand, if used in combination with some other information, the methods should provide improved detection performance. One example would be combining clock bias estimate monitoring with one of the comparison methods.

Monitoring of the user clock bias estimate from a Kalman filter mechanized in the user receiver would be an appropriate means of detecting an abnormally large satellite range error if the user clock stability is known. With a very stable user clock, a clock bias estimate that remains small would indicate a normal state, whereas an estimate that has a large value would indicate an abnormal state. For a user clock that is not very stable, the knowledge of the clock drift should be taken into consideration. In any case, the detection performance of this method is about at the same level as the comparison methods.

Without taking into account the effects of receiver clock instability, Figure 4 shows the improved performance of the combined method. The figure shows that even though both the comparison methods and the user clock bias estimate monitoring method cannot perform very well when applied individually, their combined use can significantly improve performance.

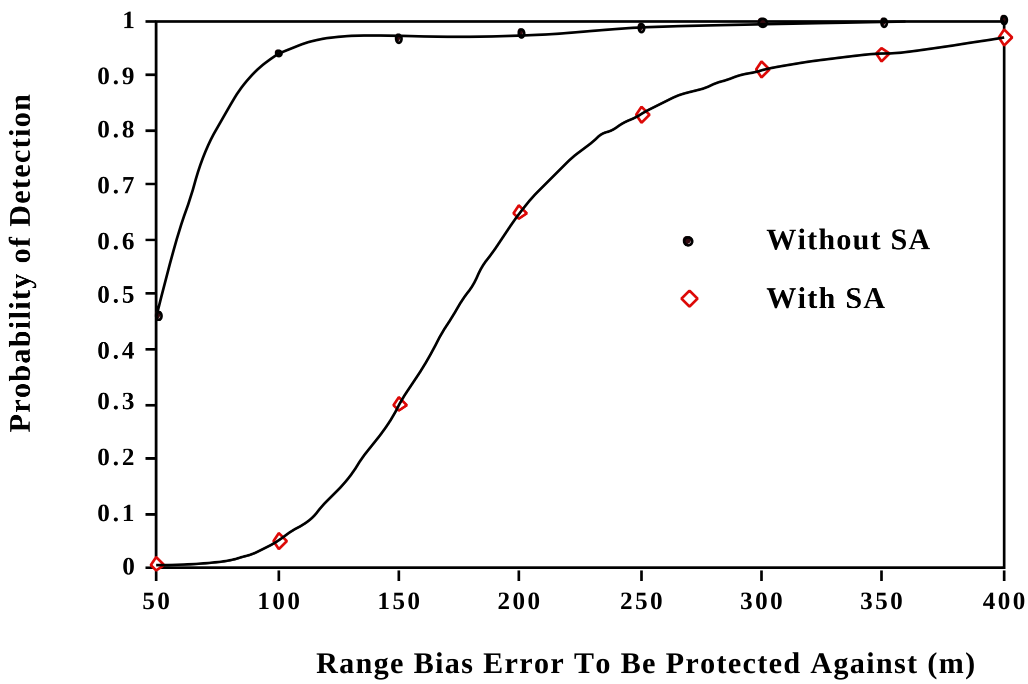


Figure 4. Detection Performance of the Combined Method at Chicago (Probability of false alarm = 0.004)

SUMMARY

The analysis contained in this paper reveals important characteristics of two comparison methods, namely, that all the observations based on five satellite ranging sources in the two methods are on a single dimensional space and that the two methods are mathematically equivalent. It was construed from the analysis that the same characteristics are possessed by any other user receiver evaluation method that depends only upon the pseudorange measurements from five satellites.

With a specific definition of a normal and an abnormal state, a closed form solution is obtained for the decision threshold as a function of probability of false alarm, and the probability of detection is then determined as a function of the magnitude of the pseudorange error to be protected against.

The numerical results show that the comparison methods alone would hardly be adequate in monitoring GPS signal performance even in the absence of selective availability (SA) if our goal is to detect a range bias error larger than a certain magnitude with a sufficiently high probability. A significant improvement was observed, however, when the methods are used in combination with the user clock bias estimate monitoring method. In the presence of SA, the combined method does not appear adequate to satisfy the integrity for nonprecision approach. In the absence of SA, the combined method could have some potential in detecting user position error that exceeds the accuracy requirement for nonprecision approach.

ACKNOWLEDGMENTS

The author wishes to express appreciation to Mr. Ronald Braff and Mr. Curtis Shively, The MITRE Corporation, for their encouragement and many helpful suggestions throughout the study and its documentation. The author also thanks Dr. M. Bakry El-Arini for his thorough review of its technical content.

REFERENCES

1. Braff, R., Shively, C., and Zeltser, M., *Radionavigation System Integrity and Reliability*, Proceedings of the IEEE, Special Issue on Global Navigation Systems, Vol. 71, No. 10, October 1983.

2. Lee, Y. C., *Analysis of Range Comparison Method as a Means to Provide GPS Integrity in User Receiver*, The MITRE Corporation Memorandum W46-M4380, 25 February 1986.

3. Lee, Y. C., *Analysis of Position Comparison Method as a Means to Provide GPS Integrity in User Receiver*, The MITRE Corporation Memorandum W46-M4422, 4 April 1986.

APPENDIX: MATHEMATICAL ANALYSIS OF THE RANGE COMPARISON METHOD USING FIVE SATELLITES

This appendix analyzes the range comparison method assuming five (5) satellites are in view of the user. In this method, the user receiver first estimates user position and clock bias using four satellites at a time. Next, each of the five 4-satellite navigation solutions is used to predict the pseudorange to the fifth satellite unused for that particular solution. Each of the predicted pseudoranges is then compared with the corresponding measured pseudorange. The quantities observed from the comparisons are mathematically related to the satellite range errors in the following.

Referring to Figure 1, vectors formed by user position, satellite position and the center of the earth satisfy the equation

$$\bar{R}_u = \bar{R}_i - \bar{D}_i \tag{1}$$

With $\bar{e}_i$ defined as the unit vector from user to satellite, Eq. (1) becomes

$$\bar{e}_i \; \bar{R}_u = \bar{e}_i \; \bar{R}_i - D_i \tag{2}$$

The range D_i to the satellite can be expressed as

$$D_i = \rho_i - B_u - B_i \tag{3}$$

where ρ_i = the pseudorange to the i^{th} satellite,
B_u = the range equivalent of the user clock offset,
B_i = the range equivalent of the i^{th} satellite clock offset.

Eq. (3) combined with Eq. (2) then gives

$$\bar{e}_i \; \bar{R}_u - B_u = \bar{e}_i \; \bar{R}_i - \rho_i + B_i \tag{4}$$

Repeating Eq. (4) for i = 1, 2, 3, and 4, one obtains linear equations of the form

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} & 1 \\ e_{21} & e_{22} & e_{23} & 1 \\ e_{31} & e_{32} & e_{33} & 1 \\ e_{41} & e_{42} & e_{43} & 1 \end{bmatrix} \begin{bmatrix} x_u \\ y_u \\ z_u \\ -B_u \end{bmatrix} = \begin{bmatrix} \bar{e}_1 \; \bar{R}_1 - \rho_1 + B_1 \\ \bar{e}_2 \; \bar{R}_2 - \rho_2 + B_2 \\ \bar{e}_3 \; \bar{R}_3 - \rho_3 + B_3 \\ \bar{e}_4 \; \bar{R}_4 - \rho_4 + B_4 \end{bmatrix} \tag{5}$$

where $(x_u, y_u, z_u, - B_u)^T$ is the user position and clock bias vector to be determined, and ρ_i's, B_i's and R_i's are quantities either measured or calculated.

Because of the great distances between user and satellites, it may reasonably be assumed that errors in the direction cosines e_{i1}, e_{i2}, and e_{i3} that constitute the matrix on the left-hand side of the above equation are negligible. Therefore, the solution errors can be expressed as a function of the errors introduced in the right-hand side. The error in each element on the right-hand side is to be defined as ε_i. That is,

$$\varepsilon_i = (\bar{e}_i \ \bar{R}_i - \rho_i + B_i)_{measured\ /calculated} - (\bar{e}_i \ \bar{R}_i - \rho_i + B_i)_{true} \tag{6}$$

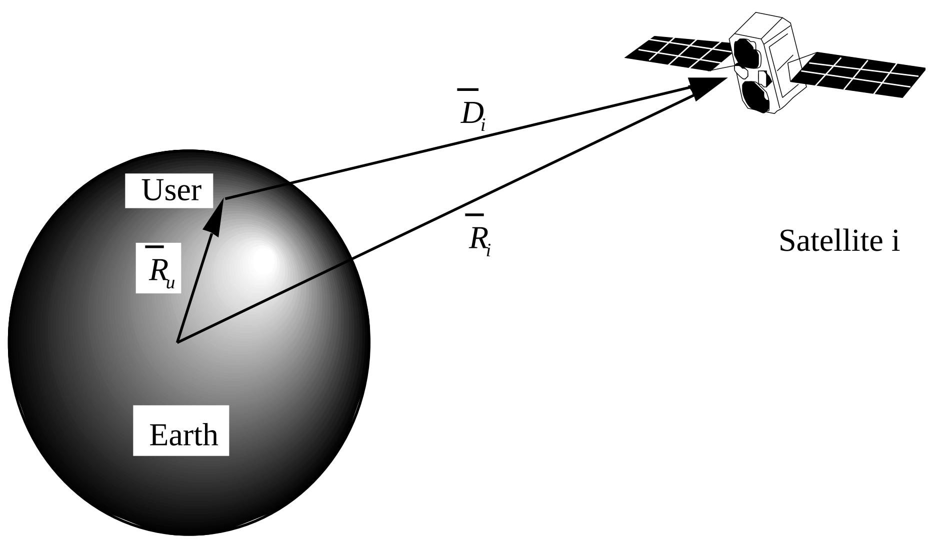
ε_i represents imprecise knowledge of propagation delay, satellite position and satellite clock error as well as Selective Availability, receiver noise, and errors due to a satellite malfunction.

Defining the resulting solution error vector $\Delta\hat{X}_u$ as

$$\Delta\hat{X}_u \equiv (\Delta x_u, \Delta y_u, \Delta z_u, -\Delta B_u)^T, \tag{7}$$

the following equation holds:

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} & 1 \\ e_{21} & e_{22} & e_{23} & 1 \\ e_{31} & e_{32} & e_{33} & 1 \\ e_{41} & e_{42} & e_{43} & 1 \end{bmatrix} \Delta\hat{X}_u = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix} \tag{8}$$



$\bar{R}_u$: the vector from the center of the earth to the user

$\bar{D}_i$: the vector from the user to the i^{th} satellite

$\bar{R}_i$: the vector from the center of the earth to the i^{th} satellite

Figure A-1. Geometry for User Position and Clock Bias Estimation

Further defining

$$G_{ui} \equiv \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ : & : & : & : \\ e_{i-1,1} & e_{i-1,2} & e_{i-1,3} & e_{i-1,4} \\ e_{i+1,1} & e_{i+1,2} & e_{i+1,3} & e_{i+1,4} \\ : & : & : & : \\ e_{51} & e_{52} & e_{53} & e_{54} \end{bmatrix} \text{ and } \underline{\varepsilon}_i \equiv \begin{bmatrix} \varepsilon_1 \\ : \\ \varepsilon_{i-1} \\ \varepsilon_{i+1} \\ : \\ \varepsilon_5 \end{bmatrix} \qquad (9)$$

Eq. (8) can be rewritten as

$$G_{u5} \quad \Delta\hat{X}_u = \underline{\varepsilon_5} \qquad (10)$$

Therefore,

$$\Delta\hat{X}_u = G_{u5}{}^{-1} \; \underline{\varepsilon_5} \qquad (11)$$

Therefore, the difference between the measured and predicted pseudoranges to the fifth satellite is

$$\begin{aligned} D_{m5} &= \hat{\rho}_5 - \rho_5 \\ &= \bar{e}_5 \quad \Delta\bar{R}_u - \Delta B_u - \varepsilon_5 \\ &= \begin{bmatrix} e_{51} & e_{52} & e_{53} & 1 \end{bmatrix} \Delta\hat{X}_u - \varepsilon_5 \end{aligned} \qquad (12)$$

Combining (11) and (12), we get

$$D_{m5} = \begin{bmatrix} e_{51} & e_{52} & e_{53} & 1 \end{bmatrix} G_{u5}{}^{-1} \; \underline{\varepsilon_5} - \varepsilon_5 \qquad (13)$$

Equation (13) is an expression for the difference between predicted and measured pseudoranges to the fifth satellite. Repeating the same procedure for the pseudoranges to the other four satellites, four more equations in a similar form are obtained:

$$D_{mi} = \begin{bmatrix} e_{i1} & e_{i2} & e_{i3} & 1 \end{bmatrix} G_{ui}{}^{-1} \; \underline{\varepsilon_i} - \varepsilon_i \qquad (i = 1, 2, .., 5) \qquad (14)$$

Define matrices $\underline{\varepsilon}$ and Q_i as

$$\underline{\varepsilon} \equiv \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix}, \quad Q_i \equiv \begin{bmatrix} 1 & 0 & & & .. & & 0 \\ : & : & : & : & : & : & : \\ 0 & .. & 1 & 0 & .. & .. & 0 \\ 0 & .. & .. & 0 & 1 & .. & 0 \\ : & : & : & : & : & : & : \\ 0 & .. & & 0 & .. & 0 & 1 \end{bmatrix} \qquad (15)$$

↑

where Q_i is a 4 by 5 matrix with all zeros in its i^{th} column.

Then the following relationship holds:

$$\underline{\varepsilon_i} = Q_i \, \underline{\varepsilon} \qquad (16)$$

Equation (14) can then be rewritten as

$$\begin{aligned} V_1\underline{\varepsilon} &- \varepsilon_1 = D_{m1} \\ V_2\underline{\varepsilon} &- \varepsilon_2 = D_{m2} \\ V_3\underline{\varepsilon} &- \varepsilon_3 = D_{m3} \\ V_4\underline{\varepsilon} &- \varepsilon_4 = D_{m4} \\ V_5\underline{\varepsilon} &- \varepsilon_5 = D_{m5} \end{aligned} \qquad (17)$$

where V_i is a row matrix given by

$$V_i = \begin{bmatrix} e_{i1} & e_{i2} & e_{i3} & 1 \end{bmatrix} G_{ui}^{-1} \; Q_i$$

Equation (17) can be put in a matrix form as follows:

$$(V - I) \; \underline{\varepsilon} = \underline{D}_m \qquad (18)$$

where $V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}$, $\underline{D}_m = \begin{bmatrix} D_{m1} \\ D_{m2} \\ D_{m3} \\ D_{m4} \\ D_{m5} \end{bmatrix}$

and I: 5 x 5 identity matrix

If matrix (V - I) were nonsingular, the $\underline{\varepsilon}$ could be computed from (18). It turns out, however, that matrix (V - I) is always singular regardless of the satellite geometry. Moreover, as will be shown below, all the row vectors of the matrix are identical except for the magnitude. That is, (18) always is of the form

$$\begin{aligned} A_1 \varepsilon_1 + A_2 \varepsilon_2 + A_3 \varepsilon_3 + A_4 \varepsilon_4 + A_5 \varepsilon_5 &= D_{m1} \\ a_2 A_1 \varepsilon_1 + a_2 A_2 \varepsilon_2 + a_2 A_3 \varepsilon_3 + a_2 A_4 \varepsilon_4 + a_2 A_5 \varepsilon_5 &= a_2 D_{m1} \\ a_3 A_1 \varepsilon_1 + a_3 A_2 \varepsilon_2 + a_3 A_3 \varepsilon_3 + a_3 A_4 \varepsilon_4 + a_3 A_5 \varepsilon_5 &= a_3 D_{m1} \\ a_4 A_1 \varepsilon_1 + a_4 A_2 \varepsilon_2 + a_4 A_3 \varepsilon_3 + a_4 A_4 \varepsilon_4 + a_4 A_5 \varepsilon_5 &= a_4 D_{m1} \\ a_5 A_1 \varepsilon_1 + a_5 A_2 \varepsilon_2 + a_5 A_3 \varepsilon_3 + a_5 A_4 \varepsilon_4 + a_5 A_5 \varepsilon_5 &= a_5 D_{m1} \end{aligned} \qquad (19)$$

where A_i's and a_i's depend solely on the satellite geometry at the time of the pseudorange measurements. This can be proven as follows:

First, matrix V is examined.

The first row of matrix V is given by

$$V_1 = \begin{bmatrix} e_{11} & e_{12} & e_{13} & 1 \end{bmatrix} \begin{bmatrix} e_{21} & e_{22} & e_{23} & 1 \\ e_{31} & e_{32} & e_{33} & 1 \\ e_{41} & e_{42} & e_{43} & 1 \\ e_{51} & e_{52} & e_{53} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e_{11} & e_{12} & e_{13} & 1 \end{bmatrix} \frac{1}{\Delta_1} \begin{bmatrix} A_{111} & A_{121} & A_{131} & A_{141} \\ A_{112} & A_{122} & A_{132} & A_{142} \\ A_{113} & A_{123} & A_{133} & A_{143} \\ A_{114} & A_{124} & A_{134} & A_{144} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e_{11} & e_{12} & e_{13} & 1 \end{bmatrix} \frac{1}{\Delta_1} \begin{bmatrix} 0 & A_{111} & A_{121} & A_{131} & A_{141} \\ 0 & A_{112} & A_{122} & A_{132} & A_{142} \\ 0 & A_{113} & A_{123} & A_{133} & A_{143} \\ 0 & A_{114} & A_{124} & A_{134} & A_{144} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{1}{\Delta_1}\left(e_{11}A_{111} + e_{12}A_{112} + e_{13}A_{113} + A_{114}\right) \\ \frac{1}{\Delta_1}\left(e_{11}A_{121} + e_{12}A_{122} + e_{13}A_{123} + A_{124}\right) \\ \frac{1}{\Delta_1}\left(e_{11}A_{131} + e_{12}A_{132} + e_{13}A_{133} + A_{134}\right) \\ \frac{1}{\Delta_1}\left(e_{11}A_{141} + e_{12}A_{142} + e_{13}A_{143} + A_{144}\right) \end{bmatrix}^T \qquad (20)$$

where A_{1ij} is the cofactor of the i^{th} row j^{th} column element of matrix G_{ul} and Δ_l is the determinant of the same matrix (G_{u1} was defined in (9)).

The matrix elements in (20) can be simplified by noting the following:

$$e_{11}A_{111} + e_{12}A_{112} + e_{13}A_{113} + A_{114} = \begin{vmatrix} e_{11} & e_{12} & e_{13} & 1 \\ e_{31} & e_{32} & e_{33} & 1 \\ e_{41} & e_{42} & e_{43} & 1 \\ e_{51} & e_{52} & e_{53} & 1 \end{vmatrix} = \Delta_2$$

$$e_{11}A_{121} + e_{12}A_{122} + e_{13}A_{123} + A_{124} \equiv \begin{vmatrix} e_{21} & e_{22} & e_{23} & 1 \\ e_{11} & e_{12} & e_{13} & 1 \\ e_{41} & e_{42} & e_{43} & 1 \\ e_{51} & e_{52} & e_{53} & 1 \end{vmatrix} = (-1) \begin{vmatrix} e_{11} & e_{12} & e_{13} & 1 \\ e_{21} & e_{22} & e_{23} & 1 \\ e_{41} & e_{42} & e_{43} & 1 \\ e_{51} & e_{52} & e_{53} & 1 \end{vmatrix} = (-1)\ \Delta_3$$

Similarly,

$$\begin{aligned} e_{11} A_{131} + e_{12} A_{132} + e_{13} A_{133} + A_{134} &= \Delta_4 \\ e_{11} A_{141} + e_{12} A_{142} + e_{13} A_{143} + A_{144} &= (-1)\ \Delta_5 \end{aligned}$$

Therefore,

$$V_1 = \begin{bmatrix} 0 & \frac{\Delta_2}{\Delta_1} & \frac{-\Delta_3}{\Delta_1} & \frac{\Delta_4}{\Delta_1} & \frac{-\Delta_5}{\Delta_1} \end{bmatrix} \tag{21}$$

Taking the same steps for the remaining rows of matrix V, one finds matrix (V - I) in the following form:

$$V - I = \begin{bmatrix} -1 & \frac{\Delta_2}{\Delta_1} & -\frac{\Delta_3}{\Delta_1} & \frac{\Delta_4}{\Delta_1} & -\frac{\Delta_5}{\Delta_1} \\ \frac{\Delta_1}{\Delta_2} & -1 & \frac{\Delta_3}{\Delta_2} & -\frac{\Delta_4}{\Delta_2} & \frac{\Delta_5}{\Delta_2} \\ -\frac{\Delta_1}{\Delta_3} & \frac{\Delta_2}{\Delta_3} & -1 & \frac{\Delta_4}{\Delta_3} & -\frac{\Delta_5}{\Delta_3} \\ \frac{\Delta_1}{\Delta_4} & -\frac{\Delta_2}{\Delta_4} & \frac{\Delta_3}{\Delta_4} & -1 & \frac{\Delta_5}{\Delta_4} \\ -\frac{\Delta_1}{\Delta_5} & \frac{\Delta_2}{\Delta_5} & -\frac{\Delta_3}{\Delta_5} & \frac{\Delta_4}{\Delta_5} & -1 \end{bmatrix} \tag{22}$$

Proof of (19) is completed by noting that each row of matrix (V - I) is the vector

$$\begin{bmatrix} \Delta_1 & -\Delta_2 & \Delta_3 & -\Delta_4 & \Delta_5 \end{bmatrix} \text{divided by} (-1)^i\ \Delta_i\ .$$

With an appropriate normalization, every single equation of (19) can be reduced to

$$C_1\,\varepsilon_1 + C_2\,\varepsilon_2 + C_3\,\varepsilon_3 + C_4\,\varepsilon_4 + C_5\,\varepsilon_5 = D_m \tag{23}$$

That is, information that can be obtained with the range comparison method is, at best, one linear combination of the five unknown satellite clock offset values. This means that once predicted and measured pseudoranges are compared for one satellite, comparisons for all the other satellites do not generate any more new information.

The analysis in this appendix can be summarized as follows.

- The difference between the predicted and measured pseudoranges for each satellite can be expressed as a linear combination of the five unknown satellite clock offset values.

- The linear equations resulting from the comparisons are identical for each of the five comparisons. This means that, once predicted and measured pseudoranges are compared for one satellite, comparisons for all the remaining satellites provide no new information.

- Therefore, by this method it is only conceivable to detect a situation where one or more satellites are malfunctioning; it is not possible to identify the bad satellite(s).